DOCUMENT RESUME

ED 433 238 SE 062 776

AUTHOR	Truran, John M., Ed.; Truran, Kathleen M., Ed.
TITLE	Making the Difference. Proceedings of the Annual Conference of the Mathematics Education Research Group of Australasia Incorporated (22nd, Adelaide, South Australia, July 4-7, 1999).
INSTITUTION	Mathematics Education Research Group of Australasia, (Australia).
ISBN	ISBN-0-9596844-9-2
PUB DATE	1999-00-00
NOTE	589p.
AVAILABLE FROM	MERGA, Publications Officer, 23/219 Kissing Point Road, Turramurra, New South Wales 2074, Australia ($85 Australian currency).
PUB TYPE	Collected Works - Proceedings (021)
EDRS PRICE	EDRS Price MF03 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS	Algebra; Arithmetic; *Educational Research; Educational Technology; Elementary Secondary Education; Evaluation; Foreign Countries; Mathematics Achievement; *Mathematics Education; Problem Solving; Statistics; *Teaching Methods
IDENTIFIERS	*Australasia

ABSTRACT

Papers from the 22nd Annual Conference of the Mathematics Education Research Group of Australasia Incorporated include the following: (1) "Making Sense of Primary Mathematics" (Gillian M. Boulton-Lewis); (2) "Seeking a Rationale for Particular Classroom Tasks and Activity" (Peter Sullivan); (3) "Research in Mathematics Education in Australia: What it Was--What it Is--What it Might Be" (J.P. Keeves); (4) "Successful Percent Problem Solving for Year 8 Students Using the Proportional Number Line Method" (Shelley Dole); (5) "Cumulative and Exploratory Talk in a Collaborative Learning Classroom" (Mary Barnes); (6) "A Case Study of Teacher Endorsement of an Integrated Learning System" (Annette R. Baturo, Campbell J. McRobbie, Tom J. Cooper, and Gillian C. Kidman); (7) "Enactivism and Mathematics Education" (Andy Begg); (8) "The Writing of Explanations and Justifications in Mathematics: Differences and Dilemmas" (Brenda Bicknell); (9) "The Mathematical Achievement of Children in the Count Me In Too Program" (Janette Bobis and Peter Gould); (10) "Mathematics, ICT and Effective Teaching" (Rod Bramald and Steve Higgins); (11) "Mathematics Teachers and the VCE: Broadening the Educational Landscape" (Christine Brew, Gilah Leder, and Glen Rowley); (12) "An Exploration of Gender Differences in Subject Choice and Belief among Year 12 Students" (Peter C. Brinkworth); (13) "Speaking with Authority in Episodes of Mathematical Discourse" (Raymond A.J. Brown and Peter Renshaw); (14) "Do Games Help the Learning of Probability?" (Tim Burgess); (15) "Invented Algorithms: Teachers Face the Challenge" (Anne Buzeika); (16) "Developing Performance Assessment Tasks in Mathematics: A Case Study" (Rosemary A. Callingham); (17) "Children's Informal Composite and Truncated Partitioning Strategies" (Kathy Charles and Rod Nason); (18) "Jumping to Conclusions: Data Interpretation by Young Adults" (Helen L. Chick); (19) "Conceptual Modelling of Functions by an Experienced Teacher" (Mohan Chinnappan and Mike Thomas); (20) "The Use of Concept Mapping Procedure to Characterize Teachers' Mathematical Content Knowledge" (Mohan Chinnappan, Michael Lawson, and Rod Nason); (21) "Equals, Expressions,

Equations, and the Meaning of Variable: A Teaching Experiment" (Tom J. Cooper, Anne M. Williams, and Annette R. Baturo); (22) "Assessing Diagram Quality: Making a Difference to Representation" (Carmel M. Diezmann); (23) "Enhanced Mathematics Learning: Does Technology Make a Difference?" (Katherine Dix); (24) "Of Course I Can['t] Do Mathematics: Ethnicity and the Stereotyping of Mathematics" (Helen J. Forgasz, Gilah C. Leder, and Tasos Barkatsas); (25) "Student Participation: Phenomenon and Analysis" (Patricia A. Forster); (26) "A Tale of Two Cities: When Mathematics, Computers, and Students Meet" (Peter Galbraith, Chris Haines, and Mike Pemberton); (27) "Technology, Mathematics, and People: Interactions in a Community of Practice" (Peter Galbraith, Peter Renshaw, Merrilyn Goos, and Vince Geiger); (28) "Teachers Exploring Numeracy Learning and Teaching in the Early Years of Schooling" (Ann Gervasoni); (29) "Understanding Mathematical Text through Peer Explanations" (Merrilyn Goos); (30) "Emerging Themes in Statistics Education" (Sharon Gunn); (31) "Mental Addition and Subtraction Strategies: Two Case Studies" (Ann Heirdsfield); (32) "The Development of a Framework of Growth Points to Monitor Students' Comprehension of Algebra in Grades 7-9" (Marj Horne); (33) "Influences on Secondary Mathematics Curriculum in Victoria" (John Horwood); (34) "Deaf Students Solving of Arithmetic Word Problems" (Merv Hyde, Des Power, and Robyn Zevenbergen); (35) "Mathematical Knowledge in a Mathematics Teaching Episode" (Clive Kanes); (36) "Learning Styles in Secondary Mathematics Classrooms" (Stephen Keast); (37) "Grade 4, 6 and 8 Students' Strategies in Area Measurement" (Gillian C. Kidman); (38) "Agency/Numeracy: A Poststructuralist Analysis of the Relationship between Classroom Mathematics and Numeracy in New Times" (Mary Klein); (39) "Assessment of Students' Understanding in Geometry: The Difficulties in Writing Good Questions" (Christine Lawrie and John Pegg); (40) "Extended Common Assessment Tasks in VCE Mathematics: Validity, Reliability and other Issues" (David Leigh-Lancaster and Ken Rowe); (41) "Free Problem-Posing: Year 3/4 Students Constructing Problems for Friends to Solve" (Tom Lowrie); (42) "Children's Generated Word Problems: A Case Study" (K.J. Maguire); (43) "The Learning Networks of Isolated Teachers: The Use of Audiographics Communication Technology to Build a Virtual Mathematics Department" (H. Elaine Mayo and Kevin Hannah); (44) "An Analysis of the Teacher's Proactive Role in Supporting the Development of Symbolizations" (Kay McClain and Paul Cobb); (45) "Learning Calculus with Supercalculators" (Barry McCrae, Gary Asp, and Margaret Kendal); (46) "Teacher Mediation of Student Constructions of Algebra Knowledge" (Brenda Menzel and David Clarke); (47) "How to Teach Generalisations in Mathematics?" (Michael Mitchelmore); (48) "The Conjunction Fallacy and Longitudinal Development of Chance Expression" (Jonathan B. Moritz and Jane M. Watson); (49) "Perceptions of Mathematical Understanding" (Judith A. Mousley); (50) "Graphics Calculators in the Public Examination of Calculus: Misuses and Misconceptions" (Ute Mueller and Patricia A. Forster); (51) "Secondary Mathematics Teachers' Responses to Computers and Their Beliefs about the Role of Computers in Their Teaching" (Stephen Norton); (52) "Visual Representations in First Year Statistics" (Lynne Outhred and Pamela F. Shaw); (53) "An Analysis of Year 12 Students' Performances on Basic Algebra Questions" (John Pegg and Robyn Hadfield); (54) "Some Language Issues in the Assessment of Secondary Mathematics" (Bob Perry, Peter Howard, and Brian Miller); (55) "Computer Algebra Systems Facilitate Positive Learning Strategies" (Robyn Pierce); (56) "The Conflict between Teachers' Beliefs and Classroom Practices" (Mal Shield); (57) "Understanding Decimals: The Path to Expertise" (Kaye Stacey and Vicki Steinle); (58) "Accessibility of Applications Tasks" (Gloria Stillman); (59) "Choices of Computational Strategies" (Paul Swan and Jack Bana); (60) "Reducing the Incidence of Mathematical Misconceptions in 'Middle Band' Students" (Philip Swedosh); (61) "Children's Understanding of the Number System" (Noel Thomas and Joanne

Mulligan); (62) "Differences between the Sexes in Mathematics Achievement in Australia" (Tilahun Mengesha Afrassa and J.P. Keeves); (63) "The Cross-Cultural Perspective of Teachers' Beliefs and Their Influence on Teaching Practices: A Case Study of Two Teachers Teaching Secondary Mathematics in Australia and Indonesia" (Theresia Tirta Gondoseputro); (64) "Graphics Calculators in Victorian Secondary Schools: Teacher Perceptions of Use" (Patrick Tobin, Alla Routitsky, and Peter Jones); (65) "Cabri Geometry: A Catalyst for Growth in Geometric Understanding" (Jill Vincent and Barry McCrae); (66) "The Relationship between the Purported Use of Assessment Techniques and Beliefs about the Uses of Assessment" (Elizabeth Warren and Steven Nisbet); (67) "Longitudinal Understanding of Conditional Probability by School Students" (Jane M. Watson and Jonathan B. Moritz); (68) "Mathematics: More Time, More Ticks!" (Martin Watts, Sue Miller, and Leo Crameri); (69) "Teacher Action Theories and the Use of Group Work in Upper Primary Mathematics Classrooms" (Allan White); (70) "Exploring Diversity: Year 2 Students' Responses to Questions Concerning Simple 2D Shapes" (Jacquelyn Whitland and John Pegg); (71) "Novice Students' Conceptual Knowledge of Statistical Hypothesis Testing" (Anne M. Williams); and (72) "Students' Optimism, Pessimism and Achievement in Mathematics: A Longitudinal Study" (Shirley M. Yates). (ASK)

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THE MATHEMATICS EDUCATION RESEARCH GROUP OF
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Making the Difference

Proceedings of the Twenty-second Annual Conference of
The Mathematics Education Research Group of Australasia Incorporated,
held at Adelaide, South Australia, 4–7 July, 1999

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John M. Truran
Kathleen M. Truran
(editors)

MERGA 22
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Copyright ©	1999, The Mathematics Education Research Group of Australasia Incorporated
Published at	Sydney, New South Wales
Published by	The Mathematics Education Research Group of Australasia Incorporated
Editors	John M. Truran, Kathleen M. Truran
Title	*Making the Difference*. Proceedings of the Twenty-second Annual Conference of The Mathematics Education Research Group of Australasia Incorporated, held at Adelaide, South Australia, 4–7, July, 1999
Desk-top Publishing	Painters prints Publishing, 82 Springbank Road, Panorama, South Australia, 5041
Printed by	Image & Copy Centre, The University of Adelaide
ISBN	0-9596844-9-2

Copies of these Proceedings are for sale at the following prices which include Postage and Packing: Australia, $A60.00; New Zealand, $A60.00; Rest of World, $A 85.00

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PREFACE

The Twenty-Second Annual Conference of the Mathematics Education Research Group of Australasia (MERGA) was held in Adelaide, South Australia, 4–7 July, 1999. The Conference attracted papers from researchers in Australia, New Zealand, the United Kingdom and the United States of America. Contributions from a large number of established researchers—many with substantial international reputations—as well as a significant number of beginning researchers were included in the programme.

The Proceedings of the Conference have been prepared after a refereeing process which conforms with current practices for ensuring that the quality of the individual papers is of a high standard, and constitutes scholarly research which is providing an addition to knowledge. The Proceedings are published by MERGA and are available for purchase from the MERGA Publications Officer, whose address is given elsewhere in this publication. Research studies reported at the conference encompassed a variety of mathematics education related topics and made use of a wide range of research methodologies and theoretical perspectives. Published presentations were organised into Keynote Addresses, Research Reports, and Abstracts of Short Communications. In accepting Research Reports for the Conference, every effort was made to be as inclusive as possible of the research needs and interests of members of the mathematics education research community.

Refereeing processes for the conference were overseen by an Editorial Committee comprising the Co-Editors, John Truran and Kath Truran (University of South Australia), Brian Sherman and David Wither (University of Adelaide) and Peter Brinkworth (The Flinders University of South Australia).

Potential referees were selected from the MERGA membership list by the Co-Editors, and approved by the Editorial Committee. The normal criterion for being considered as a referee was having a minimum of two refereed papers accepted at two earlier MERGA conferences. This criterion excluded a small number of very distinguished members, and in such cases their extensive non-MERGA publications were taken as much more than appropriate.

Authors who wished to submit a Research Report were required to submit the full text of their paper to double-blind peer reviewing by three referees. As far as was possible the research interests of referees were taken into account when papers were allocated for review. Referees were asked to heed guidelines for published papers consistent with those set out in the call for papers and to provide their written comments in a standard format under the following suggested headings:

1. Statement of problem/issues and discussion of its significance
2. Literature review and theoretical framework
3. Methodology and data analysis (where applicable)
4. Discussion
5. Conclusions and implications
6. General comments on the paper

All referees' reports were read by at least one of the Co-Editors, and in normal cases the majority decision obtained. In a few cases both Co-Editors read all the reports and also the original paper before making a final decision. In a very small number of cases further advice was sought from members of the Editorial Committee or to the Vice-President (Research). In all cases, the final decision was made by the Co-Editors.

Some Research Reports were submitted as Early Bird papers, which were refereed and accepted in the usual way. In cases where revision of the paper was recommended, authors were invited to resubmit, and the resubmitted articles were re-refereed in the usual way, often by the original referees.

Some Research Reports were submitted as entries for the Practical Implications Award. Refereeing of these papers was organised by the Vice-President (Research) of MERGA, and involved being reviewed by at least four referees chosen by the Vice-President (Research). Similar criteria were used for choosing these referees, and for recommending whether a paper should be accepted for presentation as a Research Report if it was not selected for the Award.

Abstracts for Short Communications were reviewed by the Editorial Committee to ensure their relevance to the Conference.

The aims of the Co-Editors in adopting the methods indicated above were to facilitate high quality feedback to authors and achieve best possible assurance that the conference proceedings conformed to MERGA research guidelines. It was especially hoped that beginning researchers would benefit from the feedback received.

We want to thank all those who generously agreed to act as referees for the conference proceedings—without their support both the conference and the mathematics education research community would have been the poorer.

The School of Education, Magill Campus, University of South Australia, made a room and other facilities available to act as a conference office. We should like to thank Ms Jane Harper, Head of School, for her support and encouragement. We are also grateful to Professor Kevin Marjoribanks, Chair of the Graduate School of Education, University of Adelaide, for lending us one of the School's computers to assist our work.

This year we made use of the services of a commercial desk-top publisher in order to achieve a greater uniformity of style and unity of presentation. There were several difficulties in the publishing process which led to the Co-ordinator's overseeing the publishing. This was done against the Editors' wishes, and against the advice of the MERGA Executive. As a result, the Editors did not have the level of oversight of the production process which they would have wished and not all papers have been published in conformity with the styles set down or at the standard to which they were prepared. The Editors apologise deeply for inaccuracies which have occurred, but are unable to accept the responsibility.

Finally, we thank Brian Sherman, David Wither and Peter Brinkworth for the constructive help and assistance which they gave us. We also than the MERGA Executive for their confidence in and support for our work, and particularly Clive Kanes for his encouragement during the many long dark nights of our souls.

John & Kath Truran
Co-Editors
Adelaide, South Australia
June, 1999

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Robyn Zevenbergen	Griffith University

Contents

Preface iii
List of Referees v
Table of Contents vii

KEYNOTE PAPERS

Gillian M. Boulton-Lewis
Making Sense of Primary Mathematics 3

Peter Sullivan
Seeking a Rationale for Particular Classroom Tasks and Activity 15

J. P. Keeves
Research in Mathematics Education in Australia: What it was- What it is -What it might be 29

PRACTICAL IMPLICATIONS AWARDS PAPER

Shelley Dole
Successful Percent Problem Solving for Year 8 Students using the Proportional Number Line Method 43

RESEARCH REPORTS

Mary Barnes
Cumulative and Exploratory Talk in a Collaborative Learning Classroom 53

Annette R. Baturo, Campbell J. McRobbie, Tom J. Cooper & Gillian C. Kidman
A Case Study of Teacher Endorsement of an Integrated Learning System 60

Andy Begg
Enactivism and Mathematics Education 68

Brenda Bicknell
The Writing of Explanations And Justifications in Mathematics: Differences and dilemmas 75

Janette Bobis & Peter Gould
The Mathematical Achievement of Children in the Count Me In Too Program 84

Rod Bramald & Steve Higgins
Mathematics, ICT and Effective Teaching 91

Christine Brew, Gilah Leder & Glen Rowley
Mathematics Teachers and the VCE: Broadening the Educational Landscape 98

Peter C Brinkworth
An Exploration of Gender Differences in Subject Choice and Belief among Year 12 Students 105

Raymond A J Brown & Peter Renshaw
Speaking with Auithority in Episodes of Mathematical Discourse 113

Tim Burgess
Do Games Help the Learning of Probability? 120

Anne Buzeika
Invented Algorithms: Teachers Face the Challenge 128

Rosemary A. Callingham
Developing Performance Assessment Tasks in Mathematics: A case study 135

Kathy Charles & Rod Nason
Children's Informal Composite and Truncated Partitioning Strategies 143

Helen L. Chick
Jumping to Conclusions: Data interpretation by young adults 151

Mohan Chinnappan & Mike Thomas
Conceptual Modelling of Functions by an Experienced Teacher 159

Mohan Chinnappan, Michael Lawson & Rod Nason
The Use of Concept Mapping Procedure to Characterise Teachers' Mathematical Content Knowledge 167

Tom J. Cooper, Anne M. Williams & Annette R. Baturo
Equals, Expressions, Equations, and the Meaning of Variable: A teaching experiment 177

Carmel M Diezmann
Assessing Diagram Quality: making a difference to representation 185

Katherine Dix
Enhanced Mathematics Learning: Does technology make a difference? 192

Helen J. Forgasz, Gilah C. Leder & Tasos Barkatsas
Of Course I Can[`t] do Mathematics: ethnicity and the stereotyping of mathematics 200

Patricia A. Forster
Student Participation: Phenomenon and analysis 207

Peter Galbraith, Chris Haines & Mike Pemberton
A Tale of Two Cities: When mathematics, computers, and students meet ... 215

Peter Galbraith, Peter Renshaw, Merrilyn Goos & Vince Geiger
Technology, Mathematics, and People: interactions in a community of practice ... 223

Ann Gervasoni
Teachers Exploring Numeracy Learning & Teaching in the Early Years of Schooling ... 231

Merrilyn Goos
Understanding Mathematical Text through Peer Explanations ... 238

Sharon Gunn
Emerging Themes in Statistics Education ... 246

Ann Heirdsfield
Mental Addition and Subtraction Strategies: two case studies ... 253

Marj Horne
The Development of a Framework of Growth Points to Monitor Students' Comprehension of Algebra in Grades 7-9 ... 261

John Horwood
Influences on Secondary Mathematics Curriculum in Victoria ... 269

Merv Hyde, Des Power & Robyn Zevenbergen
Deaf Students Solving of Arithmetic Word Problems ... 275

Clive Kanes
Mathematical Knowledge in a Mathematics Teaching Episode ... 283

Stephen Keast
Learning Styles in Secondary Mathematics Classrooms ... 291

Gillian C. Kidman
Grade 4, 6 and 8 Students' Strategies in Area Measurement ... 298

Mary Klein
Agency/Numeracy: A poststructuralist analysis of the relationship between classroom mathematics and numeracy in new times ... 306

Christine Lawrie & John Pegg
Assessment of Students' Understanding in Geometry: The difficulties in writing good questions ... 312

David Leigh-Lancaster & Ken Rowe
Extended Common Assessment Tasks in VCE Mathematics: validity, reliability and other issues ... 320

Tom Lowrie
Free Problem-Posing: Year 3/4 students constructing problems for friends to solve. 328

K. J. Maguire
Children's Generated Word Problems: a case study 336

H. Elaine Mayo & Kevin Hannah
The Learning Networks of Isolated Teachers: the use of audiographics communication technology to build a virtual mathematics department 343

Kay McClain & Paul Cobb
An Analysis of the Teacher's Proactive Role in Supporting the Development of Symbolizations 351

Barry McCrae, Gary Asp & Margaret Kendal
Learning Calculus with Supercalculators 359

Brenda Menzel & David Clarke
Teacher Mediation of Student Constructions of Algebra Knowledge 365

Michael Mitchelmore
How to Teach Generalisations in Mathematics? 373

Jonathan B. Moritz & Jane M. Watson
The Conjunction Fallacy and Longitudinal Development of Chance Expression 380

Judith A. Mousley
Perceptions of Mathematical Understanding 388

Ute Mueller & Patricia A. Forster
Graphics Calculators in the Public Examination of Calculus: misuses and misconceptions 396

Stephen Norton
Secondary Mathematics Teachers' Responses to Computers and Their Beliefs about the Role of Computers in Their Teaching 404

Lynne Outhred & Pamela F. Shaw
Visual Representations in First Year Statistics 411

John Pegg & Robyn Hadfield
An Analysis of Year 12 Students' Performances on Basic Algebra Questions 418

Bob Perry, Peter Howard & Brian Miller
Some Language Issues in the Assessment of Secondary Mathematics 425

Robyn Pierce
Computer Algebra Systems Facilitate Positive Learning Strategies 431

Mal Shield
The Conflict between Teachers' Beliefs and Classroom Practices 439

Kaye Stacey & Vicki Steinle
Understanding Decimals: The path to expertise 446

Gloria Stillman
Accessibility of Applications Tasks 454

Paul Swan & Jack Bana
Choices of Computational Strategies 462

Philip Swedosh
Reducing the Incidence of Mathematical Misconceptions in 'Middle Band' Students 469

Noel Thomas & Joanne Mulligan
Children's Understanding of the Number System 477

Tilahun Mengesha Afrassa & J. P. Keeves
Differences between the Sexes in Mathematics Achievement in Australia 485

Theresia Tirta Gondoseputro
The Cross-Cultural Perspective of Teachers' Beliefs and Their Influence on Teaching Practices: A case study of two teachers teaching secondary mathematics in Australia and Indonesia 494

Patrick Tobin, Alla Routitsky and Peter Jones
Graphics Calculators in Victorian Secondary Schools: teacher perceptions of use 502

Jill Vincent & Barry McCrae
Cabri Geometry: A catalyst for growth in geometric understanding 507

Elizabeth Warren & Steven Nisbet
The Relationship between the Purported Use of Assessment Techniques and Beliefs about the Uses of Assessment 515

Jane M. Watson & Jonathan B. Moritz
Longitudinal Understanding of Conditional Probability by School Students 522

Martin Watts, Sue Miller & Leo Crameri
Mathematics: more time, more ticks! 530

Allan White
Teacher Action Theories and the Use of Group Work in Upper Primary Mathematics Classrooms 538

Jacquelyn Whitland & John Pegg
Exploring Diversity: Year 2 students' responses to questions concerning simple 2D shapes 546

Anne M. Williams
Novice Student's Conceptual Knowledge of Statistical Hypothesis Testing 554

Shirley M. Yates
Students' Optimism, Pessimism and Achievement in Mathematics: A longitudinal study 561

Abstracts of Short Communications 571

Author Index and Topic Index 579

KEYNOTE PAPERS

MERGA 22
adelaide 99

MAKING SENSE OF PRIMARY MATHEMATICS

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In this address I consider the research undertaken by me and by others which is concerned with how primary school children make sense of mathematics. The focus is on the strategies children use, the effect of processes and materials used by teachers, and the demand that these make on capacity to process information. The issues examined include the use of concrete representations, aspects of number, length and time measurement, and the transition from arithmetic to algebra. The results suggest that the strategies and materials that teachers choose often do more harm than good unless they are used thoughtfully and carefully. On the basis of the results I argue for mathematics to be taught in the most meaningful, straightforward, interesting, way possible.

I am delighted to have been invited to present this paper. It has provided me with the stimulus to review research I have carried out in early mathematics learning and to attempt to make explicit the beliefs and assumptions underlying what I have done and concluded. Some of you might not be so delighted because it is minimally contentious and to a certain extent challenges conventional wisdom.

For more than 20 years I have been concerned with how and why children make sense of mathematics. I no longer teach students to teach mathematics, rather, my concern now is generally with how people learn and that perspective will no doubt shape what I have to say. During those twenty or more years I have seen many innovations and, recycling of the same, and heard confident pronouncements about materials and methods that would improve children's mathematical learning. In fact none of this is new. The debate about how best to teach mathematics and whether or not concrete materials help or hinder the process has a long history (Szendrei, 1996). I have been caught up in believing some of the ideas myself but I have always been sceptical and therefore I have constantly questioned the value of what I, and others, have been doing. This perspective might in a sense be destructive, if you believe in something strongly enough you will usually make it work even if the approach is not perfect, but is that really the most sensible behaviour? So a great deal of the research that I discuss here is of a questioning nature and some of it at the microlevel. It focuses on what children should be able to learn at particular ages and why, what they actually know, and how they make sense of mathematics as a result of, or despite, what is occurring in classrooms or mandated by curricula. I have come to believe that the most important ingredient in teaching anybody anything is to find out what they already know and to persuade them, through activity and discussion, to be interested in changing their conceptions for something more useful and effective. I am a moderate constructivist and probably a latter day phenomenographer. Basically I believe that we need to know what each child knows at any point in time in order to teach effectively. Then we should go about the process in the most meaningful, straightforward, interesting way possible.

There is not really a great deal of content in the primary mathematics curriculum. The real objective in those years, in my opinion, should be to have children understand the basic concepts. This is not achieved by chopping the content up into manageable and often meaningless segments and then trying to support performance with materials that might hinder rather than facilitate the process. Some good mathematics teaching occurs in many classrooms but in others problems are caused which permanently influence children's beliefs and conceptions about learning mathematics and the motivation to do so. Let me give you an example of how this can occur although such a situation would probably not arise in quite such a stark way today.

Many years ago my eldest son came home from school quite upset. When I asked him what was the matter he said that his 'tables had fallen out of his pocket.' 'What do you mean fallen out of your pocket I asked?' 'Well the teacher gave us these tables to learn on a piece of paper and I've lost them.' 'That's no problem, I said, you can use this box of matches to work them out.' Quite a while and much grumbling later he had found out that you could actually derive these things yourself. Before that he was being lead to believe that multiplication tables came from some external authority, that all you had to do was learn them by repetition and give them back, that you could not do anything about them yourself and that there was no meaning or interest in them.

Let me give you a few historical examples of some other problems I have seen. I taught young children and teachers in New Guinea over a number of years. When I first went there in the 60's someone had sold the idea of Cuisenaire rods as a way of teaching number to New Guinean children. It had been demonstrated elsewhere that children could find answers to problems well beyond what one would normally expect of them using the structure of those little coloured rods. It seems that no one really questioned then whether children related what they were doing to any knowledge of the number system as long as they could come up with the right answer. I did not know anything at that time about cognitive capacity or processing demand although I had a vague notion of Piaget's stages. However it seemed to me that using Cuisenaire rods to teach early number concepts was imposing a lot of unnecessary information on children who were learning in a third language and whose counting system did not go much beyond '20' in a base 5 system (that is using a couple of number names in combination to arrive at '*10*' which was the fingers of a hand). So I mostly used the blocks to improve their knowledge of colour names, comparative lengths, and their fine motor skills. I set about approaching numbers and the number system through counting and grouping objects using the base ten system of ordinal and cardinal numbers in English. When I came across these blocks in the Northern Territory in the 80's I suggested they be used for creative activities or paving sections of the floor. This does not mean that I think Cuisenaire rods cannot be useful in teaching more advanced concepts such as fractions but they need to be used in an informed and consistent manner.

The next wave of materials to sweep over us in New Guinea, and subsequently in other parts of Australia, were those devised by Dienes. He came to Papua New Guinea and persuaded the educational hierarchy that use of his Logic and arithmetical materials would teach children to think logically, understand the number system and even to learn algebra with ease. We were lead to believe that his Multibase blocks (which I think actually had their origin in Montessori's beads) were particularly appropriate because there was a number of counting systems in the children's first languages that were in bases other than ten. The work with sets and Logic blocks was great fun but did not seem to lead directly anywhere, and the Multibase blocks in base 10 were only useful if they were used very carefully and consistently, and deliberately related to place value and algorithmic processes As described later they need to be used in the same way with other children to be of any use. The blocks in other bases and the algebra materials were very difficult even for the teachers to make sense of and were quickly abandoned. Again it seemed to me that the path to understanding basic mathematics was being deliberately obfuscated by a clutter of peripheral concepts that increased the processing load, and in most cases caused complete confusion, unless children chose to bypass them all and make their own sense of things.

To a certain extent Dienes (cf. 1960) justified his materials and procedures on the basis of developmental theories such as those of Bruner and Piaget. This caused some of us to believe that if we understood Piaget's work more thoroughly we would be able to devise curriculum content and teaching procedures that would be more effective for most children.

Whilst the notion of stages has been most illuminating in generally understanding what students might be capable of learning at particular ages and has underpinned programs such as the Nuffield Project, it has also been restrictive. It limited, in particular, what people believed could be taught to young children and it did not take into account what individual children might make sense of, and hence the knowledge they might acquire, depending on their capacity to process information and social and cultural factors.

Then came the work of the Neo-Piagetians and the belief that processing capacity at various ages, and the demand of tasks, could be quantified and that these seemingly exact formulae could be applied to children's development and hence to curriculum content to devise more exact matches between them. These theories have been useful and have provided a basis for explaining why some of the material based approaches I described above have not always worked as well as they should. It has provided a theoretical framework for some of my research but it does not allow us to explain fully the idiosyncratic way in which most children do or do not make sense of their environment and do or do not acquire mathematical knowledge.

What emerges generally from the Neo-Piagetian work above is that working memory (or short term memory as it is alternatively or additionally described) has a very limited capacity. In an information processing model of memory this is where conscious thinking occurs. The limited capacity of working memory puts constraints on how much information can be considered at any one time without overloading that capacity. This is particularly the case with young children, where items of knowledge are separate rather than chunked. In some of the research I describe it is obvious that the strategies and materials that are used are likely to overload capacity unless teachers firstly ensure that knowledge of them becomes automatic through practice. If that occurs then some freeing up of working memory will occur and they can be used more effectively.

USE OF CONCRETE REPRESENTATIONS

Teachers are generally persuaded to use concrete materials as representations of mathematical concepts because, according to Halford and Boulton-Lewis (1992), they believe, or have been lead to believe, that they reduce learning effort and serve as memory aids, increase flexibility of thinking, facilitate retrieval of information from memory, mediate transfer between tasks and situations, indirectly facilitate transition to higher levels of abstraction and can be used to predict unknown facts. There are however some disadvantages in their use. Mapping from a concrete representation to a concept imposes a processing load on working memory, particularly if the representation is unfamiliar, and this can interfere with understanding a concept. If a representation is poor it can generate incorrect information. If the teacher does not understand the materials well, uses them infrequently or uses them inappropriately, they also cause problems. If the climate in the classroom is one where they appear to be remedial devices for children with problems then they are not useful either because most children will not want to be seen to need them.

Most of the arithmetic concepts taught in the early years of school, such as addition and subtraction operations, require two way mappings if materials are used. That is, if concrete representations such as MAB (Multibase Arithmetic Blocks) are used they must be mapped into the concept of a quantity and the verbal name for it, and then into the symbolic place value representation of the number. The procedure is not usually as well ordered as this. The child is often expected to start with symbols alone or quantities expressed in word problems, to map the symbols into concrete representations, to perform the operation with the concrete materials and to map the resulting quantity back into symbols (written or verbal). We described (Halford & Boulton-Lewis, 1992) the mappings required for 324 minus 179 using MAB blocks. I will illustrate this process in the presentation to demonstrate why I believe that probably the most difficult approach to two or three digit subtraction, in terms of the processing load, is to use concrete materials, and perhaps counting, to support limited place value knowledge, whilst trying to understand and perform an algorithmic process. The child can end up with a huge confusing mess of blocks, not to mention other confusion, which is a positive deterrent to their use. Alternatively if a child has a good understanding of the basic addition operation, and of place value, then mental strategies should make less of a processing demand because neither the written algorithmic procedure, nor concrete representations of the numbers, need to be considered. These mental strategies, however, at some stage need to be related to written algorithms and calculator procedures.

ASPECTS OF NUMBER LEARNING

Fortuitously I have recently had a paper published in the Journal of Mathematical Behavior (Boulton-Lewis, 1998) which summarises, a little after the event, research I have conducted over the last ten years or so into children's understanding of various aspects of number. If you are interested in following up the details of any of those studies you will find the references listed there. The data were derived from structured interviews and analysed keeping cognitive processing demand in mind. I have proposed that an explanation for some of the difficulties experienced with concrete representations, as well as some of the strategies that teachers and children use, is that insufficient account has been taken of the processing load they impose, over and above the processing load of the basic operation, that their use initially entails. I have also suggested that we do not sufficiently take into account children's existing understanding and their preferred ways of making sense of new material. I will also touch on some research in children's understanding of length measurement and time where it seems that children make sense of these measures themselves with reference to common tools (c.f. Szendrei, 1996) as opposed to educational materials. Then I will briefly describe research in prealgebra and early algebra learning from a similar perspective. The intent of most of all this research was to find how children interpreted symbols and operations, how they represented them with concrete materials and what strategies they used to solve problems.

Place Value

We focused first on understanding of place value by children aged 6 to 8 years in the first three years of school (Boulton-Lewis & Halford, 1992). Amongst other results we found error patterns for reading and interpreting digits in numbers such as 12, 13, 16, and 18 in years 1 and 2 and these were not surprising because of the conflict between the ways in which they are spoken and written in English. Generally there was more success across all years in interpreting and representing numbers such as 20, 30, 40 and even 23 and 27 than the 'teens' numbers. When the number of items of information that must be taken into account to read, understand and represent a number such as 13, as compared with 30, is considered it appears that at least one extra processing step is required. The figure illustrating this can be found in Boulton-Lewis and Halford (1992) and will be discussed in the presentation.

Because of the known difficulties with 'teens' numbers, curriculum guidelines often recommend teaching decade and other numbers before introducing them. Whilst this might avoid problems with 'teens' it fragments the ordinal sequence and probably stops children coming to grips with the meaning of the place value system and the ways in which it is read. A more meaningful approach, we suggested, despite the processing load, is to teach the numbers in normal sequence and to encourage the children and teachers to really talk about these strange numbers and how you represent, read and write them. Nothing new I hear you thinking but does this really happen in classrooms?

Pretest for 20	
Interviewer (I)	'What's this number?'
Child (C)	'Oh, twenty, that's easy. Two tens.' Placed two MABs under 2.
I	'How many tens do you have?'
C	'Two tens and no ones.'
I	'Why do we write the 2 here and the 0 there?'
C	'Ooh because if you put that number there (0 before 2) it would make 2
Posttest for 20	
C	'This is going to be fun, I'll do all different things. Now ten' (One MAB ten). He then put out 6 counters, 2 sticks, and one MAB unit.
I	'What do you have there?'
C	'20…1, 2, 3, 4, 5, 6, 7, 8, 9 (Counted all single materials), 9.'
I	'Nine, so nine and ten make twenty? Are you certain?'
C	'Yes.'
I	'How many tens?'
C	'One.'
I	'How many ones?': 'Nine.'
I	'What does "20" tell you?'
C	'Twenty.'

The most unexpected result with regard to place value was that in years 1 and 2 only half the children changed from using less to more efficient materials to represent tens in numbers. Some children changed from using more to less efficient representations. Such behaviour is illustrated by the following protocols for one child at the beginning and end of year 2.

So what was happening here? The teachers in years 1 and 2 did encourage children to use a mix of materials to represent numbers and that might have caused the change. As a result, by not making strong mappings between digits in the tens place of a number, the spoken number name, and the materials that most efficiently represent 'ten', knowledge of these materials would not become automatic. Hence this would increase the processing load of using such numbers in operations because the child would need to count or think of most of the number in units. There was no problem with such by year 3 but the result suggests that strategies and materials that teachers use with the best intentions can cause problems.

Subtraction

Subtraction was the focus of our next series of studies These studies were conducted with children in years 1, 2 and 3 and subsequently with children in years 4, 5, and 6. We found, in a preliminary study (Boulton-Lewis, 1993a), a sequence of solution procedures similar to that found by Carpenter and Moser (1982, 1984) that is proceeding from material to verbal to mental strategies. By year 3 not many children chose to use written algorithms. They rarely used concrete materials and preferred to use mental strategies including recall of facts and place value explanations and were more successful when they did. This suggested that children did not understand the algorithmic procedures and saw no connection between them and their successful mental strategies. I suggested above that the most difficult way to perform written algorithms was with the use of unfamiliar concrete materials and these results support that proposition. The children were not sufficiently familiar with any concrete representation to know the relations between it and a quantity, numeration or operations. However children demonstrated knowledge of place value and the structure of numbers which they could have been assisted to relate to algorithmic procedures.

We also identified some teaching and curriculum procedures that were causing confusion. These included backward counting, the recommended sequence in the curriculum for teaching algorithms, and children's inability to 'read' symbols such as '-' and '13'.

The backward counting procedure with and without concrete materials increases the processing demand of a subtraction task. It does this because, in counting backwards to subtract, some of the information must be processed in parallel whereas when counting forward the information can be processed serially. It can also be confusing because in order to arrive at the correct solution an extra count must be made at the end of the sequence or else the count must not include the first number of the minuend. Some examples of the kind of confusion that occurred are as follows.

1. 8-3=6 (using counters) I counted backwards 3 places 8, 7, 6 (year 1)
2. 16-8=10 (with fingers and toes and no knowledge of place value) 15, 14, 13, 12, 11 (on right hand) 10 (on left hand), 10, I was right (year 2).

It was recommended in the curriculum guidelines at the time, in years 2 and 3 in Queensland, that 2 and 3 digit subtraction be taught initially without any decomposition. It was also recommended that children should be encouraged to use a range of concrete materials and counting strategies as required. By the end of year 3 it seemed that children had practiced so many subtraction items where no decomposition was required that they could see no use for concrete materials and did not really understand place value either. They often verbalised the numbers in the tens and hundreds columns as units and seemed to think of them as separate single digit operations. Below are two examples of children solving 31-6. The first shows only partial understanding of place value and inefficient use of materials. The second shows good mental strategies and no use of an algorithm.

1. The child read 31-6 correctly, wrote 31, said you can't do this, you can't take 6 from 1.' Asked if materials would help she put out 3 MAB tens and 1 unit, changed 2 tens for units, counted off 6, forgot the other ten and answered 15.
2. The child wrote the operation as a vertical algorithm, did not use it and said 'I took 6 from 30 and that is 4 and then I added 1 on for the 1 in 31 so that's 25.'

The preceding results might have been peculiar to the teaching in that particular school therefore a further study using similar tasks and procedures was undertaken with a sample of children in years 1, 2, and 3 in three schools. The results indicated that children across the three schools were choosing, where possible to recall or solve subtraction tasks orally/mentally without materials. They did this both correctly and incorrectly. When they did use materials the most frequent strategy was forward counting. In a few cases where they chose to use written algorithms they usually did not use materials and mostly used 'buggy procedures' when decomposition was required. Their use of mental strategies was more successful and apparently not related to use of written algorithms. A MANOVA was undertaken for the most frequently used strategies early and late in each year. The details of these can be found in Boulton-Lewis (1993b), however the general developmental sequence was again from material (using objects and counting) to verbal (using only oral counting) to mental (using known number facts and place value) as found in other places by Carpenter and Moser (1982, 1984) and De Corte and Verschaffel (1987). However we found that materials were used consistently throughout the three years, that children tried to recall and use number facts as early as possible, and that by year 3 the mental explanations were mostly based on knowledge of place value and decomposition of tens and hundreds, and were used in preference to written algorithms. In summary the children did not use written algorithms with any confidence or success. Neither were they making connections between these, their mental strategies, and the materials that the teachers wanted them to use. The results were similar to those that Hart (1989) found in the United Kingdom. It is suggested that this is in part because the demand imposed by concrete materials was not considered and addressed. We also believe that teaching subtraction operations for numbers with 2 and 3 digits without the need for decomposition, until they are performed automatically without understanding, obviates the need to consider place value and to use materials that would, if used properly, facilitate understanding.

A further study was conducted for subtraction in years 4, 5, and 6 (Boulton-Lewis, Wilss & Mutch, 1996a). The general frequency of strategy use, in decreasing order, was from material to mental to written strategies in all years. This was similar to the sequence found for years 1 to 3 except that there was now a move to using written procedures. Materials were used more at the beginning of each year and this could have been due to the perceived difficulty of increasingly larger multidigit subtraction items, increasing familiarity with the materials and increasing capacity to process information. This sequence would have been influenced by teachers' behaviour and expectations however the tendency to adopt mental strategies continued despite, or perhaps because of, what teachers do. There was a decline in the use of materials in year 6 and this was probably due to better overall understanding of the subtraction process and recognition that the task can be performed mentally or with a written algorithm. On the basis of the results of this study and earlier work it was suggested that if algorithms were treated minimally in the early years and if much greater use was made of mental strategies, with and without concrete materials, then a range of algorithms could be explored later with greater understanding, and as an enjoyable way of relating to and confirming mental procedures. It has been suggested in the literature that children should be introduced to a range of algorithmic procedures, presumably, so that they can develop a deeper understanding of the process. For example working from left to right to solve subtraction could be tried. Evans (1990) observed second and third grade children working on two digit subtraction algorithms and noted that two thirds began working in the tens column. Baroody (1987) advocated this procedure for carrying out mental computations and Fuson (1990) suggested that subtraction taught from either direction is viable. Decomposition is typically taught to American [and Australian] children as a right to left process. We suggested that interest in the use of algorithms is probably killed when children must rely on the support of concrete materials and are hindered in understanding the process by the demand that makes initially on processing capacity. Children in this study were also using calculators to link mental and written procedures but their use declined over the three years as facts and procedures became better known.

Addition

A study of addition (Boulton-Lewis & Tait, 1994) was undertaken with the same sample of year 1, 2 and 3 children as those for the subtraction research described in Boulton-Lewis (1993b). The results showed similarly that children generally chose to use material strategies and count until year 2, to use the verbal strategies of recall of facts and derived facts until year 2, and then to use place value explanations (mental) or joining of sets of objects and counting (material) as frequently at least as written algorithms in year 3, with or without concrete materials despite the fact that they were being 'taught' to use the written algorithm. When errors were analysed it was apparent that they occurred when children did not understand place value, chose to use written algorithms without materials or without understanding, used place value explanations without materials or just used large amounts of discrete materials. The results indicated, just as they did for subtraction, that children used verbal and mental strategies in preference to formal algorithms and did not want to use materials unless they could not perform the task in any other way. The results are encouraging in that children are making sense of the operations using their own strategies. They apparently reduce the processing load by using no materials or materials only. The results are not so good if the objective as proposed in the curriculum guidelines is to have children relate their prior knowledge of the addition operation to the recommended formal written procedures.

LENGTH AND TIME MEASUREMENT

Length

We (Boulton-Lewis, Wilss & Mutch, 1996b) also interviewed children in years 1, 2 and 3 to determine their use of strategies and devices for measuring length. They were given a range of materials (both common tools and educational materials as described by Szendrei, 1996) and three tasks that required respectively, comparison or direct measurement, direct measurement, and indirect measurement or consideration of size and number of match sticks fixed to cards. They were asked to explain their procedures as they worked on tasks. Our results showed that the most preferred strategy was to use a standard measuring device (common tool) such as a 30 cm ruler initially in an arbitrary way in all years. It was the most frequently chosen device in years 1 and 3 and the second most frequently chosen in year 2 despite the fact that teachers were encouraging children to use arbitrary devices such as hand spans, paper clips, or other discrete objects as recommended in the curriculum. Visual perception, sometimes with counting, was used quite heavily in year 1 and less so in subsequent years.

On the basis of theories of cognitive development, and what we knew about learning in and out of school, we questioned the recommendations in curriculum documents about teaching length measuring to young children. It is suggested that children should initially undertake a range of measurement activities with arbitrary units so that they will see the need for standard units. An analysis of the processing load of such tasks suggests that the procedure would defeat the purpose of what it is intended to achieve. In order to realise that arbitrary units are not reliable children must reconcile the varying lengths and numbers of arbitrary units and reason transitively. Alternatively to use a standard device such as a 30 cm ruler to make a pair by pair direct comparison of lengths of objects is a less demanding task. In addition in earlier research I found (Boulton-Lewis, 1987) that children could successfully measure using a ruler before they could devise a strategy using arbitrary units and before they could reason transitively. Using a standard device also has the advantage that it is a common tool and children see adults using such measures, therefore it appears to be part of meaningful real world activity. Our results confirmed these predictions and showed that children apparently prefer to use a standard measuring device even if they do not understand it fully or use it accurately. We suggested that children should be encouraged to measure directly and indirectly with standard and arbitrary measures from the first years of school. To avoid the kinds of errors of usage associated with standard measures described in the literature such as incorrect alignment of the ruler and the object to be measured, focussing on the numbers on the ruler rather than the standard unit of measurement, starting at a point other than zero, and leaving gaps between rulers, teachers should discuss explicitly with children how to use standard measures. It seems that these results also provide a strong case for the more complex concept of the construction of, and need for, a standard measuring device to be left until children are about 8 or 9 years of age. It also suggests that measuring activities would provide more interest for children if they were allowed to use standard measures in more or less real world situations.

Time

Children from years 1-6 were tested for their ability to read and record analogue and digital times (Boulton-Lewis, Wilss & Mutch, 1997). The children in year 6 were also asked to describe their strategies. Based on a review of the literature and cognitive theories a sequence for acquisition of time concepts was proposed. This was hour, half hour, quarter hour, five minute and minute times. It was proposed that times after the half hour would be more difficult and digital times would be learned sooner. This sequence was confirmed for years 1-3 but irregularities occurred in years 4-6 where it was obvious that there was greater individual difference in ability to read and record analogue times. Although formal instruction for reading time is not recommended until year 2 there was an improvement in reading time from years 1-3 and most children could read hour times in year 1. Minute times proved to be the most difficult although it was recommended in the curriculum that these be taught in year 2 along with hour times. The majority of children achieved success earlier for digital reading than analogue reading. This did not necessarily mean that they understood what they were reading. By year 4 it was apparent that understanding of both representations of time was developing as children were starting to use strategies such as calculating based on 60 minutes and referring to the analogue clock when explaining the reading and recording of digital times. In years 4-6 a major error was in reading the hour hand for analogue times and this suggests the need for explicit instruction. The results show that children know more about time at an earlier age than teachers and curricula give them credit for, that strategies could be investigated to determine those which produce greater understanding and that teachers should not assume that all children understand all aspects of reading and recording time. The results made eminent sense in terms of the difficulties involved, the information processing analysis and the kind of time representation that children see being used around them.

TRANSITION FROM ARITHMETIC TO ALGEBRA

We have carried on the research in the use of materials and strategies and applied it to prealgebra and early algebra learning in the last year of primary school and the first two years of secondary school. Although this exceeds the brief that I have, to talk about primary school mathematics, I want to mention briefly some of our results. Some of the problems that we found in understanding prealgebra and algebra concepts relate to the use of concrete representations and poor understanding of basic arithmetic operations originating earlier in the primary years. When we undertook a pilot study of the use of concrete materials to teach algebra in year 8 (Boulton-Lewis, Cooper, Atweh, Pillay, Wilss & Mutch, 1997a) we found that students did not use the procedures taught to them for concrete representations, no students used materials voluntarily, and the preferred mode of thinking to solve unknowns in equations, on both pre and posttests, was to use arithmetic strategies mentally both correctly and incorrectly. We explained the lack of use of materials primarily on the basis of the processing load associated with them. In the school where we undertook this study we found that a range of representations and strategies had been recommended to teachers to facilitate the teaching of algebra. We believe that some of them would be useful and that some are questionable (for example the use of different coloured counters to represent directed numbers and variables in equations). MacGregor and Stacey (1995) asserted that 'many recommendations in the pedagogical literature [for teaching algebra] ... have no supporting research background' (p82). They found for example that the use of patterns in primary and later years as a foundation for algebra did not lead to the understanding that might be expected. We believe that as with all use of representations the relative cognitive demands must be considered. As with other materials they need to be used in a consistent explicit way in which the concepts related to the materials are abstracted and discussed. We saw them being used in a random and indiscriminate fashion on the implicit assumption that you could use a representation once to represent an aspect of algebra, use something quite different the next time, and expect the children to abstract and integrate the algebraic concepts themselves. We discuss a range of these materials and strategies in (Boulton-Lewis, Cooper, Atweh, Pillay, Wilss & Mutch, 1997b). I will consider one of these here, that is, the use of tables of variables to teach functional relationships between two variables. It has been argued that introducing algebra in this way facilitates its use as a language for expressing relationships. However it requires students to understand the concept of variable, and the syntax of algebra [or arithmetic at least], to take into account the two operations at once and the relationship between them. and in some cases to relate these to concrete representations. It is obviously a more demanding task than the use of a linear equation with one unknown and it is no wonder that MacGregor and Stacey found that many students had difficulty generating algebraic rules from patterns and tables. It is even less surprising when one considers the faulty knowledge of necessary arithmetic operations that we found with year 7 students.

Subsequent to the pilot study we undertook a three year study of the transition from arithmetic to algebra. Only the results for 51 year 7 students in three primary schools are considered here (Boulton-Lewis et al., 1997b). This part of the study provided information about knowledge of operations and operational laws, equals, pronumerals (unknowns), variables and solution strategies for linear equations. It was assumed that the students would have been taught about operations and equals as part of the curriculum in arithmetic but that any knowledge of variables and linear equations would only have been derived intuitively from their knowledge of arithmetic. We found that most of the students had sufficient understanding of the basic binary operations, or sequences of them for subtraction, multiplication and addition, in that order to be able to use them as a basis for algebra. Some students did not have sufficient understanding of division. Only two thirds of the group had sufficient understanding of the inverses of multiplication/division and addition/subtraction. About 20-25% of the sample had sufficient understanding of the correct order of arithmetical operations to allow them to apply this satisfactorily to learning linear equations. With regard to the 'equals' sign in a completed equation only half the students knew that it meant both sides were equal. For an incomplete equation, with a series of operations, all the students believed it meant find the answer. This means that in subsequent algebra learning most would want to find the answer after the sign and at least half would need to learn the concept of equivalence. More than half the sample could solve an equation with □ as an unknown number or knew intuitively that it was like an *x* or *y* despite no explicit instruction in variables. The majority of students knew what *x* was in an equation. When asked to use concrete materials most of the students 'illustrated' arithmetical solutions.

We proposed that the most accessible route to algebra could be through solving simple linear equations arithmetically to find unknowns. We know that this runs counter to most current thinking but it fits with Sfard's operational perspective of algebra (Sfard & Linchevski, 1994) and has the advantage that the transition can be made without the need for concrete representations except in the early stages of learning about numbers and arithmetic and perhaps to illustrate the concept of variable. It is also in keeping with the developmental sequence proposed by Halford and Boulton-Lewis (1992). We have explored a two path model for the transition in other papers (eg. Pillay, Wilss & Boulton-Lewis, 1998).

CONCLUSION

In conclusion I have a few simple suggestions for making mathematics more meaningful for children in primary school and beyond. As I stated at the beginning it seems to me that there is not a great deal of essential content to master and that we should aim for real understanding of the basic material rather than coverage and activities that often do not seem to make any sense. I believe that some curriculum and teaching decisions are giving children the wrong beliefs and conceptions about learning mathematics. We also seem, except in the best classrooms, to be depriving children of motivation by suggesting that many aspects of mathematics are derived from some external authority, are not directly accessible by children themselves and have little to do with any worthwhile activity in the real world. We could remove the artificial constraints from curricula. That is we could set general targets for children to meet but make it clear that they should be encouraged to go beyond these and not be forced to perform meaningless tasks such as 2 and 3 digit subtraction without decomposition. We could also give mathematics real meaning by setting up situations such as those described by Resnick, Bill and Lesgold (1992). They argued that human mental functioning must be understood as fundamentally situation-specific and context-dependent. They assessed the early mathematical knowledge that children brought to school and designed a programme that drew on that knowledge, developed children's trust in their own knowledge, used correct formal notation from the start to link that knowledge to the formal language of mathematics, and introduced key mathematical concepts as quickly as possible (including multi digit subtraction in the first year). At the same time they allowed children to use manipulatives or expanded notation as they chose, encouraged everyday problem solving, and caused children to talk about mathematics rather than just do it. The teaching involved a great deal of discussion guided by the teacher in what they described as a 'kind of cognitive apprenticeship approach'. This obviously assisted the children to use a range of strategies effectively because the year 1 children in their programme, who were not restricted to a limited range of numbers, performed significantly better on standardised tests than a similar group of children who were taught in a more conventional way. So I am suggesting that we rethink what we are doing, remove complicating strategies and unnecessary materials and teach so that all aspects of mathematics in the early years are seen as part of a meaningful system. We should also keep in mind the demand that various operations and strategies will make on children's capacity to process information but at the same time recognise that motivation, provided by ownership of real problems and acknowledgment of children's existing mathematical knowledge, might allow children to explore and make better sense of mathematics by themselves with a little help.

I can still remember the delight evidenced by another of my sons, aged six, when he 'borrowed' his father's measuring tape, climbed without permission to the top of a two metre cement wall and crawled along the top measuring lengths in centicubes (he was using them at school) and used numbers way beyond those prescribed in the curriculum. This was an exciting and meaningful activity for him in more ways than one. I was pleased he survived the experience and learned from it. I am not suggesting that we encourage such dangerous activities but I would like to see children believe they can take some control of what they learn in primary mathematics.

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SEEKING A RATIONALE FOR PARTICULAR CLASSROOM TASKS AND ACTIVITY

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This paper traces the evolution of some research projects that relate to the use of a particular type of classroom task as the focus of the learners' mathematical activity. The tasks used are, on one hand, open-ended to allow opportunities for thinking and creativity, and, on the other hand, content specific to ensure that the focus of pupils' activity is not just mathematics generally but the specific content that the classroom program seeks to address. The research grew out of a particular view of learning and teaching, consideration of the motivation of the learners, a perspective on mathematics as the content focus of the learning, and a recognition of constraints operating in classrooms. These perspectives are summarised first, after which there is a discussion of characteristics of the classroom tasks, and some reports of research into their use. Some implications for teacher education are suggested.

Underlying this article is a fundamental assumption is that views on teaching and learning, perspectives on mathematics, and classroom tasks used are directly linked. The following provides a brief elaboration of issues that inform a particular approach to classroom tasks and explains what implications are drawn from each of these for practice.

The article includes:
- a discussion of learning and teaching,
- a summary of some research on motivation of students;
- elaboration of mathematics as the focus of the learning;
- consideration of constraints on classroom processes;
- factors influencing the structure of classroom tasks;
- description of a particular form, termed content specific open-ended tasks;
- a summary of some research into the use of such tasks; and
- some implications for teacher education.

LEARNING AND TEACHING

The tasks described are intended to build on a social constructivist view of learning, to acknowledge views on learning and thinking derived from studies of expertise, to accommodate aspects of how students can be supported in their learning growth, and a recognition that the orientation of the learner is central to the learning.

A social constructivist view, as summarised by Ernest (1994), recognises that knowing is active, "individual and personal, and that it is based on previously constructed knowledge" (p. 2), and that the knowledge is not fixed, rather it is socially negotiated, and is sought and expressed through language.

Ernest listed among the pedagogical implications that there is a need to be sensitive to learners' previous constructions, to seek to identify errors and misconceptions, to foster metacognitive techniques, and to acknowledge social contexts of learners and content. Other implications seem to be that experiences that allow learners to think and create for themselves, and to have opportunity to discuss their interpretations and develop shared meanings will be productive. In other words, the learners' active and social construction of concepts is important.

A view, sometimes seen as competing with this but which is seen here as complementary, was outlined by Bransford, Brown & Cocking (1999) who argued that we should draw on models of learning and thinking from studies of expertise. They suggested that experts have well organised systematic knowledge that is organised to support understanding.

They have fluent access to knowledge that is generally contextualised and embedded in a conceptual framework. A similar perspective was outlined by Owen and Sweller (1988), who explained that experts draw on a large store of knowledge based on their experience. This is interpreted to mean that students need explicit experiences of a range of mathematical concepts, opportunities to link new knowledge to existing conceptual frameworks, and practice at accessing that information. In other words, planned experiences with key mathematical concepts are necessary.

The role of the teacher in each of these is central. As Lerman (1998) said "the metaphor of students as passive recipients of a body of knowledge is terribly limited: so too is the metaphor of students as all-powerful constructors of their own knowledge, and indeed of their own identities" (p. 70). Lerman argued that social relationships associated with classroom learning must also be considered, including roles for informed peers and the teacher. A useful way of thinking about a role for the teacher was proposed by Vygotsky (1978). He described the zone of proximal development to be the "distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined by problem solving under adult guidance or in collaboration with more capable peers" (p. 86). Vygotsky noted that learning is essentially social, and is the process by which children "grow into the intellectual life of those around them" (p. 88). This defines a key role for expert teachers in supporting learners, and identifies an active dimension to those teachers' role.

Lerman argued that the zone of proximal development is within the learning activity rather than separated from it, and that the task is an essential prompt for learners' activity. He suggested that the zone may actually be associated with the classroom, or the group, rather than necessarily the individual learner.

The contentious issue is how this zone can be identified and activated. One metaphor may be a climbing wall, in which holds are colour coded to indicate difficulty. The particular colour indicates where you should climb and is matched to your desired challenge. If you use a hold of any colour the climb is too easy. Perhaps the colour helps you to scale more challenging paths than would be possible without the guide.

Wood (1989) considered the gaps between what is known, and what is recognised, create space for learning: "By means of a number of 'scaffolding functions', one more knowledgeable than the learner may help to bridge such gaps, essentially by activating problem solving in the child" (p. 60). The role of the teacher in identifying blockages, prompts, supports, challenges and pathways is important, but so is the task. The task can help create awareness of potential learnings, and can even be phrased in such a way as to act as hand holds.

The implications for teaching are that with suitable tasks at the appropriate level of challenge, teachers can support pupils in their learning, by providing any necessary support and direction.

MOTIVATION

A unifying characteristic of each of these implications for classroom experience is the centrality of the learners. Directly related to this is the need for learners to take at least some responsibility for their own learning, a key aspect of which seems to be their motivation or disposition to learn.

In a review of research on motivation, Middleton and Spanais (1999) identified five over arching findings. First, it appears that students who are successful are more likely to perceive engagement in mathematics as worthwhile, particularly if the success is attributable to their ability and effort. The perception that lack of success is due to the lack of ability can inhibit students' motivation to learn. This may operate differentially based on gender (e.g., Fennema & Leder, 1990), and culture (Stigler & Stephenson, 1994).

The second result is that motivation towards mathematics is developed early, is learned, is stable, but is amenable to influence by teachers. It seems that the same applies to students' conceptions of mathematics and the way it is learned (McDonough, 1998).

Third, assisting students to develop intrinsic motivation to learning mathematics is superior to providing extrinsic rewards. McComb and Pope (1994) argued that this is particularly important for "hard to reach" students. Mellin-Olsen (1981), in particular, linked motivation to learn with the quality of learning. He described an I-rationale for learning that arises from a view of the school as a way of improving social prospects which he said has a tendency to produce instrumental learning. An S-rationale is based on a view of what is significant knowledge to the student. Relational learning can result if students see some purpose, either in interest, through perceived usefulness, or even as a goal to some other end, for the learning.

This is linked to the fourth of the Middleton and Spanias (1999) findings that inequities exist in the way some groups of students have been taught to view mathematics.

Fifth, achievement motivation in mathematics can be effected through careful instructional design. This relates to the Middleton's earlier (1995) research that identified arousal, personal control, and interest as important in determining the motivation of students to undertake particular activities.

Middleton and Spanais concluded that:

> when individuals engage in tasks in which they are motivated intrinsically they tend to exhibit a number of pedagogically desirable behaviours including increased time on task, persistence in the face of failure, more elaborative processing, the monitoring of comprehension, and selection of more difficult tasks, greater creativity and risk taking, selection of deeper and more efficient performance and learning strategies, and choice of activity in the absence of extrinsic reward. (1999, p. 66)

The implications of this for classroom teaching are that pupils need opportunities for success, that tasks need to present an appropriate level of challenge and difficulty for the students, that increasing students' sense of control, interest, and arousal can help elicit intrinsic motivation, that classroom tasks in which the students choose to participate will be more productive, and that there is a need for consideration of the social context in which the learning takes place.

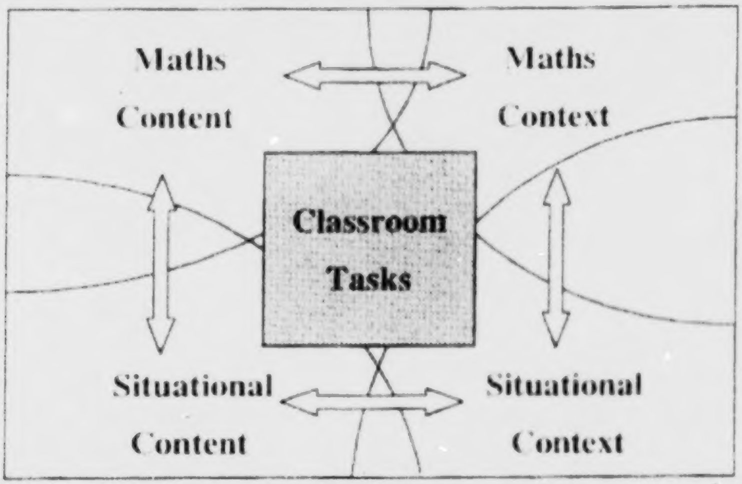
This latter issue is discussed in more detail below, and relates to difficulties experienced by groups of students who are different from the mainstream either socio economically or culturally, and the way that schooling practices are not always compatible with the way such students present and respond.

MATHEMATICS AS THE FOCUS OF THE LEARNING

Another perspective that underlies these classroom tasks is the need to make the mathematics to be learned explicit. There are many tensions associated with this focus. It is widely recognised that situating classroom tasks in contexts within the experience of the students can enhance both the capacity and orientation to engage in working on the tasks (e.g., Australian Education Council, 1991). One tension arises because of the need for students to be able to extract key mathematical elements so that skills and strategies developed can be transferred to other contexts.

The following diagram aims to clarify this notion of the orientation of the tasks. There are four relevant aspects to any problem or task based on a particular situational context, as presented in the following figure:

Figure 1
Relationships between Aspects of Classroom Tasks.



To clarify the intention, the elements are elaborated for the following problem:

> *If each person were to shake hands with each other person in the room once and only once, how many handshakes would take place? (Stacey & Southwell, 1983, p. 38)*

The *mathematical content* of this problem is about recognising that there is a pattern, recognising the n(n-1)/2 relationship, manipulating the symbols, substituting, calculating, and like actions.

The *situational content* is about knowing what is a handshake. This issue is perhaps most relevant to those students with different cultural or linguistic backgrounds, and is sometimes overlooked. Even though this context is probably clear enough, consider how it may be viewed by children who may never shake hands, to women who may shake hands less or greet each other in other ways, to Thai people or to Maoris who may at times use other forms of greeting. This perhaps suggests the possibility of multiple interpretations of particular contexts.

The *situational context* is about people in a room, that you do not shake hands with yourself, and the key variable is the number of people. What we normally think of as real life contexts are included under this heading.

The *mathematical context* is about the concept of trial and error, of strategies such as making the problem simpler, and making a table. Further, a key issue is that we want students to recognise that this context is the same as the mathematical context associated with calculating the number of lines needed to join *n* points on a circle, and of calculating the number of different pairs for table tennis chosen from *n* students. This latter aspect is seen here as the focus for mathematical learning and is often blurred by focus on either the situational context or the mathematical content.

The issue of what is appropriate mathematical content is at the forefront of the "Math Wars" currently raging in the United States. For example, one of the contributors to the debate is a group called *Mathematically Correct*. On their website, a paper by Stevenson (1998) says:

> In our view the NCTM Standards present a vague, somewhat grandiose, readily misinterpreted view of what American children should learn in mathematics. (p. 1)

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> In many cases schools elect to emphasise students' general attitudes and fail to provide them with opportunities to learn fundamental mathematics skills. (p. 1)

Likewise, Allen (1998) argued:

> ... mathematics is given structure and coherence by the bones and sinews supplied by definitions, postulates and proof. Make no mistake, a person's problem solving ability depends on how much mathematics he (sic) understands. ... A student who understands the mathematical theory underlying his solution has a master key that will open many locks. (p. 2).

On the one hand, these opinions highlight the need for the curriculum, and classroom tasks, to address specific mathematics content explicitly. On the other hand, just focusing on the mathematical content is too narrow as is elaborated below. In other words, even though the focus may be on particular concepts, the enacting of these concepts requires attention to generalising, problem solving, creating, linking, and synthesising, and these contextual processes are as much a part of the mathematics curriculum as the specific content.

The implications for classroom tasks in this is that there are key ideas in mathematics which should form the focus of the mathematics curriculum and classroom tasks, and the role of the teacher is to frame a program which builds on the mathematical concepts the students have. Situational contexts can help give the students access to the mathematics, but these should not be the end goal of the learning experience nor should they be allowed to blur either the mathematics content or the mathematical context. Providing classroom tasks that allow consideration of the mathematical context is seen as a key element of the teachers' role. Open-ended tasks, because of the possibility of multiple responses and the formation of generalisations, directly promote thinking about the mathematical contexts.

CONSTRAINTS ON CLASSROOM PROCESSES

A further consideration in structuring classroom tasks is related to classroom processes. A helpful framework suggested by the research of Yackel and Cobb (1996), and further elaborated by Herscovics and Schwarz (in press), delineates two complementary norms of activity in mathematics classrooms. *Mathematical norms* refer to the principles, generalisations, processes and products which form the basis of the mathematics curriculum and which serve as the tools of so much other learning. *Socio-cultural norms* refer the usual practices, organisational routines and modes of communication that impact on approaches to learning, types of responses valued, views about legitimacy of knowledge produced, responsibility of individual learners, and acceptance of risk-taking and errors. These socio-cultural norms are too often under-emphasised.

One aspect of the socio-cultural norms is the very real constraints which students naturally impose on the mathematics teacher. It appears that the reaction of students is foremost in the thinking of teachers in planning, during teaching, and in post lesson reviews. Teachers seek to avoid negative student reactions (Shroyer, 1982), to adhere to stated plans (Peterson & Clark, 1978), to minimize risks (Marland, 1986), and to adjust their teaching to the preferred learning style of the students, who exert indirect but real influence over the teacher (Larson, 1983).

Doyle (1986) noted that this influence of students on teachers tends to be negative. He reported that pupils tended to misbehave during tasks that involved higher-order processes such as understanding, reasoning, and problem formulation because they could not anticipate the response expected by the teacher, and the risk of failure was high. In contrast, students worked efficiently on tasks involving recall of algorithms. Doyle argued that, on the one hand, students tried to reduce their risk of failure by seeking to increase the explicitness of task requirements and to reduce the level of accountability, thereby narrowing the demands of a task. Teachers, on the other hand, tended to react by selecting tasks that were familiar and easy.

Desforges and Cockburn (1987) reached similar conclusions. They examined the influence of classrooms processes on mathematics teaching in particular. Among their findings were that students were not interested in each other's opinions and quickly became fidgety during class discussions. Classroom order was seen as a negotiated contract, and teachers were understandably reluctant to threaten the cooperation of students. Children liked the worksheets, which reinforced the teachers' approach. The more the content moved toward unfamiliar work, the more difficult teaching became. Any classroom teaching strategy should seek to address this directly.

Included among the implications for classroom processes are that reducing the censure commonly associated with failure, and therefore the liability associated with risk, is a key. This can perhaps be achieved by explicit attention to classroom sociocultural processes, along with consideration of tasks for which the risk is reduced by, for example, the allowing the possibility of multiple correct responses. It may also be helpful for teachers to emphasise the way that discussion contributes to learning for the students, and the way in which they can benefit from hearing the perspectives of their peers.

A different but highly relevant aspect of these socio-cultural processes is related to the biases inherent the classroom, and the cultural biases inherent in so called progressive teaching methodologies. As Lerman (1998) commented, classroom discourses both "distribute powerlessness and powerfulness" (p. 76). Zevenbergen (1998) referred to the literature outside mathematics education that has recognised the distinct differences in patterns of language use of socially-disadvantaged students and that of the formal school, but noted that only minimal work on this has been carried out in the field of mathematical learning. Zevenbergen argued that procedural processes can contribute to student disadvantage, and noted that pedagogy often renders invisible cultural norms through which meaning is conveyed. Students from backgrounds where there are discontinuities between linguistic registers and societal aspirations of home and school have to decode aspects of classroom processes. For children whose cultural norms are similar to those embedded within mainstream pedagogical practice, the mathematics is usually more accessible while the converse is true for those students whose culture does not fit the dominant classroom routines. Zevenbergen proposed that one solution that appears to enable these students to gain access to the lesson content is to make socio-cultural norms of pedagogy explicit to teachers and, through them, also to children.

While the full implications for this are not discussed here, it is clear that tasks that allow teachers to discuss with pupils the goals and purpose of their activity, that can be accessed in a variety of ways, and that allow for a wide range of legitimate classroom activity, will facilitate an environment in which the teacher can address these other aspects of socio-cultural processes.

STRUCTURING CLASSROOM TASKS

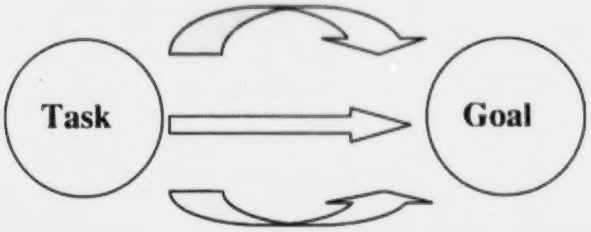
The key argument here is that the above considerations cannot be addressed by emphasising productive classroom strategies such as discussion and group work if the tasks set are restricted to those that only focus on mathematical content or that overemphasise situational contexts. At least some tasks must also address mathematical contexts, as described above, as well as addressing the orientation of the learner and providing opportunities for teachers to address the socio-cultural norms of classroom activity.

Christiansen and Walther (1986) also argued that the tasks posed are critical. They explained that the task set, and the associated activity, serve as the interaction between teacher and learner. They argued that non-routine tasks, because of the interplay between different aspects of learning, provide optimal conditions for cognitive development in which new knowledge is constructed relationally and items of earlier knowledge are recognised and evaluated. It follows that the best tasks are those that provide appropriate contexts and complexity, are open so that they may stimulate construction of cognitive networks; thinking, creativity and reflection, and address significant mathematical topics explicitly.

Sullivan, Warren and White (in press) developed a characterisation of tasks based on the task, the activity, and the goal. They defined the *task* as the statement presented to students that serves as the prompt for their work. Their *activity* is the thoughts and actions in which they engage in response to the prompt. The *goal* is the results the students seek as a product of their activity in response to the task statement. Each has the potential to be open or closed. *Closed* implies there is only one acceptable pathway, response, approach, or line of reasoning. *Open* refers to the existence of more than one (preferably many more than one) possible pathways, responses, approaches or lines of reasoning.

It is assumed here that at least some classroom tasks should stimulate, or at least allow, open activity — that is, students will ordinarily follow different approaches to the task goal. Open activity is seen as fostering some of the more important aspects of learning mathematics, specifically investigating, creating, problematising, communicating, and thinking, as distinct from merely recalling procedures. The following is a representation of this process.

Figure 2
Link between task, goal and activity



This assumption on the openness of open *activity* underlies most literature on problem solving and investigations (see, for example, Anderson, 1996). There is, though, considerable debate about the significance of the openness of the *task* and/or the *goal* (see, for example, Pehkonen, 1997; Wiliam, 1998)

Tasks that have open statements contribute to a broad and rich curriculum and have potential to make a significant contribution to mathematical learning. Nevertheless the tasks discussed here have a focus on specific aspects of the mathematics curriculum. This content specific nature of the tasks implies that the statements need to be closed. The openness in the tasks discussed here arise from the openness of the goals, and the term *open-ended* is used to describe tasks that have such goals.

In other words, the tasks have closed statements, promote open activity, and have open goals or ends.

It should be noted that even though the task statement might be closed, this does not mean that all students must start working on the task in the same place or in the same way.

Indeed the very openness of the goals allows for students of different understanding and backgrounds to approach the tasks in different ways. This point is elaborated further below.

CONTENT SPECIFIC OPEN-ENDED MATHEMATICAL TASKS

It is argued that in order to address the various perspectives on learning, learners, mathematics, and classroom processes, the choice of the task is critical. It is suggested that one option is to consider content specific open-ended tasks.

The term *content specific open-ended tasks* can be illustrated by means of some examples:

> *A number has been rounded off to 5.6. What might be the number?*
>
> *Draw as many triangles (on square paper) as you can with an area of six square units.*
>
> *The mean height of four people in this room is 155cm. You are one of those people. Who are the other three?*
>
> *A ladder reaches 10 metres up a wall. How long might be the ladder, and what angle might it make with the wall?*
>
> *What has the same mass as ten $2 coins?*
>
> *A train takes 1 minute to go past a signal. How long might be the train and how fast might it be travelling?*
>
> *What are some functions that have a turning point at (1,2)?*
>
> *Find two objects with the same mass but different volume.*
>
> *At what times do the hands of a clock make an angle of 90°?*

(For a wide range of such tasks at the primary level, see Sullivan & Lilburn, 1997.)

To explore ways in which such tasks might contribute to a classroom program, it may be helpful to contrast corresponding closed and open-ended tasks. An example of a conventional closed item might be:

> *A rectangle is 10 m long and 5 m wide. What is the perimeter and area?*
>
> A comparable open-ended task is:
>
> *If the perimeter of a rectangle is 30 m, what might be the area?*

This latter open-ended task is different from the conventional perimeter and area question in two major ways. First, it requires a higher level of thinking and engagement than does the conventional question. Traditionally mathematical questions require students to repeat a procedure or recall an algorithm. The open-ended task has the potential to engage students in constructive thinking by requiring them to contrast the related concepts of perimeter and area and to think about relationships for themselves. Another advantage of the open-ended task over the conventional item is that the need for thinking by individual students is made clear to them. The students cannot rely on remembering a rule or simply manipulating formulas. They must think about the concepts, their meaning and the links between them.

Second, the question has more than one possible appropriate answer. Some students might give a single response, others might produce many appropriate answers, and there may be some who will make general statements. The following are some examples of the different levels at which students might respond to the open-ended question:

Type of response	**Sample response**
Single	56m^2
Many	14m^2, 26m^2.36m^2, 44m^2, 50m^2, 54m^2, 56m^2 and even 40.25m^2, 56.25m^2, etc.
General	Area is $l \times w$ where $l + w = 15$

The openness of such questions offers significant benefits to classroom teachers because of their potential for students at different stages of development to respond at their own level. In particular, the openness of the goals allows easier entry to the task, and also allows for ready extension using similar task statements.

It is proposed that among other advantages for classroom teachers are that open-ended tasks are suited to group learning, and they focus the attention of students onto aspects of mathematics such as generalising, and identifying patterns and relationships. This can include introducing new concepts such as fractional measurements used above. Such tasks allow students to be creative, to work with others in responding to set tasks, and to recognise that problems can have multiple solutions. An important feature of the tasks is that learning occurs as an outcome of the students' explorations and thinking, rather than as a result of listening to the teacher (Sullivan & Clarke, 1991). A further characteristic is that such tasks can be created by teachers for any level, for any topic.

It is interesting to note that this very task was criticised by Wu (1994). Wu's views are liberally quoted on the *Mathematically Correct* website (see, for example, Becker & Jacobs, 1998). Wu claimed that the students would be unable to find complete solutions and therefore the task should not be used in lower levels. This seems to be a somewhat restricted and counter-productive view. There is an implication that students should not work on problems unless they are able to produce one complete solution. It ignores the critical fact that students can learn about mathematics from their explorations of situations. Further, a key component of such an approach to teaching is that, in the review of students' work, other students have the opportunity to hear a range of approaches and responses, and so learn that there are multiple answers and scope for further thinking.

Stevenson (1998) also specifically criticised an example of such a content specific open-ended task. The example that concerned him, when paraphrased into Australian language, would read:

I have some coins in my pocket. I put three of the coins in my hand. How much money might I have in my hand?

Stevenson claimed that such problems are "likely to be of little help to the majority of teachers, teaching the majority of our children" (p. 1).

While this particular open-ended task does not necessarily require a higher level of thinking and engagement than its closed alternatives, it still allows the possibility of practice at addition. It shares the advantage of the area task in that it allows multiple answers and ready extension, as well as the potential for use of pattern. I identified 32 different solution totals. The context could also lead on to investigations like:

> *Assuming coins have the same chance of being drawn, what is the expected amount you would have in three coins?*
>
> *Assuming coins have the same chance of being drawn, how could you model this problem?*
>
> *Making no assumptions about the coins, how could you model this problem.*

In summary, it is argued that content specific open-ended tasks have the potential to provide students with opportunities for the very experiences that seem likely to contribute to a rich classroom curriculum.

SOME RESEARCH INTO THE USE OF CONTENT SPECIFIC OPEN-ENDED TASKS

Some research projects have sought to explore the way that students respond to open-ended tasks, and the way that such tasks can be used in classrooms.

Sullivan, Clarke and Wallbridge (1991) asked pupils at a range of grade levels a variety of open-ended tasks. The tasks were posed in various contexts, with some reponding individually, and others working in groups. Sullivan, Clarke and Wallbridge found that:

- pupils can respond to such tasks at a range of levels especially if prompted by the wording of the task;
- the quality of response within a single grade level varied, and student responses improved in higher grades;
- students at all levels gave better answers to questions which drew on content that had been learnt some years previously;
- students can learn from the activity of working on such tasks;
- students often gave only one response even if they could have given more; and
- the degree of difficulty of open-ended tasks varied markedly from similar closed tasks, with open-ended tasks often being more difficult.

They proposed that programs that used predominantly open-ended tasks may need to be supplemented with specific closed skill practice, and that teacher reviews after open-ended activity may need to be explicit for students generally to gain the benefit of their explorations.

Clarke, Sullivan and Spandel (1992) compared responses of students from a range of secondary levels to open-ended tasks drawn from four discipline areas. They found that:

- the inclination to give single responses (or the reluctance to give multiple responses) is a product of schooling, and not peculiar to mathematics;
- the explicit request of multiple responses produces a significant increase in the response level in all academic contexts;
- the capability to give multiple responses increases significantly with year level, except in the case of English;
- that gender-related differences in response level are evident, and where these exist they favour girls; and
- students in both year 7 and 10 possessed a comparable reluctance to provide multiple responses; but when multiple responses were explicitly requested, there was a significant increase in the proportion of multiple responses offered by all pupils in all four academic domains.

Clarke, Sullivan and Spandel recommended that inferences concerning student learning drawn from student performance on open-ended tasks should take into account students' beliefs concerning appropriate answers in academic contexts or risk mistaking student inclination for student capability.

Sullivan, Bourke and Scott (1997) conducted a detailed investigation of a classroom implementation of a program based solely on content specific open-ended tasks. The program was planned with the teacher to ensure that the tasks were suitable for the class, and was delivered in a way that was compatible with the intentions of the study. Most tasks posed were open-ended, and there were few teacher explanations. Generally the students engaged in personal constructive mathematical activity and there were few management or organisational difficulties created by the approach. Observation of individual students and interviews confirmed these impressions and indicated that teaching based on open-ended tasks is suitable both for students who are confident at mathematics and for those who lack confidence. Students showed significant improvement on a test of closed tasks based on the content of the program and the improvement was maintained after the program. Students also showed an overall improvement on the open-ended tasks although the tendency to give multiple correct responses was not maintained after the program.

Sullivan, Warren and White (in press), using methods based on the work of Hay (in press), compared the responses of large numbers of Year 8 students at comparable closed and open-ended tasks. They found that students responded well to some tasks but poorly on others. In most cases the responses were comparable with results on similar tasks previously published.

One of their goals was to explore the effect of situational context on the students' responses. For the closed tasks, the context seemed to help for one task, but made no difference in tasks that included diagrams. It is possible that the diagrams performed a similar role to the context for closed tasks. For the open-ended tasks, the context helped for two tasks, but seemed to make a third task more difficult.

Another of the goals of Sullivan, Warren and White was to compare responses to closed and open-ended tasks. They used a specific prompt to seek multiple responses. In two task types, students found the open-ended tasks more difficult suggesting that these required thinking above and beyond that required for the corresponding closed tasks. In the other case the open-ended task was easier. An analysis of responses and a breakdown of elements of the tasks seemed to explain these differences. The open-ended tasks that were more difficult required students to link two concepts and to use such links to conjecture and generalise. Such open-ended tasks may serve the role of stimulating students' thinking to higher levels.

Both the context and the open-endedness seemed to affect the focus and student response to the tasks. Sullivan, Warren and White suggested that each of the type of tasks could contribute productively to classroom programs. In some cases the open-ended tasks may serve as a useful preliminary exploration of topics; in other cases, they may be better left until later.

Stephens and Sullivan (1997) reported on results from a project that sought to examine a range of aspects of performance assessment in mathematics (see also Beesey, Clarke, Clarke, Stephens & Sullivan, 1998). Open-ended tasks were used as one component of a number of their performance assessments. They reported that teachers were able to apply the scoring rubric to the tasks, that the scoring allowed the tasks and student performance to be evaluated, and that independent judgments could be made validly.

Sullivan, Warren and White (in press) also argued that both closed and open-ended tasks could be used productively in assessment of student learning in that they require different responses from conventional closed questions and reveal different and important information. Where students give multiple responses, this tells the teacher much about the thinking of the students. Where students give some correct and some incorrect responses this reveals information about the quality of understanding of the students. Because the open-ended tasks reveal different information from the closed tasks, they have potential to enrich assessments.

SUMMARY OF IMPLICATIONS FOR TEACHING

It has been argued that the tasks teachers pose as the basis of classroom activity are directly linked to the type of thinking and engagement of learners. A range of considerations for the design of tasks has been described, and it is proposed that content specific open-ended tasks have potential to address many of these background considerations.

The following summarises the teaching implications for classroom tasks indicated above. Classroom tasks should ideally allow teachers to:

- be sensitive to learners' previous constructions;
- identify errors and misconceptions;

- foster self awareness of learning processes;
- acknowledge social contexts;
- allow opportunities for learners to think and create;
- provide source material for discussion;
- exercise leadership of the learning and provide expert advice;
- allow practice at accessing knowledge;
- foster linking new knowledge to previous conceptual frameworks;
- encourage learners to take responsibility for their learning;
- ensure opportunities for success;
- include an appropriate level of challenge;
- increase students' sense of control and choice, interest, and arousal;
- explicitly focus on specific mathematics concepts;
- emphasise key ideas of mathematics in focusing the curriculum;
- attend to problem solving;
- establish a classroom ethos where students can take risks;
- change the nature of right/wrong as the measure of success;
- create opportunity to discuss students' goals and purpose of activity;
- allow a wide range of legitimate activity;
- consider pupils' perspectives of learning and learning environments; and
- collect rich assessment information.

As has been discussed above, content specific open-ended tasks have potential to address most of these ideals explicitly, and certainly allow the teacher to use the tasks as a springboard.

IMPLICATIONS FOR TEACHER EDUCATION

It is relevant to consider the implications for teacher education. This article has outlined an example of how results from research can be used to inform particular approaches to teaching, provide the tools for evaluating effectiveness of classroom strategies, clarify goals and purpose of particular strategies, and facilitate the study of practice. Since content specific open-ended tasks are simple to create, the approach allows prospective and current teachers significant control over the tasks they use in their teaching.

A further issue is such tasks provide a vehicle for considering the management of the complexity of teaching. Mousley and Sullivan (1997) argued that teacher educators should introduce prospective teachers to a range of dilemmas they will face in classrooms, and highlight both awareness of the tensions and the need to seek reconciliation of tensions to allow action. Open-ended tasks provide suitable source material for this.

Mousley and Sullivan proposed a framework in which prospective teachers would consider that choices are not so much about which poles of dilemmas to emphasise, or even about determining which poles are compatible with conventional wisdom, but are about how to use the dilemmas themselves to strengthen both curriculum and pedagogy. Mousley and Sullivan suggested that it is better that teachers become conscious of such choices, and be able to use alternative approaches to advantage. Rather than seeing different processes as dichotomous, teachers can become aware of the continua associated with each dilemma and move along these strategically, according to situational factors operating at particular times.

Mousley, Sullivan and Mousley (1997) used consideration of dilemmas as a central principle in their use of technology to provide case study material for teacher education students. They created a CD based resource that uses an open-ended task to raise problematic teaching situations and enhance awareness that there are dilemmas to be confronted. The basic proposition which underlies the resource is that the study of particular exemplars of quality practice can stimulate reflection on key components of teaching. It is expected that groups of users will investigate and discuss focus questions that aim to draw out varied pedagogical beliefs and to stimulate sensitive responses.

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Endnote

[1] The assistance and comments of Phil Clarkson, Marj Horne, Andrea McDonough, and Judy Mousley on drafts of this paper are gratefully acknowledged.

RESEARCH IN MATHEMATICS EDUCATION IN AUSTRALIA: WHAT IT WAS - WHAT IT IS - WHAT IT MIGHT BE

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I am greatly honoured to have been invited to speak at this MERGA Conference in Adelaide, and I regret that this is the first of the 22 conferences that I have attended. I have chosen as the title of my talk today one that is derived from the title of books by F.W. Westaway. When I started teaching mathematics and science in Adelaide schools a little over 50 years ago, I was guided by the Mathematical Association booklets and books by Westaway on '*Craftmanship in the Teaching of Elementary Mathematics*' and '*Science Teaching - What it was - What it is - What it Might Be*'. These publications were built on the collective wisdom of countless mathematics and science teachers at universities and schools in England. Today, guidance is drawn from the findings of Australian research in the field of mathematics education that emphasizes not only the teaching of mathematics but also the learning and application of mathematics in schools and a world of science and technology.

The contribution of research to mathematics education is relatively recent. Only 30 years ago, 500 copies of book reporting the findings of research including the relationship between curricular time and achievement that I had written were pulped, with only a few copies being sent overseas, and the remainder destroyed. This was done because a Director General of Education who had reduced the time given to mathematics in the curriculum of the schools in his state could not approve a publication (Keeves, 1968) that reported on the relationships between mathematics achievement and curricular time. Research could not be reported for open debate if the findings were contrary to declared policy.

Eight years later in April 1976, a senior bureaucrat in Canberra directed that a chapter that I had written should be removed from a report of a research study before publication (Keeves and Bourke, 1976). A colleague challenged this decision with a letter to a newspaper and was led to resign from his position in a research institute. The chapter was concerned with the very low performance in numeracy and literacy of 10 and 14-year-old Aboriginal students in Australian schools. I pay tribute today to this colleague - John Foyster. In the discussion that followed his resignation, John wondered what he could do in the months ahead while he was seeking employment. The suggestion emerged that he should run a conference that would provide opportunities for free and open discussion of research in mathematics education leading to the dissemination and publication of research papers, and perhaps to the establishment of an association concerned with research in mathematics education. John Foyster's initiatives led within a few months to the first of the MERGA conferences, to the establishment of MERGA as an association, and subsequently to its publication program. This was the climate at that time for research in Australian education and mathematics education in particular.

Several changes have taken place that have supported the growth of research in mathematics education in Australia. However, the role of the Mathematics Education Research Group of Australia with its annual conferences and the publication of the papers presented, together with the reviews of research into mathematics education in Australia that are published in a four-year-cycle (Briggs, 1984; Blane and Leder, 1988; Atweh and Watson, 1992; Atweh, Owens and Sullivan, 1996) and the widely acclaimed international journal, the *Mathematics Educational Research Journal*, have led the way in the development of research in mathematics education in this country. The Australian Association of Mathematics Teachers (AAMT) which was formed from the Mathematics Associations in each of the Australian states, has also published in their journal *The Australian Mathematics Teacher*, the findings of research in a form relevant to teachers and has held biennial conferences to maintain communication between the states. In addition, the Mathematics Education Lecturers Association (MELA) has conducted annual conferences for its members with an interest in research, which are now run in collaboration with MERGA.

Research in the area of mathematics education in Australia is both active and well directed to the problems of learning mathematics at all levels of education. It has been greatly strengthened during the past decade by the inclusion of the former teachers' colleges and colleges of advanced education into the university system, with the expectation that members of the staffs of all universities should engage in research. A significant development in recent years has been the appointment of professors of mathematics education at several universities who provide leadership for the sizeable numbers of university staff, and graduate students who are engaged in research into the problems of the teaching and learning of mathematics. The higher degree students who are working in the field of mathematics education have had, in the main, extensive teaching experience, and as a consequence the studies they undertake are concerned with the practical problems of learning and teaching in Australian schools.

Australian research in mathematics education is not subordinate to research conducted in Britain or the United States as it once was. It is conducted with considerable vigour and commitment, but greatly benefits from regular contact with fellow research workers in other parts of the world. Moreover, Australian research workers in the field of mathematics education are well represented at international conferences, are contributors to international reviews of research, publish widely in international research journals and are strongly listed in overseas research citations.

MATHEMATICS EDUCATION - TODAY

The present-day setting which contains the strong program of research activity that has been built up over the past two decades has provided both the foundations and the support for mathematics education in schools that in Australia is undergoing a marked transformation. The regeneration is seen in both the *National Statement and Profile on Mathematics for Australian Schools* (Australian Education Council, 1990; Curriculum Corporation, 1994a, 1994b). The autonomy given to the schools in the mid-1970s and early 1980s in Australia for school-based curriculum development was both inefficient, ineffective, and substantial and widespread changes were required. The need for change came also from the impact of technology, through computers and calculating devices, on how mathematics was being used in industry, commerce and daily life, and the opening up of new fields of applied mathematics. Moreover, the opportunities available through computers for the storage and analysis of large amounts of data required that the ideas of probability and statistics should be taught to all students across all levels of schooling. The move towards a national curriculum in mathematics, as in other areas of the curriculum was perhaps the only way to introduce change effectively. In the late 1980s the Commonwealth Minister of Education took the necessary initiatives to rectify the situation before it was too late.

The work on the development of the *Statement* and *Profile* on mathematics was undertaken with very substantial collaboration and consensus by curriculum specialists, administrators, teachers and teacher educators form across the country. The manner in which the*Statement* and *Profile* were produced is a remarkable collaborative achievement underpinned by scholarly work and research. Nevertheless, the development has been plagued by controversy and beset by fierce contestation in many of the curriculum areas including Mathematics. Unfortunately, there are sometimes obvious shortcomings in the publications produced. However, there seems general agreement that these could and should be rectified, with the profiles made more specific and more useful. While some Australian states would seem not to endorse the *Statement* and *Profile* and examination agencies, supported by university academics whose scholarly domains are under threat, have blocked acceptance of the *Statement* and *Profile* at the highest level of schooling, there is little doubt that the movement towards a national curriculum in mathematics along with parallel development in other areas of the curriculum is established and is a marked advance in Australian education.

The changes that have occurred in this area of mathematics for Australian schools have been inspired and supported by the findings of research and the advancement of knowledge, although research in the fields of the mathematics curriculum and mathematics education was barely recognized as a domain of inquiry in the mid-1960s. The developments that have occurred in Australia are innovative and are a marked advance in a world context with the following key characteristics.

1. The *Statement* and *Profile* on mathematics introduce new content and new skills that arise from advances in knowledge, technology and the changes in the uses of mathematics in industry, commerce and daily life.

2. The developmental sequence of the *Statement* and *Profile* on mathematics order hierarchically both the curriculum content and the learning outcomes in terms of bands and levels in order to attempt to breakdown the former lock-step age and grade structure that existed in Australian education in the mid-1960s.

3. The *Profile* on mathematics that accompanies the curricular statement specifies in detail the performance outcomes for each aspect of the mathematics curriculum in terms that emphasize and guide the assessment of student learning.

4. The *Statement* and *Profile* on mathematics were developed to provide a framework rather than a detailed syllabus and were designed to permit individual students and class groups to progress systematically through an ordered curriculum at an appropriate pace.

5. The *Statement* and *Profile* on mathematics emphasize consistent and regulated learning sequences with (a) an underlying scale of development in mathematics learning and (b) bench marks or standards of performance that should be attained by students as their learning advances.

Some similar developments have occurred in Britain and the United States with respect to the specification of performance outcomes and the identification of attainment levels and bench marks. However, the *Statement* and *Profile* on mathematics forms a coherent and carefully planned curriculum structure for mathematics teaching and learning across all levels of schooling, with a latent scale of achievement to facilitate the assessment of learning and to guide instruction. The tasks and items on the profile can, as a consequence, be validated against a scale of achievement, and content and topics in the statement can be reordered in an appropriate way for instruction and assessment.

THE CONTRIBUTIONS OF RESEARCH

This national curriculum in mathematics introduces into Australian schools many of the findings of research in mathematics education that has been undertaken in Australia and overseas during the past three decades. Key developments that are derived from research involve:

(a) an awareness that all students can learn mathematics, with an increased concern for gender and social equity;

(b) an acknowledgement that attitudes and values associated with the learning of mathematics are of importance, since they not only influence achievement directly, but also influence future participation in mathematics courses as well as the application of mathematics in industry, commerce, and daily life;

(c) an appreciation of the processes of mathematical inquiry, with a recognition that computers and calculators can be used for computation, thereby reducing the emphasis on the learning of algorithms for calculation;

(d) a recognition of the importance of using mathematics during learning, both for purposes of motivation as well for subsequent application to new situations;

(e) an understanding of the importance of the interrelations between educational objectives, methods of teaching, and scaled achievement outcomes in the construction of the *Statement* and *Profile* on mathematics.

It is of considerable interest to examine each of these five domains in which the findings of research have greatly influenced the *Statement* and *Profile* on mathematics and as a consequence the teaching and learning of mathematics in Australian schools. It is, however, important to emphasize that this is only a selection of the relevant research that has been carried out in Australia over the past 30 years in the field of mathematics education.

1. All Students can Learn Mathematics

The studies conducted under the auspices of the International Association for the Evaluation of Educational Achievement (IEA) on mathematics (Keeves, 1968; Rosier, 1980; Lokan et al. 1996, 1997) have identified the effects of gender and social context on learning outcomes in mathematics.

(a) *Gender and mathematics learning*

Leder (1992) undertook a major review of gender differences in mathematics learning from an international perspective. The research evidence shows (Keeves, 1973; Baker and Jones, 1993) marked differences across countries and over time indicating that such differences are likely to be related to the roles of women in society rather than to biologically based factors. Moss (1982) examined the data collected in the First and Second IEA Mathematics studies in Australia and showed that at the lower and middle secondary grade levels gender differences were very small, and that some reductions had occurred with respect to the size of gender differences in mathematics achievement between 1964 and 1978 at the upper secondary school level, where the differences were noticeably larger. Jones (1988) in a follow-up study of the 13-year-old students from the IEA Study in 1978, reported that sex of student as well as attitudes and achievement in earlier grades were found to have noticeable direct effects on participation in both general mathematics and advanced mathematics courses at Year 12 and in subsequent choice of scientific and technological careers.

The debate informed by these research findings together with the findings contained in other reviews of research in the field (Leder, 1984; Barnes, 1988; Leder and Forgasz, 1992; Barnes and Horne, 1996) have maintained interest in the area and supported action research based on the proposition that all students can do mathematics. From the time of publication in 1975 by the Schools Commission of the report *Girls, Schools and Society* (McKinnon, 1975) and a subsequent report a decade later *Girls and Tomorrow: The Challenge for Schools* (Miland, 1985) there have been programs for change to remove all aspects of discrimination associated with gender in schools particularly in the area of mathematics learning.

(b) *Social background and mathematics learning*

Medium size correlations of 0.2 and 0.4 between achievement in mathematics and various measures of home background have been reported from all three IEA Mathematics Studies (Keeves, 1968; Rosier, 1980; Lokan *et al.* 1996). Relationships were also found between social background and retention at school at the post-compulsory grades (Keeves, 1968). Deckers *et al.* (1986, 1991) have also documented the steadily increasing rates of students staying on at senior secondary school levels and undertaking some mathematics courses. Although, there has been a growing expectation that all students should be learning somemathematics at all levels of schooling, Malone (1993) has reported a decline in the popularity of the more demanding mathematics subjects in most parts of Australia. Moreover, Ainley *et al.* (1994) have shown a strong relationship between rates of participation in mathematics-type courses and social background of students at Years 11 and 12 across Australia. More recently Brinkworth and Truran (1998) have investigated the reasons given by students for continuing study or not to study mathematics at the Year 12 level.

Southwell and Khamis (1992) compared responses to attitudinal statements held by primary school students, secondary school students and primary school teachers. They reported that the view that careless errors and a lack of effort contributed to poor performance, appeared to be stronger among secondary school students and teachers than among primary school students. This finding suggested that Australian students would appear to attribute their success more to their efforts, than to their level of ability, a view that is very different from the views expressed by teachers and students in Britain and the United States. Nevertheless, the lower levels of participation and performance of students from lower socio-economic status homes would seem to be an issue warranting attention together with action research programs.

2. Studies of Attitudes and Values

The introductory strand in the *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990) is concerned with 'Attitudes and Appreciations' and states:

> An important aim of mathematics education is to develop in students positive attitudes towards mathematics and their own involvement in it, and an appreciation of mathematical activity. (Australian Education Council, 1990, p. 31)

While difficulties were clearly encountered in specifying the corresponding facets of the profile in this strand across levels, it is evident that attitudes and values are seen to have a key role in mathematics learning.

The First IEA Mathematics Study provided in 1964 the impetus to undertake an examination in Australia of students' attitudes towards mathematics. This study examined several aspects of attitude towards mathematics with well constructed Guttman type scales (Keeves, 1966). Rosier (1980) and Afrassa (1998) have presented evidence to show, at the lower secondary and pre-university levels, a decline over a 14-year period from 1964 to 1978, in students' attitudes towards the importance of mathematics, but a general increase in attitudes towards the ease of learning mathematics. Subsequently, Keeves, (1972, 1986) developed a scale concerned with liking mathematics and in a longitudinal study of change in mathematics performance during the first year of secondary schooling showed the influence of initial attitudes on later achievement and examined the factors that influenced changes in attitudes towards mathematics and achievement over time. Fraser (1980) has documented a decline in attitudes towards mathematics across year levels in Victorian lower and middle secondary schools, as well as the more favourable attitudes held by students towards mathematics than towards some other school subjects, such as social studies and art.

Research studies in the United States have reported the influence of attitudes of 'avoidance of success' and 'fear of success' on mathematics learning and Leder (1979) in a study of Australian high school students reported the following findings concerned with the learning of mathematics: (a) girls were more likely than boys to attribute sex differences in participation in a career requiring the study of mathematics to external factors than could be modified than to innate ability factors; (b) girls had on average higher levels of fear of success than boys; and (c) girls who chose to continue with intensive courses in mathematics, tended to be relatively low on fear of success.

3. New Technology and the Use of Calculators and Computers

New technology has both directly and indirectly transformed the teaching of mathematics in Australian schools and further change must be expected. The initial impetus for reform at the primary school level came from the introduction of decimal currency in the mid-1960s and the subsequent introduction of metric measures in preparation for the widespread use of calculators and computers.

From the findings of local and international studies, the *National Statement on the Use of Calculators for Mathematics in Australian Schools* (Australian Association of Mathematics Teachers, 1987) recommended that calculators should be used at every grade level in Australian schools. Subsequently, longitudinal research was carried out in six schools in Victoria from 1989 to 1993 into the use of calculators in the early years of schooling involving approximately 1000 students. This work by Stacey and Groves (1996), reported very favourably on the effects of the frequent use of calculators in the teaching of mathematics in the first five years of schooling which supported very clearly the policies and practices advanced in the *National Statement on the Use of Calculators.*

Doig, Carss, and Galbraith (1992) reviewed research into the use of computers in the teaching and learning of mathematics and observed that computers were more readily accepted than were calculators. However, they cautioned that the long-term benefits to be derived from the use of computers were unclear. Subsequently Rowe (1993) prepared a monograph on *Learning with Personal Computers* which addressed the issues involved and examined the research associated with learning through computers especially in mathematics. While the costs of providing computers for schools in less affluent areas raises problems of equity for universal computer usage, as each year passes there is more and more acceptance of widespread computer usage at all levels of schooling.

4. Problem Solving, the Processes of Inquiry and Using Mathematics

The *Statement* and *Profile* on mathematics clearly reject the approach to teaching mathematics as a collection of algorithmic procedures, or a logical system in which logical deduction plays a central role. The approach adopted, while in some places lacking in coherence and consistency, draws on many fields of research concerned with the functions of the brain, the roles of concrete experience and hypothetical thinking, problem solving, transfer and student learning. The findings from many fields of research are drawn together to make the learning of mathematics today very different from the teaching of mathematics that most critics experienced as highly intelligent students in a competitive school climate in which the learning of mathematics was only for the very able. The modern world demands that a knowledge and understanding of mathematical ideas and relationships are required by all who complete 12 years of secondary schooling. Consequently, the development of mathematical competence by individual students must occur in a systematic way across the years of schooling.

It is not possible to name the many scholars and research workers who have contributed to the new vision of mathematics learning that is presented in the *Statement* and *Profile* on mathematics. However, reference must be made to some who have made distinguished contributions and the fields in which they have worked: Piagetian theory (Dienes, 1960, 1963, 1964; Collis, 1975a, 1975b); Information processing theory (Halford, 1978a, 1978b; Halford and English, 1996; Boulton-Lewis et al, 1986; Sweller and Chandler, 1994); Constructivist theory (Leder and Gunstone, 1990; Clarke and Kessel, 1995); and Problem solving (Stacey, 1989, 1992; Rowe, 1985; Lawson and Chinnappan, 1994).

The perspective of modern cognitive science, based on neural net accounts of brain functioning provides a comprehensive view of cognition as situated action and rejects the view that cognition is merely symbol processing (Lakomski, 1999). The internalization of knowledge and the development of understanding from this perspective is built upon situated cognition and interactions with other people and physical systems. The instructional approach and the specification of curriculum content in the *Statement* and *Profile* on mathematics would appear to be developed from such perspectives which are derived from cognitive science and cognitive psychology.

5. A Scale of Performance Outcomes

Underpinning both the *Statement* and the *Profile* on mathematics that have been advanced for the teaching and learning of mathematics in Australia is the view that the learning of mathematics lies along a scale of development. This idea derives in part from the work of Biggs and Collis on the assessment of student learning in terms of the Structure of Learning Outcomes (SOLO) Taxonomy (Biggs and Collis, 1982).

Australian researchers have pioneered several aspects of the use of new measurement techniques to develop a scale of performance that is independent of the items or tasks used in assessment and independent of the samples of individual students employed to calibrate an interval scale (Andrich, 1978; Masters, 1982; Wilson and Adams, 1995). These procedures are not restricted to situations where multiple choice test items with simple right or wrong responses are scored, but extend to the use of constructed response questions in which partial credit is given for incomplete responses and to problem solving tasks where levels of performance are assessed. A further use of these scaling procedures has been to validate and calibrate the mathematics profiles for the different strands in the *Statement* and *Profile* on mathematics (Bailey, Adams and Wu, 1992).

It is the introduction of the idea of scales of development and learning into the statements and profiles in the planning of the Australian curriculum that has made the Australian National Curriculum unique in a world where new curricula are being constructed. These perspectives permit the assessment methods to be substantially widened from formal testing to programs which employ a variety of other methods including observations with detailed records in the primary school, and project work in the secondary school in the learning of mathematics. Work is, however, required to develop further appropriate scaling procedures.

This account of the research in mathematics education as it is today in Australia is highly selective. It is simply focused on what is seen to be the major development in mathematics education in Australia during the past decade, namely the preparation and publication of the *Statement* and *Profile* on mathematics. A remarkably wide range of other research studies has been conducted into the teaching and learning of mathematics in Australia. Time does not permit the presentation of the more complete overview which has been prepared and recently published (Keeves and Stacey, 1999).

There is no doubt that the research in mathematics education in Australia today is rich, varied and useful with some contributions that are both unique and powerful in a world context.

RESEARCH IN MATHEMATICS EDUCATION AS IT MIGHT BE

Research into mathematics education in Australia cannot tackle all the questions to be answered and all the problems that are seen to exist. However, it should maintain a broad coverage of the range of problem situations in order to inform effectively both curriculum planning and development in this country as well as teaching and instructional practice and the assessment of student learning. Moreover, it should build on existing research strengths in Australia, and focus on clearly identified problems, such as youth unemployment and the decline in the relative performance of boys. Furthermore, it should seek to profit from the advances and opportunities provided by new technology. In addressing the question concerned with **What it Might Be**, I would merely wish to draw attention to five areas where I think progress can be made over the coming decades.

1. Catering for Individual Differences

Forty years ago W.C. Radford (1961) drew attention to the problems of the proper handling of individual differences to break down the crippling effects of lock-step grade-level instruction. For a decade the resources of the Australian Council for Educational Research were directed towards this problem until efforts were diverted by policies of curricular freedom and school based curriculum development. Unfortunately, the curriculum guidelines provided by the *Statement* and *Profile* on mathematics are not accepted today in all parts of Australia and individual differences in rates of learning can never be appropriately handled without a coherent curriculum framework, which is too complex a task for individual schools and individual teachers to develop. Moreover, with increased mobility between the states and territories of Australia, there is a need for the curriculum frameworks to be adopted across the country.

2. The Use of Technology in Schools

New technology, including the use of computers, calculators and the Web for information storage and retrieval are no longer future developments. They exist and will continue to exist in classrooms and homes. Moreover, new information technologies will emerge over the coming decades. The issue to be addressed is one of how to use them effectively to (a) promote learning; (b) allow students greater control over their own learning; (c) provide effectively for different learning rates among individuals; and (d) inform students and teachers of new knowledge, understandings and applications as they occur.

3. Investigate Changing Conceptions of Learning

Major advances have been and are being made in the fields of cognitive science and learning theory, with respect to (a) memory and the structure of knowledge, (b) problem solving and reasoning, (c) early learning prior to attendance at school, (d) meta cognitive processes and self-regulatory capabilities, (e) participation in cultural experiences (Bransford *et al* .1999). Not only do these advances suggest new designs for learning environments, but also new modes for effective teaching. Educational research needs to undertake the task of integrating these developments for the advancement of learning. Thus educational research should become an integrative science.

4. Scales of Learning

One of the great strengths of educational research in Australia is the widespread acceptance of the idea of scales of learning, and in particular the development of procedures by which measurements can be readily made using such scales. Further research work is required to advance the effective use of these scales in the statement of curriculum objectives, in the design of teaching and learning experiences and in the assessment of learning outcomes through the curriculum profile.

5. The Study of Factors Influencing Learning and Development

With scales to measure learning and procedures to equate measurements made with different assessment instruments, the monitoring and measurement of student learning and development can be undertaken. Furthermore, analytical procedures have been advanced for the investigation of change in performance over time that go beyond the simple test and retest comparison which has uncertain and unknown reliability. These procedures pave the way for the systematic investigation of the factors that influence change, and student learning and development.

CONCLUSION

In conclusion, the establishment of the Mathematics Education Research Group of Australia opened the door for free and unfettered release and discussion of research results, and for peer assessment of research findings removed from the political control of bureaucrats. MERGA over 20 and more years has supported and widely publicized the findings of research in mathematics education in Australia. As a consequence, the program of research in mathematics education is wide, relevant, vital and with significant impact on the teaching and learning of mathematics in schools. Nevertheless, considerable efforts are required to raise the standards of performance in mathematics of students in Australian schools and to ensure that students at all levels of schooling in Australia are learning mathematical ideas and relationships that will increase their employability, their quality of life and their more effective participation in Australian society.

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PRACTICAL IMPLICATIONS AWARD

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SUCCESSFUL PERCENT PROBLEM SOLVING FOR YEAR 8 STUDENTS USING THE PROPORTIONAL NUMBER LINE METHOD

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A review of the literature to guide the development of a coherent teaching program on percent for Year 8 students revealed varied alternatives and little consensus. Working from the premise that fundamentally percent is a proportion, the proportional number line method was created to assist students experience success in percent problem solving, to also promote conceptual understanding of percent as a proportion, and to embody the multiplicative structure of percent situations. Classroom research indicated that students readily adopted the method.

BACKGROUND

The percent symbol % is arguably one of the most frequently sighted mathematical symbols in Australian society. It can be found in shops, on billboards, in newspapers, on television, and is used for a variety of purposes. Commonly in our society, the term *percent* and the percent symbol are used, for example, to convey messages about discounts and savings in commercial transactions, profits and losses in the business world, interest rates in the banking sector, statistical information in media research reports, and so on. The percent notion is firmly embedded in our society. As a mathematical topic, percent translates directly to the real world. However, percent is often misused or misunderstood when applied to the real world (Watson, 1994). Despite its pervasiveness, the percent notion may possibly be poorly understood.

As a topic within the mathematics curriculum, percent is difficult both to teach and to learn (Cole & Weisenfluh, 1974; Parker & Leinhardt, 1995; Smart, 1980). Consultation of the literature to assist in the development of a comprehensive and coherent teaching program reveals a particularly confusing and entangled picture. Percent literature relevant to instruction can be classified as either that which relates to percent concept development, or that which relates to percent calculations and percent problem solving. Suggested teaching approaches for developing the concept of percent include the following: linking percent to common and decimal fractions (e.g., Hauck, 1954), using 10x10 grids to promote mental images of percents (e.g., Bennett & Nelson, 1994; Reys, Suydam & Lindquist, 1992), investigating the special language of percent used in society (e.g., the use of such terms as 100% attendance, 200% attendance) and building estimation skills through exploration of patterns of simple percent calculations (e.g., Cooper & Irons, 1987, Glatzer, 1984), linking percent to ratio understanding (e.g., Brown & Kinney, 1973; Cooper & Irons, 1987), and studying percent expressions as statements of proportion (e.g., Schmaltz, 1977). The above approaches appear to advocate the building of percent conceptual knowledge upon other prior mathematical knowledge, namely, decimal-fractions, ratio, proportion and therefore appear to suggest that percent can be regarded as any or all of these. It could be argued, therefore, that successful development of percent knowledge from any of these suggested approaches would be dependent upon student familiarity with one or more of these prior topics. If percent can be a decimal-fraction, a ratio and a proportion, developing the concept of percent solely from one of these perspectives would appear to be a narrow approach to percent instruction.

As instructional approaches for developing the concept of percent are varied, so too are suggested methods for performing percent calculations and solving percent application problems. As stated by Ashlock, Johnson, Wilson and Jones (1983), percent application problems are of three types each which contain three elements, as follows: (i) finding a part or percent of a number (e.g., 25% of 20 is x); (ii) finding a part or percent one number is of another (e.g., x% of

15 is 5); or (iii) finding a number when a certain part or percent of that number is known (e.g., 20% of *x* is 6). In general, percent application problems give two elements in the percent statement and require students to find the third. (Throughout this report, the three types of percent application problems will be referred to as Type I, Type II and Type III problems respectively). Models, strategies and procedures for solving percent application problems, as presented in the literature, include the use of concrete materials and diagrams, such as fraction/percent overlays and elastic strips (e.g., Weibe, 1986), 10x10 grids (e.g., Bennett & Nelson, 1994; Cooper & Irons, 1987), comparison scales (e.g., Dewar, 1984; Haubner, 1992), as well as strategies for the identification of key words and mnemonic devices (e.g., Boling, 1985; McGivney & Nitschke, 1988). The solution of percent application problems using the above methods requires the use of a variety of arithmetic procedures, including common fraction, decimal-fraction, and percent conversions; whole number multiplication and division; and decimal-fraction multiplication and division. Typically, the models and strategies aim to assist students visualise the process of the calculations employed, but also appear to present a particular perspective of percent as a decimal-fraction, a ratio, or a proportion.

With such controversy in the percent literature on teaching approaches, the question arises as to which method is best, or more fundamentally, should percent be promoted as a decimal fraction, a ratio or a proportion? A comprehensive review of percent literature was undertaken by Parker and Leinhardt (1995) and they provide compelling arguments in relation to these two questions. In terms of planning instruction, they state that to date, there is no single best method for teaching percent offered in the literature. As to the meaning of percent, they state that percent is an elusive, concise concept with multiple meanings and therefore should not be regarded from any single perspective, but that "percent is fundamentally a language of privileged proportion which simplifies and condenses descriptions of multiplicative comparisons" (p. 472). In their discussion of issues pertaining to percent teaching and learning, Parker and Lienhardt appear to suggest that there are three basic components necessary for successful instruction in percent. They state that instruction must serve to assist students to read, interpret and define relationships between percent problem components; that a solid representation of percent is required; and that instruction must aim to build students' understanding of percent as a proportion.

If percent is defined fundamentally as a proportion, it follows that instruction should be based on drawing on students' prior proportional knowledge to build percent knowledge. However, the perceived difficulty of implementing a program of percent instruction from a proportional perspective is that students typically do not have a well-developed concept of proportion at the time they meet instruction in percent (Lo & Watanabe, 1997). The development of the proportion concept and proportional reasoning skills takes a long time and is dependent upon the consolidation of many other prior mathematics topics (Post, Behr & Lesh, 1988). The wisdom of a teaching program which aims to build students' percent knowledge through proportional understanding and reasoning is questionable. If a proportional approach to percent instruction is undertaken, it follows that the material presented must be accessible to students regardless of the level of their proportional knowledge, it must provide the foundation for continual and continued knowledge growth of percent and percent as a proportion, and fundamentally, it must provide students with the means to analyse, interpret and visualise percent situations.

From the literature on percent, selected literature on proportion, and other literature pertaining to teaching and learning mathematics, a new approach to percent problem solving is proposed. It is based on representing percent as a proportion, and designed to enable students to explore all types of percent situations meaningfully and to experience successful percent problem solving.

In this approach, entitled the proportional number line method, the first step is to identify the elements in a given percent problem. As previously stated, percent situations contain three elements, and in this method, they are labelled as the part, the whole, or the percent. Using this method, identification of the part, the whole and the percent as either given or needed in the problem is the initial point of entry into percent problem solving. Identification of elements in this way is similar to using a part-whole schema (e.g., Mahlios, 1988; Wolters, 1983) for identifying addition and subtraction word problems. In this case it can be seen to be a part-whole-percent schema. The purpose of the part-whole-percent schema is to provide assistance with the first stage of percent problem solving which, according to Parker and Leinhardt (1995), lies in reading, interpreting and defining the relationships within percent problems. For representing percent problems, a single vertical number line is utilised in this approach which follows a similar format to the comparison scales suggested by Dewar (1984) and Haubner (1992), where an amount is compared to the percent base of 100 simultaneously in a linear fashion on a dual-scale number line. The dual-scale number line provides a clear image of the proportional relationship of percent situations (Dewar, 1984; Haubner, 1992), and can be used to model all three types of percent situations, including increase and decrease. Once the elements of the percent situation are represented on the dual-scale number line, they can be directly translated into a proportion equation and solved. For this approach, solving the proportion equation occurs using the cross-multiply technique (Rule of Three) via a hand-held calculator. The cross-multiply calculation is utilised simply to enable students to attain solutions to given percent problems. Although it is acknowledged that the cross-multiply technique may be regarded by students as a meaningless arithmetic procedure (Cramer, Post & Currier, 1992), justification for inclusion in this approach is that the proportion equation $^{a}/_{b} = {}^{c}/_{d}$ embodies the proportionality of percent (Post, Behr & Lesh, 1988) and with appropriate instruction, the cross-multiply technique can be taught meaningfully (Robinson, 1981). It was also seen as a means to alleviate cognitive load necessary to free up mental space (Sweller, 1988), and avoid the possible creation of messy errors which can interfere with problem solving (Noddings, 1990).

Solving percent problems using the proportional number-line method occurs in four steps. The first step is the identification of the component parts of the problem situation; the second step is representation of the percent situation on a dual-scale number line; the third step is the translation of the percent situation to a proportion equation; and the final step is calculation of the proportion equation and hence determining the solution to the problem. The four steps in the proportional number line method applied to each of the three types of percent situations are exemplified as follows:

1. Percent situations contain three elements, which can be identified by using the terms *part*, *whole*, *percent*. In any percent problem, two elements are given, and solution requires finding the third. The three types of percent problems give the three different combinations of the three elements as follows:

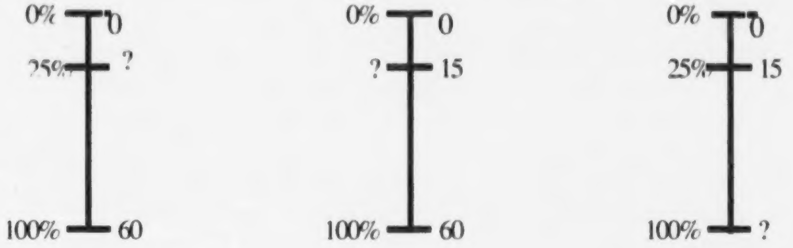
    Type I e.g., 25% of 60 = x part = x, whole=60, percent = 25%
    Type II e.g., x% of 60 = 15 part = 15, whole=60, percent = x%
    Type III e.g., 25% of x = 15 part = 15 , whole=x, percent = 25%

2. The percent situation can be represented on a dual-scale number line, which is a vertical line with the percent scale located on the left and the quantity scale located on the right. Figure 1 shows the positioning of the part, whole and percent elements on the dual-scale number for each of the three types of percent situations.

3. The percent situation can be written as a proportion equation taken directly from the number line. For each of the three percent situations displayed in Figure 1, the proportion equations are as follows: $^{25}/_{100} = {}^{?}/_{60}$ for Type I; $^{?}/_{100} = {}^{15}/_{60}$ for Type II; and $^{25}/_{200} = {}^{15}/_{?}$ for Type III.

4. The Rule of Three procedure enables solution of the proportion equation, and therefore the percent problem. Application of the Rule of Three to the proportion equations generated in Step 3 would proceed as follows: 25 x 60 ÷ 100 = 15 for Type I; 15 x 100 ÷ 60 = 25% for Type II; and 15 x 100 ÷ 25 = 60 for Type III.

Figure 1
Representation of Type I, Type II and Type III Percent Situations on the Dual-scale Number Line.

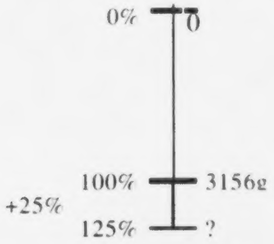


Type I Type II Type III

In the above description, the three different types of percent problems (Type I, II and III) are given to show the different placement of the elements on the number line according to the problem types. In this approach, students do not need to identify whether the percent problem is a Type I, II, or III problem; they merely need to identify elements of the problem in terms of part, whole or percent. For placement of elements on the number line, students need to recognise that the "whole" corresponds to 100% and thus these two amounts are on the same level either side of the number line. The percent and part are on the same level, with the percent located on the scale between 0% and 100% and the corresponding part at the same level on the other side of the number line.

The dual-scale number line of the proportional number line method can be seen to lend itself to representation of percent increase problems. When used in this way, it clearly exemplifies the additive and multiplicative nature of percent increase situations. Figure 2 shows representation of the following percent increase situation: *A baby's mass increased 25% in 3 months from its birthweight of 3156g.*

Figure 2
The Dual-scale Number Line Representing the Additive and Multiplicative Nature of Percent Increase Situations.

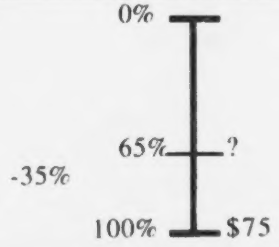


From Figure 2, it can be seen that this percent increase situation can be interpreted two ways. The baby's new mass can be seen as its original mass plus 25% more. It's new mass can also be seen as 125% of its original mass. Therefore, two calculation procedures are possible. One approach is to find 25% of the original mass and add this amount to the original mass as a two-step calculation. Another approach is to calculate 125% of the original mass (using the Rule of Three: $^{100}/_{125} = ^{3156}/_{?}$) to determine the new mass by using a one-step procedure.

The proportional number line method can also be used for percent decrease situations. In Figure 3, the number line depicts the following problem: *A shirt costing $75 was reduced 35% in a sale. How much would the shirt cost now?* To place 65% on the number line, the problem needs to be clearly interpreted using complementary percent knowledge: a 35% discount is the same as 65% of the original amount. A subtraction takes place, and thus the number line model shows the subtraction. The situation can thus be interpreted in two ways: the shirt is discounted 35%; or, the shirt now costs 65% of the original amount. Similarly, calculation can also proceed in two ways. Either, 35% of the original cost can be found, and subtracted from that cost; or 65% of the original cost can be calculated using the Rule of Three $(^{65}/_{100} = {}^{?}/_{75})$. Using a subtractive application takes percent discount problems from a two-step procedure to a one-step procedure. The number line thus organises elements within percent problems, particularly increase and decrease problems, so that interpretation can occur.

Succinctly, the proportional number line method contains a percent schema for interpreting percent situations in terms of their component parts, a representation which embodies percent as a proportion, and a structure for symbolisation of the percent situation as a proportion. The approach appears to be a simple, yet powerful way to represent the proportional nature of percent situations, and more importantly, for providing the basis for developing students' understanding of the additive (and subtractive) and multiplicative nature of percent increase and decrease.

Figure 3
The Dual-scale Number Line Representing the Subtractive and Multiplicative Nature of Percent Decrease Situations.



To explore the potential of the proportional number-line method, the approach was incorporated into a teaching program on percent and trialed with Year 8 students. As part of a larger study, a series of teaching experiments were conducted in authentic classroom situations with instruction provided during the students' timetabled mathematics classes over the period allocated to percent instruction (2 school weeks -approximately 10 lessons of 40 minutes duration). For the purposes of this report on implementation of the proportional number line method for percent problem solving with Year 8 students, results pertaining specifically to this method with one class will be presented here.

THE STUDY

Students in this study attended an inner city, single-sex secondary school. The class was non-streamed, therefore a range of ability levels were represented in the class. For the study the students were presented with a teaching program consisting of five teaching episodes on the following topics: the concept of percent; the Rule of Three, interpreting and solving percent problems; percent, common and decimal fraction equivalence; the language of percent increase and decrease. For the concept of percent episode, the language of percent used in the real world was explored and classroom activities included searching newspapers, discussing terms such as discount, profit, loss, bank interest, and also looking a common percent phrases such as 110% effort, 200% attendance, and so on. The focus of the second episode was on the Rule of Three and relating it to solving equivalent fractions (proportion equations). Two episodes were devoted to the proportional number line method. In these episodes students were presented with a variety of percent problems with which they could utilise the proportional number line method.

Some examples of the relatively routine problems presented to students were as follows:

> In a school of 875 students the measles epidemic infected 72%. How many students were infected?
>
> The school has 453 girls which is 62% of the total school population. How many students are at the school?
>
> On my Science test I got 58 out of 80. What percent did I get?

Students were also presented with more complex percent problems such as the following:

> A car dealer bought a new car from the manufacturers for $9 500. At what price must the dealer sell the car to ensure he makes a profit of 90%?
>
> Following minor goldrush and a baby boom, the population of a small town increased from 225 to 910 in two years. What was the percent increase for that town?

In the episode on equivalence the focus was to promote students' understanding of percent as a number written in a variety of equivalent forms (common fraction, decimal fraction, percent).

A variety of data collecting instruments were used in the study including pre-, post- and delayed posttests, teacher generated field notes and reflections upon program implementation, student worksamples, ad hoc interviews with students, interviews with observer, student diaries. Pre- and posttests were administered prior to and immediately after the instructional period. The delayed posttest was administered approximately eight weeks later. The pretest contained items which required students to perform percent calculations and solve percent application. The posttest followed a similar format to the pretest. The delayed posttest was a condensed version of the pre- and posttest with fewer items but of similar type to those on the pre- and posttest. The teaching program was analysed in terms of (i) its *effectiveness* in promoting students' growth of percent knowledge and permanence of that knowledge, and (ii) its efficiency of implementation in actual classrooms with respect to such factors as time, cost, resources, and preparation requirements.

RESULTS AND DISCUSSION

The class mean scores on the pre-, post- and delayed posttests were 12%, 77% and 76% respectively. Pretest results indicate that prior to instruction, students' percent problem solving and percent calculation skills on all three types of percent problems were poor. Immediately after instruction, students' performance increased considerably, and from delayed posttest scores, successful performance was maintained. Analysis of test papers revealed that the majority of students continued to draw the dual-scale number line to assist with organisation of the elements within percent problem situations. Other students dispensed with the number line and wrote proportion equations directly from the problem posed.

As part of the teaching experiment, students were required to write entries in a mathematics journal at the conclusion of lessons. Comments from students' journals relating directly to the proportion number line method included the following:

> One thing I learnt in maths today was how to work out percent problems. I found out that part means percent, well they mean the same thing.
>
> The new way we learnt to do percent problems is a lot easier than the way we learnt last year, so now I can do percent problems a lot easier and quicker.
>
> I use that way (number line method) instead of my own way because it is so easy.
> Using the new way is easier.
>
> I think I am getting better at percentage now because I find it easier to work them out with the number line.

From the above comments it appears that students valued the proportional number line method, particularly for its simplicity and ease of use, and that it enabled successful calculation of percent problems and equations. Many students appeared to regard the method as superior to their previous methods used for percent calculations as they readily abandoned their own methods and adopted the proportional number line method.

At one point during the teaching experiment, students were working on related percent activities, and one student's worksheet showed spontaneous transfer of the key processes inherent in the proportional number line method to another related activity. For this activity, the student was working on the conversion of percent amounts in order to construct a pie chart. One of the equations written on the student's worksheet was the following: ${}^{65}/_{100} = {}^{x}/_{360}$. Instruction in the proportional number line method appeared to enable this student to make connections between percent and proportion and to link this knowledge to other proportional situations.

CONCLUSION

Trialing the proportional number line method for percent problem solving with students in a real classroom has assisted in determining the contribution of this method for the development of Year 8 students' percent knowledge and problem solving skills. The study has shown that the proportional number line method assists students to perform percent calculations and solve percent problems, including percent situations relating to percent increase and decrease. The students in this study readily adopted the method, and regarded it as a better method than their own previous methods. Delayed posttest results indicated that students retained this method and could readily use it for solving percent problems after a break in percent instruction. Such results indicate the degree to which this method was internalised by the students.

The creation of the proportional number line method occurred after consulting a variety of literature on teaching and learning percent. Its purpose was to enable students to engage in successful percent problem solving to fully explore the domain of percent and also to lay the foundation for linking percent knowledge to proportional knowledge. In this study, overt knowledge of the link between percent and proportion was apparent in one student. Further research will determine the extent to which the proportional number line method promotes understanding of the fundamental proportionality of percent.

As a result of this study, it appears that the proportional number line method is a useful method for assisting Year 8 students to operate in the domain of percent. Incorporated within a teaching program on percent, the method appears to be effective in promoting the development of students' knowledge of percent and efficient in that it can be readily implemented in the classroom. It also provides the means to base the teaching of percent upon a proportional perspective and thus assist teachers to lay the foundation for broadening students' conceptual understanding of percent as a multifaceted topic.

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CUMULATIVE AND EXPLORATORY TALK IN A COLLABORATIVE LEARNING CLASSROOM

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This paper describes an approach to the analysis of talk within small groups of students working on mathematical tasks. Two typical ways of talking and thinking are identified: cumulative talk and exploratory talk. An example of each type is given, using data from a study of a Year 10 class. Changes to the nature of the talk when the teacher is present are also reported. The focus is on both the content of the talk and the interactions in the group.

INTRODUCTION

This paper reports work in progress on a study of student-student interactions during collaborative learning activities, that is, when students are working on shared tasks, with a common goal, and for most of the time independently of their teacher. I am interested in the social construction of mathematical competence and ways in which student gender interacts with this. I have chosen to focus on students at the stage where they are beginning calculus, because this is when they make key course choices which affect their post-school options and their future relationship to mathematics. These choices are predicated on students' evolving perceptions of themselves as learners of mathematics.

For the purpose of the main study I needed to find ways of analysing student-student talk, describing group processes and identifying those which may affect both the knowledge constructed by the students and their perceptions of themselves as learners. I have made use of the work of Mercer (1995) who identifies three typical ways of talking and thinking among students working in small groups: *Disputational talk*, characterised by disagreement and individualised decision-making; *Cumulative talk*, in which speakers build positively but uncritically on what others have said; and *Exploratory talk*, in which the participants engage critically but constructively with each others' ideas. In this paper I describe key characteristics of the different types of talk in the classes I observed, and discuss the roles of teacher and students in producing them.

ANALYSING TALK WITHIN STUDENT GROUPS

Research on classroom talk has tended to focus on interactions between teacher and students in whole-class learning situations. Until recently, student-student interactions were studied to a much lesser extent. Much of the research on peer interactions, especially in the USA (Webb & Palincsar, 1996) has had a process-product focus, categorising utterances such as asking questions, giving explanations, and so on, and using quantitative analysis techniques. Research in Britain has more often adopted a qualitative approach. A major influence was the work of D. Barnes and Todd (1977). Initially these researchers planned to use a quantitative approach, but abandoned the attempt because they found that the linguistic form of an utterance frequently did not match its function, utterances frequently seemed to belong in several categories, meanings were fluid and changing in response to on-going events in the conversation, and "the features the observer focusses on as being most significant are precisely those which are hardest to categorise in a reliable way" (D. Barnes & Todd, 1977, p. 17).

Analysing talk within small groups of students is not easy, because it is much less structured than teacher-led talk. As D. Barnes (1992) observes, in a teacher-led discussion the teacher decides what is relevant to a topic, how much will be said about it, and what answers are acceptable. The teacher in general also decides who will talk, when and for how long. In small group discussions, these tasks have to be accomplished by the students themselves.

They must make their own judgements about what is relevant and what is "correct", and they must also organise turn-taking. These two facets of a discussion are not independent—the same utterance may serve several functions. For example, a student reading out a question from the worksheet initiates a new topic for discussion, but this may also be interpreted as a bid by that student for the attention of the group, and perhaps even as an attempt to exert control over the group process.

To take account of this complexity in talk between students, D. Barnes and Todd (1977) distinguish between two frames of reference: 'content' and 'interaction'. As Edwards and Westgate (1987) point out, the content frame is concerned with what the group are talking about, their ideas and logic. Within this frame, the participants provide evidence of what they think (or what they want the others to believe they think). The interactional frame is concerned with how the group relate to one another. Within it, the participants provide evidence of their attitudes to one another and the relationships between them. During a group discussion, there is simultaneous interplay between these two frames.

Various schemes have been devised to categorise the content of talk. Thomas (1994) divided talk within groups into non-task-related and task-related talk. Task-related talk was divided into social (or managerial) and cognitively-oriented. Finally, cognitive talk was divided again into action (dealing with the task at hand) and reflection (dealing with explanations and understanding of ideas). A broadly similar scheme was proposed by Pirie and Schwarzenberger (1988). In view of the difficulties of categorisation noted by D. Barnes and Todd (1977), however, such schemes may be of limited value. It is better for analysis to focus on the sequence, and how the individual turns depend on one another. Similar arguments apply to coding the interactional aspects of a discussion.

To study disputational, cumulative and exploratory talk, it is therefore necessary to look at excerpts of talk larger than individual turns. The description of the structure of a lesson given by Clarke and Helme (1997) is useful here. They consider a lesson as consisting of a number of episodes, each defined by a consistent goal such as the solving of a particular problem. An episode is composed of one or more negotiative events associated with identification of, and attempts to resolve, a sub-goal. Negotiative events may be initiated by an expression of uncertainty, like asking a question. In many cases, however, the question is not explicitly stated, but is implicit in the students' actions. A negotiative event may be quite lengthy, or consist of only one or two turns.

METHOD

The data reported here are drawn from an ethnographic study of a Year 10 mathematics class in an independent coeducational school in a major Australian city. This class was chosen for the study because of the teacher's interest in, and experience in using, collaborative learning methods. A series of lessons was videotaped, and whenever the students were working in small groups, the camera was trained on one of these groups. Transcripts of the videotapes were prepared, including descriptions of actions, gestures, facial expressions or voice intonations which were judged to be relevant. Although the data gathered also included interviews, field notes, student worksheets, and other materials, the discussion here is restricted to portions of the videotapes and the transcripts which record small group discussions.

The first stage in analysing a transcript was to subdivide it into episodes and negotiative events, as defined by Clarke and Helme (1997). The transcripts were then coded for the *content* of the talk using an adaptation of the scheme devised by Thomas (1994). That is, utterances were classified as non-task related talk, task-related social talk, task-related cognitive action talk, and task-related cognitive reflective talk. A study of the coding revealed that the character of the talk tended to be generally consistent within an episode. The consistency of the goal within an episode appears to result in consistency in the pattern of talk. Within each episode, other features of the talk were noted, such as repetition, contradiction, interruption, and completion of a sentence by another speaker.

Study of the *interactional* frame made use of aspects of the exercise of power identified by M. Barnes (1998). These included initiating a negotiative event or non-task talk; rejecting or ignoring an attempted initiation; claiming the attention of the group by interrupting another, or by use of gesture or emphatic speech; introducing, rejecting or accepting a suggestion; and handing over control of the discourse by passing a worksheet or other essential resource to another student.

The shift from one episode to another is characterised by a change of goals—the task on which the group is working may change, and the teacher may be absent or present. These changes influence both the content and the interactional patterns of the talk, and hence the level of talk (disputational, cumulative or exploratory).

My main interest is in the exercise of power within collaborative groups, as this may be relevant for the study of gender effects. The interaction frame in particular may illuminate attempts to dominate, control, or influence the group process.

ANALYSIS

The data analysed so far contain no episodes of disputational talk. Occasional pairs of turns in which one student contradicted another without giving an explanation are better described as part of cumulative talk. It was clear on these occasions that the first student had made a slip, and realised this when attention was drawn to it. No explanation was given, because none was needed, as everyone tacitly acknowledged the correction.

Episodes 1 and 2 below show examples of cumulative and exploratory talk respectively, while Episode 3 illustrates the effect of the teacher's presence on the group interaction. All are chosen from the same lesson, and relate to the same group of three students.

An explanation of notation used in the transcript is given below:

Symbol	*Meaning*
—	interruption by another person
-	self-interruption
//	simultaneous speech
(...)	indecipherable utterance
(what)	the best guess for an indistinct utterance
[ ]	observations from videotape or field notes

Episode 1: An Example of Cumulative Talk

1. Selena: How do you write a relationship for this? [Mike leans forward to look, as Selena points to the worksheet.]
2. Mike: T varies, (T equals) one over n. [Jacqui writes.]
3. Jacqui: What's happening is T varies as one on n.
4. Mike: T inversely as n. [reading from worksheet] Calculate T equals five, T equals ten, T equals twenty. [Selena nods]
5. Mike: Although - oh yeah, that's right. Oh golly. So T equals k over n.
6. Jacqui: [writes] And then all you've got to do is substitute //the values in
7. Mike: //one of them in, and find the k.
8. Jacqui: What?
9. Mike: Find the k.
10. Jacqui: Yeah. So you've got to - What numbers is she going to put in?
11. Mike: Sixty and nine, twenty and um [pause]
12. Jacqui: Is that equal? [Gets out ruler, leans over worksheet to measure accurately.] Thirty and ten.
13. . . . [Brief interruption - talk about what another group is doing]
14. Mike: Here we go. So we've got, thirty and ten.
15. Jacqui: Thirty and ten. This is the n. So if we substitute ten for it, it will pass //through thirty
16. Mike: //So the time is - Yep.

17. Jacqui: equals k on—
18. Mike: ten.
19. Jacqui: ten.
20. Mike: k equals, three hundred?
21. Jacqui: [pauses to think] Yep. Okay, so k equals three hundred.
22. Mike: So we've got T equals three hundred over n, is our equation?
23. Jacqui: Yep.

Content: Nearly all of this episode was directly concerned with the task at hand—finding the equation of the graph. The only exceptions were the brief off-task interruption (13), not given in detail here, involving talk about what another group was doing, and Mike's "Here we go" (14) which brought them back to the task (coded as task-related social). The talk progressed steadily towards finding the equation. Mike and Jacqui spoke confidently: "What's happening is ..." (3) and "All you've got to do ..." (6). At no stage did they go back over what had already been done. Selena, however, took no part in this episode other than to ask the initial question. She was new to the class at the time of this lesson, and the ideas and techniques of inverse variation were unfamiliar to her.

Interaction: This is a good example of cumulative talk. Each participant in the discussion was building on what the other said and neither played a dominant role. They frequently repeated what the other had just said (14, 15 and 20, 21) or paraphrased it (2, 3, and the first part of 4). On occasions, one completed a sentence begun by the other (6–7 and 17–18) which is a form of affirmation. There were numerous expressions of agreement but at no point was there any disagreement.

Episode 2: An Example of Exploratory Talk

24. [Mike hands the worksheet to Jacqui. She turns it over, and sees what is on the other side.]
25. Jacqui: Oh. [Reads] "Explain what is happening in question three and why." [turns the paper over again to look at the question on the first side, looks at the others, sits back]
26. Jacqui: Like — [gestures as if to say "what does that mean?". Selena laughs.]
27. Mike: — Like you're taking away two and they're becoming negative, I don't know?
28. Jacqui: ( )
29. Selena: Just uh —
30. Jacqui: — You're taking away //the time
31. Selena: //time, from the time.
32. Jacqui: You're taking away the number of workers by the time and then minusing it from something else.
33. Mike: It's the time for a certain //amount of workers
34. Selena: //Time take time —
35. Mike: — minus the time for a certain other amount of workers.
36. Jacqui: [picks up her pen and begins to write.] So it might be taking away.
37. Mike: Do they mean what is happening with the answers or just the questions?
38. Jacqui: [Reads from sheet again] "Explain what is happening in question three and why."
39. Mike: Yeah, yeah.
40. Jacqui: And then what does it mean. Yes, okay. It's taking—
41. Selena: —It's the difference of time.
42. Jacqui: [to Selena] Like, the number of workers multiplied by the time.
43. Mike: Taking away the number //of
44. Selena: //the time of a certain —
45. Mike: — Yeah, the time for a certain —
46. Selena: — for a certain number of workers from uh, the time for a certain number of workers. [Selena looks to Mike and smiles. They use similar hand gestures to emphasise what they are saying.]
47. Mike: We know what we mean.
48. Jacqui: [pause while she writes down the answer] workers from //blah blah, the same thing.
49. Selena: //Another certain —
50. Mike: — Another certain amount of workers.

Content: Here, the students were focused on the task all the time—there was no off-task talk, and organisational matters were managed without the need to talk. As they struggled to make sense of the question and find an explanation, their talk was nearly all reflective. At the start they all seemed unsure of what was wanted. They spoke hesitantly as they sought for words to explain their ideas, and occasionally used gestures to help them. Uncertainty was expressed in gesture and tone of voice (26) as well as in words, such as "I don't know" (27) and "it might be" (36) and by direct questions "Do they mean …" (37). The discussion did not move forward steadily, as in the previous example. The group several times went back again over what they had already said—for example, they re-read the question in an attempt to understand what was wanted. Statements made near the beginning were repeated later on.

Interaction: At the start, Mike handed over the role of manager and scribe to Jacqui by passing her the worksheet. By hesitating, Jacqui (26) and then Mike (27, 37) gave others a chance to speak. Occasionally they completed one another's statements (as in 30-31), but later Mike persisted with what he was saying despite interruption (33–35 and 44-46). In places one or more of the students corrected a mistake. For example, Jacqui's statement (32) was corrected by Selena and Mike (33–35) and Mike's statement (43) was corrected by Selena (44). At this point in the lesson, Selena appeared to be more confident than the others—she did more interrupting, correcting and explaining. Compared with the previous episode, the difference in Selena's participation warrants further examination and explanation.

Gesture seemed to be more frequent in exploratory talk. The students were less certain about the ideas they wanted to convey, and were groping towards ways to express them. In such circumstances, they may have found words inadequate to express their tentative thoughts, and supplemented them with gestures (see Reeve and Reynolds, 1999).

Episode 3: The Effect of the Teacher's Presence

51. Mike: [Teacher joins the group] Miss _____, this is pretty challenging, I've decided.
52. Teacher: What've you got?
53. Jacqui: [gesturing towards another group] They haven't even got the equation.
54. Selena: Five. [She and Mike are completing the last part of question 2.]
55. Teacher: Did you get the equation?
56. Mike: Oh yeah.
57. Jacqui: We think we got it.
58. Teacher: So what's happening here, with these values?
59. Mike: //(…)
60. Jacqui: //They're decreasing as the, [points to graph] that increases —
61. Mike: — You see this —
62. Jacqui: — in other words, like that — [gestures again]
63. Mike: — this is a realistic graph because it's actually not going to cut the axes —
64. Jacqui: — so as n increases, T decreases, which sounds right.
65. Teacher: So what would happen if I had, six thousand workers?
66. Jacqui: It would be taking not very time - not very much time.
67. Jacqui: Well, it may not be realistic, because, like, the problem isn't defined because it's like ten workers planting one tree, or ten workers planting ten trees. So it could cause complications if it works out technically.
68. Teacher: So just give me an interpretation of the general trend here, what's happening?
69. Mike: Well, as, as —
70. Jacqui: — It starts off being —
71. Mike: — T increases, n decreases, so it's inverse variation.
72. Teacher: Okay, fine, and [pause] these differences here, some of them are approximately the same, like the difference between twenty and thirty is ten and between thirty and forty is ten. Whereas these ones are not the same, like - as they were before. Can you try and um, // work out why?
73. Jacqui: //It's because it's not a straight line, it's not actually like as one varies as one and two and two and three [gesturing with pencil in the air] it's got a curve to it.
74. Teacher: Okay, very good. All right, can you fill out the rest of it, and turn and look over the other side. Oh, you, you're pretty much doing that?

Content: The teacher's interaction with the group changed the nature of the talk. Up to this point in the lesson, the group had found the task routine and had been working through it in a straightforward manner, without need for explanation or reflection. The teacher's questions prompted reflection, and in particular thought about the possible real world interpretation of the model they had been investigating. This helped to establish the discussion at a more exploratory level.

Interaction: There are features of the interaction of teacher and students in this informal setting which would be rare in whole-class discussion. Mike, not the teacher, initiated the discussion, and he did so by making an ironic remark (51). Then the teacher's question (52) was largely ignored—Jacqui talked about the progress of another group (53) while Mike and Selena finished off what they were working on (54). The teacher had to rephrase and re-focus her question (55) to get a response. Finally, Jacqui interrupted the teacher (73), beginning to answer the question before the teacher had finished. Thus the small size of the group permitted an informality that would be unusual in whole-class teaching, and to some extent served to counteract the powerful role of the teacher.

On the other hand, as the episode progressed it began more and more to resemble the traditional Initiation-Response-Evaluation (I-R-E) pattern characteristic of classroom talk (Sinclair & Coulthard, 1975). The teacher initiated nearly every negotiative event (52, 55, 58, 65, 68, 72) and towards the end evaluated the responses before continuing (72, 74). It is clear that the presence of the teacher completely altered the dynamics of the interaction. Jacqui and Mike appear to be competing with one another to gain the teacher's attention. In her presence, their goals no longer are to complete the task they have been set, but to impress their teacher with what they understand. Jacqui (60, 62, 64) wanted to make one point, and Mike (59, 61, 63) had something different to say. Each persisted, in spite of being interrupted several times by the other. The same began to happen again in 69–71, but Mike gained the floor and the teacher accepted his explanation, so Jacqui did not have a chance to complete what she began to say. At the next opportunity, Jacqui interrupted the teacher to get her response in first and demonstrate her understanding (73). It is noticeable that Selena took no part at all in this competition for attention.

DISCUSSION

The absence of disputational talk may be a consequence of the maturity of the students (a Year 10 class studying an accelerated mathematics course) and the care which the class teacher had taken to train the students in group work, making clear her expectations that they would listen to and value other people's contributions, and explain and justify their own assertions.

What Outcomes May We Expect from Cumulative Talk?

In cumulative talk, as we have seen, students generally have the satisfaction of success. They find that they agree with their colleagues, and confirm and validate one another's statements. We may expect this to result in the affective outcomes of reassurance, increased confidence and rapport within the group. Another expected outcome is the consolidation of material already learned. Cumulative talk is generally the result of students working on tasks that they can achieve without much hesitation or uncertainty, so it is unlikely that they will be dealing with material that is unfamiliar, or constructing knowledge that is really new to them. At least in mathematics, a high incidence of cumulative talk may be a sign that the tasks set are not challenging enough. While such tasks may have the positive outcomes described above, and may also be stepping stones to more demanding activities, they should not be used too frequently.

What Outcomes May We Expect from Exploratory Talk?

Usually we would expect exploratory talk to lead to the generation of new knowledge and understanding. Mercer (1995) argues that in planning collaborative activities in the mathematics classroom, we should be aiming to promote exploratory talk:

> Exploratory talk, by incorporating both conflict and the open sharing of ideas, represents the more 'visible' pursuit of rational consensus through conversation. More than the other two types, it is like the kind of talk which has been found to be most effective for solving problems through collaborative activity. (Mercer, 1995, p. 105)

If exploratory talk is successful, it achieves important positive goals, but if it is unsuccessful it may create frustration and generate negative attitudes. It is important, however, not to be too quick in deciding that exploratory talk has not resulted in the generation of new knowledge—the first outcome may be to spur students to think more about the problem or topic, leading to later valuable breakthroughs.

CONCLUSION

The approach described above appears to be an effective way of dealing with the complexity of classroom talk. The consistency of type of talk observed within episodes suggests that the flow of classroom events may effectively be described at this level. This may prove to be a useful technique for data reduction where there are many transcripts to be analysed. At the same time, the possibility of looking in more detail at events occurring within an episode is always available. This paper has identified some key features of exploratory and cumulative talk which will be used in later analysis.

The value of this approach is demonstrated by the fact that it has permitted observations of specific factors which may tend to encourage or support exploratory talk. The examples given above demonstrate the key role played by the teacher in collaborative learning. Carefully chosen teacher questions, either asked in person, or incorporated in the written questions on the worksheet, can have the effect of shifting the talk from cumulative to exploratory. Teachers, however, need whenever possible to pose a question and then move away, leaving the students to think about it, rather than risk being drawn into an evaluative role which removes responsibility from the students.

Questions posed by thoughtful students can also provide a spur to exploratory talk. But for this to happen, the students need to be encouraged to listen carefully to other people's questions and take them seriously. It may also be important to encourage students to think about what reflective or exploratory questions they can ask.

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A CASE STUDY OF TEACHER ENDORSEMENT OF AN INTEGRATED LEARNING SYSTEM

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Eight classes (Years 4 to 7) from a Queensland primary school trialed an integrated learning system (ILS) as a means of remediating students' mathematics learning problems. At the end of the trial, the teachers were asked whether they would recommend the system to other schools. Endorsement appeared to be related to the computer knowledge of the teachers and concomitant experiences of the students, the extent of integration of the ILS sessions in classroom teaching, and the pedagogical beliefs of the teachers.

Over the last two decades, Australian educational authorities have striven to ensure that schools are preparing their students for the challenges and demands of the modern computer age. Queensland aims to have, by 2001, at least four computers in every classroom, and teachers competent in the use of technology. However, the proliferation of computers in schools has not been accompanied by an integration of technology into classroom instruction (Bracey, 1994; Fuglestad, 1996; Hadfield, Maddux, & Love, 1997; Jones, 1998), a condition that appears to stem from teacher anxiety (Kristiansen, 1992; Turner, 1995), a clash in pedagogical beliefs (Chu & Spires, 1991), lack of adequate inservice (Castner, 1998), and problems associated with computers (Pacey, 1992; Yelland, 1997).

One of the by-products of the growth of information technology in education has been the computer-based integrated learning system (ILS) which includes extensive courseware plus management software. An ILS has three essential components, namely, *substantial course content*, *aggregated learner record system*, and a *management system* which "will update student records, interpret learner responses to the task in hand and provide performance feedback to the learner and teacher" (Underwood, Cavendish, Dowling, Fogeiman, & Lawson, 1996, p. 33). This paper reports on a study to determine reactions of teachers at one school to the core numeracy courses in an ILS.

STRUCTURE OF THE ILS

The ILS trialed by the school was a comprehensive instructional system powerful enough for complex courses. According to the manufacturer, its courses were designed to foster the development of foundation skills and concepts and to promote the use of higher-order thinking skills. It should be noted that the manufacturers endorsed the system only as a tool for teachers to use to consolidate already introduced material and to diagnose student difficulties. They argued that it is the teachers' role to introduce the material to be practised on the ILS, and to remediate the difficulties identified by the ILS. Thus, they contended that the effectiveness of the ILS depends on the quality of teacher input and that any evaluation of the ILS should take into account the role of the teacher in relation to the program.

The ILS in this study is a closed system, that is, the curriculum content and the learning sequences were not designed to be changed or added to by either the tutor or the learner (Underwood et al., 1996). In the core numeracy component, basic mathematics material is categorised in *strands* (e.g., fractions, measurement) with each strand incorporating several hierarchical levels through which students need to progress. With each level, tasks are supplied via electronic worksheets (designed in America but appropriate for Australian schools) which are presented in random order to students. When students achieve high mastery (80%) at one level, the ILS automatically raises them to the next level. The random nature of the presentation is to ensure that task performance correctly represents level. The worksheets vary in quality, but many are reasonably attractive in their presentation and creative in the way they probe understanding, particularly with the use of 2-D representations of appropriate teaching materials in mathematics (e.g., Multi-base Arithmetic Blocks, Place Value Charts, fraction and decimal diagrams). There are some speed games to drill number facts. As well, there are on-line student resources in the form of a *Help* icon (provides answer), a *Tutorial* icon (provides information on how to do a task), a *Toolbox* Icon (provides calculators, rulers, tape measures, etc.), a *Reference* icon (provides definitions), and an *Audio* icon (reads text passages to the students through earphones). Worksheets can be printed to provide off-computer activity.

However, there are aspects of the ILS that cause some difficulties. For example, the use of the Help and Tutorial icons automatically grades performance as incorrect, insufficient task variety in some domains causes repetition of examples (a feature that often lead to student boredom and frustration), and novel presentation or solution formats of some tasks were difficult for students to interpret (e.g., the units digit must be typed first in solutions to number facts). Generally, questions are closed and performance is based on speed (with time delays leading to the ILS defaulting to incorrect).

LEARNING AND THE ILS

ILS's tend not to meet many of the characteristics of effective software and this is true of the particular ILS reported on in this paper. In its core numeracy course, the ILS operates as a tutor, with the students passive and the computer in control. This is contrary to modern views about learning with computers (Sivin-Kachala, Bialo, & Langford, 1997) and learning of mathematics (Kennedy & Tipps, 1997), particularly with respect to higher cognitive functioning (Carnine, 1993; Riel, 1994), investigations, and the construction of knowledge links (Wiburg, 1995). The random nature of the worksheet delivery means that the ILS does not provide sequences of activities that can change misconceptions (as argued by Sivin-Kachala et al., 1997). The ILS is used by one student at a time so there is no place for groups (an important component of effective use of technology according to Sivin-Kachala et al., 1997).

The ILS does not meet any of the five criteria put forward by National Council of Teachers of Mathematics (Kennedy & Tipps, 1997) for the effective teaching of mathematics. For example, the ILS: (a) requires students to work individually and thus does not build mathematical communities; (b) is the sole authority for correct answers and therefore does not encourage logic and mathematical evidence as verification; (c) encourages memorisation of facts and procedures rather than mathematical reasoning; (d) emphasises mechanistic answer-finding (precise answers in a precise order) rather than conjecturing, inventing, and problems solving; and (e) treats mathematics as a body of isolated concepts and procedures rather than as a connected schema.

Furthermore, the ILS generates and presents tasks at random, thus giving no continuity or logic between questions. Whilst it does provide students with feedback on the correctness of their responses, a process that is desirable for effective learning (Sivin-Kachala et al., 1997), its worksheet nature makes it susceptible to the same pedagogical flaws that were identified by Erlwanger (1975) in the Individually Prescribed Instructional (IPI) packages that proliferated in the US in the 1970s (Baturo, Cooper, & McRobbie, 1998). Whilst there is very little evidence of the ILS's improving student learning (e.g., Becker, 1992), they are nevertheless reasonably popular in many schools in Queensland. The exploration of the reasons for this is the focus of the study described in this paper.

METHOD

There were four cases in the study, with each case consisting of the two teachers and two classes from each of Years 4, 5, 6 and 7 in a large primary school. In Years 4, 5 and 7, the two classes were combined in one large room (with the two 2 teachers team teaching); the year 6 classes were in different buildings. Each class had approximately 30 students, and had been provided with 3 computers, 3 ILS systems, and a printer. The students were generally from low socioeconomic backgrounds and few students had computers at home. The Years 4, 6 and 7 teachers had placed the computers on one side of their classrooms whilst the Year 5 teachers had placed them in a withdrawal room adjacent to the classroom. Initially, all computers were stand-alone systems which meant that students had to use the same computer for each of their allocated sessions and teachers had to access all computers to print reports. Because of these restrictions, a peer-to-peer network system was set up to replace the stand-alone system. This meant that any student could use any available computer, and reports for any student could be obtained from any computer. All classes were using the ILS for the first time.

Data were gathered by interviews and observations. Two semi-structured interviews were developed for school administrators, computer coordinators, teachers, teacher-aides and technical staff (where available). The first interview (beginning of the study) focused on logistics and management of the ILS, beliefs about teaching and learning mathematics and the role of the ILS, existing use of computers by students, and perceptions of students' likes, dislikes and preferences with respect to the activities within the ILS's core mathematics course. The second interview (end of the study) focused on changes in logistics, management, perceptions of students' attitudes and performance, and teachers' recommendations for the ILS. Observations were made of classrooms and students on the ILS system. Where convenient, teachers and students were questioned as to the reasons for any actions that attracted interest.

Each classroom was visited for six days - two days at the start of the study (in which the first interview was administered), two days in the middle of the study, and two days at the end of the study (in which the second interview was administered). The teachers were interviewed outside of their class using audiotapes; field notes were kept of observations.

The interviews were transcribed into protocols and the field notes were restructured and summarised. The data for each of the cases were combined, restructured, and studied for commonalities; characteristics of the cases which related to the possible impact of the ILS were identified, and responses for each case were summarised into tables. Finally, data on categories were related to teachers' evaluations (endorsements) of the ILS to identify factors that appeared to influence the effectiveness of the ILS (as perceived by the teachers).

RESULTS

The cases were similar in some aspects. For example, each case had 6 computer (stand alone, then peer-to-peer), 6 ILS programs, a roster system that enabled each student to have three 15-minute sessions per week, the prior computing experience of the students was either minimal (e.g., restricted to game playing) or nonexistent, and close supervision in terms of teaching advice was difficult to provide because of the teaching commitments with the remainder of the class. Therefore, supervision in all cases tended to be limited to providing superficial help (e.g., how to enter/exit the ILS; what key or icon to press; how to interpret a particular task). This section focuses on the differences between the cases.

Year 4 Case

The two Year 4 teachers had virtually no prior computing knowledge but were committed to providing mathematics computer experiences for their students. However, there were no other computers in the classroom to provide other computer activity (and the 6 computers were fully used for the ILS). These teachers were also committed to providing remediation for students with mathematics learning difficulties but were able to provide superficial remediation only. Furthermore, although they had received ILS inservice before beginning to use the ILS in their classroom, they initially found it difficult and time consuming to print reports and to decipher the reports in terms of students' progress and consequently were unable to utilise this aspect of the ILS to inform their mathematics teaching with respect to remediation. However, with the support of the Year 5 teacher who coordinated the ILS program in the school, they were eventually able to make use of the reports and integrate the ILS activity and feedback from the reports into their mathematics teaching. These teachers did not set up a reward system possibly because their students were enthusiastic users of the ILS and did not require enticement to persevere ("they riot if they don't get a turn"). The teachers liked the way the ILS presented mathematics, they thought that their students were benefiting by using it, cognitively and affectively, and were happy to endorse it as an effective teaching resource for primary schools.

Year 5 Case

One of the Year 5 teachers was the school's ILS coordinator and she ran the program in the double Year 5 class. She had used the system in other primary schools for three years and her prior experience had alerted her to potential problems (e.g., students' waning enthusiasm over time, and technical procedures) with respect to using the ILS. To offset the former problem, she instituted a system of rewards to ensure that the students remained on task. Moreover, she was facile with the reporting system (particularly those that described common student errors) and was aware of, and used, the extra worksheets designed to reinforce the tasks encountered in the sessions. As well as this means of keeping track of students' progress, the students were taught to record their overall result at the end of their ILS session and to note their errors. These records and notes were then given to the teachers who arranged individual remediation when required. Thus, this case provided substantial off-computer class and individual mathematics follow-up for their students, even although, unlike the other classes, the Year 5 computers were located in a withdrawal room adjacent to the classroom.

The other computing knowledge of both Year 5 teachers was limited to word processing and spreadsheet programs and, whilst they would have liked to provide these computer experiences to their students, they had no other computers with which to do so. The Year 5 teachers approved of the way the ILS interacted with the students. They believed that their students liked the ILS and were benefiting from working on it and consequently strongly endorsed it.

Year 6 Case

Unlike the other cases, the two Year 6 teachers were in different rooms. Teacher A had considerable knowledge and experience of how to use computers in classrooms whilst Teacher B had knowledge of the ILS from a previous school he'd taught at. Each had another computer with which to do other mathematics computing (games, spreadsheets, problem solving) with their students. They disliked the way in which the ILS presented mathematics and interacted with the students - Teacher A firmly ("wouldn't be doing it if I had the choice") and Teacher B strongly ("really appalled by the 1970s concept … thought we got rid of this years ago"). They felt that the ILS was not value for money because it was not interactive enough, it was demeaning ("another way to show failure"), and could be replaced by much better software. They felt that they had not been adequately inserviced to enable optimal use of the ILS, and were pedagogically opposed to what they *could* use. Neither integrated the ILS activities into their other mathematics teaching; they saw it as "extra" revision for their students and felt there was little transfer from the ILS to other mathematics.

Teacher A kept the trial going to the end although he felt that students were ambivalent towards it ("some kids enjoy the speed games") and not greatly benefiting from it. He wanted to use the three ILS computers for other computing activity. Teacher B did not complete the trial. He stopped rostering students onto the ILS and began to use the computers for what he saw as more creative endeavours (e.g., problem-solving games and simulations, multi-media based projects). He felt that the ILS was conflicting with the way he wanted to set up his classroom (which he saw as student-centred and creative). He believed that it was not useful for the students and reported that his students were bored with it. Neither teacher endorsed the system.

Year 7 Case

Apart from the six computers dedicated to the ILS, these teachers had six other high powered computers in their classroom. With these they ran an extensive and creative program of computing activities including spreadsheet and data bases, playing problem-solving games and simulations, and conducting Internet research and multi-media projects (and publishing a class newspaper). Thus, they had reasonably extensive knowledge and experience of how to use computers in classrooms for mathematics, including the use of the ILS. Their other computer work was designed to fit in with their teaching program but the ILS program could not be modified to do so. Consequently, they did not like the roster system because they felt that it interfered too much with their program of teaching ("lost students at important times"). They saw the ILS as an "extra to classwork" and thus did not follow up on the ILS reports of students' learning difficulties. Whilst they were reasonably satisfied with the ILS in terms of its presentation of mathematics, they did not like the way it interacted with the students ("it directs them too much … kids like to be in control of the computers"). They believed that their students were not benefiting from it and did not like it, although a student survey showed that many of the students did, in fact, like it and thought that the ILS was helping them learn mathematics. This phenomenon appeared to be a pedagogical clash between relational (Skemp, 1987) teachers and instrumental students. Neither teacher finished the trial and neither endorsed the system.

CASE CHARACTERISTICS

From an analysis of the data, categories of characteristics common to the cases emerged and were classified as *operational*, *user*, and *teacher belief* characteristics. Operational characteristics incorporated the number, location, and set-up of the computers and the way in which the students were supervised. User characteristics included the teachers' prior general computing knowledge, the students' prior computer experiences, follow-up off-computer activities (i.e., the extent to which the ILS activities and reports were utilised in everyday classroom mathematics teaching), and the organisation of any rewards to support student effort on the ILS. The students' prior computer experiences were classified as *nil* if restricted to games, *limited* if included the use of wider mathematics software, *creative* if used for problem-solving activities and/or spreadsheets/data bases, or *extensive* if involved multi-media. The teacher-belief characteristics included the teachers' attitude to the pedagogy underlying the ILS, satisfaction with the delivery of ILS instruction, satisfaction with ILS inservice, and perception of the benefit students were receiving from the ILS.

Looking across the four cases, it appears that teachers' pedagogical beliefs, teachers' prior computing knowledge, and students' prior computer experiences were the factors that most affected endorsement of the ILS. The Years 4 and 5 teachers (who endorsed the ILS) had limited computer knowledge, provided their students with limited computer experiences, tried to provide follow-up off computer activities to reinforce ILS activities or to re-teach mathematics problems reported by the ILS, were compatible with the mathematics pedagogy inherent in the ILS, were satisfied with the ILS's delivery of mathematics activities and with the inservice provided, and believed the ILS was of positive

Table 1 provides a summary of the characteristics in relation to endorsement.

Table 1
Characteristics That Influenced Endorsement With Respect to the Cases.

	Cases			
	Year 4	Year 5	Year 6	Year 7
Characteristics	Endorsed		Not endorsed	
Operational				
Number of computers	6 (all ILS)	6 (all ILS)	8 (6 ILS)	12 (6 ILS)
Computer location	Classroom	Withdrawal room	Classroom	Classroom
Supervision	Superficial	Superficial	Superficial	Superficial
User				
Teacher computer knowledge	Nil	Limited	Extensive	Extensive
Student computer experience	Nil	Limited	Creative	Creative
Follow-up of results	Partial	Extensive	None	None
Rewards	None	Some	None	None
Teacher belief				
Pedagogy	Compatible	Compatible	Incompatible	Incompatible
Delivery	Satisfied	Satisfied	Unsatisfied	Unsatisfied
Inservice	Satisfied	Satisfied	Unsatisfied	Unsatisfied
Benefit	Positive	Positive	Negative	Negative

benefit to their students. On the other hand, the Years 6 and 7 teachers, who did not endorse the ILS, had provided (or wanted to provide) their students with creative computing experiences, were opposed to the pedagogy inherent in the ILS, were not satisfied with the ILS's delivery of activities nor with the inservice provided, and could not perceive any improved mathematical performance in their students.

DISCUSSION AND CONCLUSIONS

The four cases show a relationship between endorsement, student computer knowledge, teacher computer experience, and teachers' beliefs (pedagogy, delivery, inservice and benefit) about the ILS. Differences in teachers' beliefs about the pedagogy of the ILS appeared to be very significant. The Years 4 and 5 teachers (who endorsed the ILS) approved of how the ILS operated in relation to their students. The Years 6 and 7 teachers (who did not endorse the ILS) had concerns with the ILS even at the start of the trial. As the Year 5 coordinating teacher stated, commenting on what was happening in the school, "The first thing is, it won't work if teachers do not believe in it!"

Teachers' beliefs in relation to the benefit of the ILS to their students appeared to be related to their knowledge of the educational uses of computers in classrooms, and not necessarily to knowledge of what their students believed. The Years 6 and 7 teachers had more extensive computer knowledge than the Years 4 and 5 teachers had. The Years 6 and 7 teachers believed that students should not be passive with respect to computers and mathematics teaching, and they wanted to provide their students with more creative, active and problem-solving computer experiences than that provided by the ILS. They believed that their students were not happy with the ILS and were not benefiting from it. However, teachers' perceptions of their students' feelings about the ILS were not always in harmony with their students' responses to a survey with respect to their likes and dislikes about the ILS. However, this error in perception did not stop the Year 7 teachers deciding not to use the ILS in the future. On the other hand, the Years 4 and 5 teachers, with more limited computer knowledge, liked the ILS because it required little from them in terms of organising what to do with the software. For students in Years 4 and 5, whose computer experience had been limited, the ILS appeared to be an exciting experience with its varied displays and use of mouse and earphones.

Teachers' beliefs about the appropriateness of the ILS's interactions with their students appeared to be related to their beliefs about how mathematics should be effectively taught. The ILS in this study places students in a passive role in the learning process, providing practice worksheets in random order. As Baturo, Cooper and McRobbie (1998) argued, this tended to result in learning that is limited to syntactic knowledge. Thus, the ILS does not place students in an environment which the literature argues should be effective in teaching students mathematics (e.g., Kennedy & Tipps, 1997). The Years 6 and 7 teachers agreed with this, but the Years 4 and 5 did not. In fact, the Years 4 and 5 teachers argued in support of mathematics teaching approaches that reflected the ILS; that is, approaches based on instrumental learning (Skemp, 1977) and transmission models of teaching. As well, their attitude to computing, through preference or lack of knowledge, also appeared to reflect the approach embedded in the ILS, that is, a tutor-tutee relationship where students are passive and the computer is in control. The impression was that the mathematics classroom practices of the Years 4 and 5 teachers reflected the ILS; that is, involved repetitive practice on exercises with the teacher in control and the students in a passive role.

Policy within Queensland schools aims to increase teachers' knowledge of classroom uses of computing. This may have an effect on the Years 4 and 5 teachers' continued endorsement of the ILS. However, if their beliefs about teaching mathematics stay the same, they are likely to continue to use the ILS. As many Queensland teachers share the Years 4 and 5 teachers' practices with respect to teaching mathematics, this could explain the continued popularity of the ILS in Queensland schools.

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ENACTIVISM AND MATHEMATICS EDUCATION

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The development of learning theories has continued since the acceptance of constructivism with enactivism being one emerging theory. Some influences on this development include criticisms of constructivism, ideas about Cartesian dichotomies, consideration of non-cognitive knowing, notions from phenomenology, and the neural biological work which emphasises evolutionary or Darwinian notions and systems theory. This paper, based on the literature, puts some of these interrelated influences together to introduce enactivism.

INTRODUCTION

In educational research there is a constant consideration of theories that provide possible frameworks for studies. This paper considers some of the criticisms of constructivism and some of the notions that underpin enactivism as part of this process. It is not a summary of enactivism but an attempt to provide an alternative way to make sense of what happens in classrooms and to tie together more coherently some of the research about learning, teaching and curriculum.

In the Anglo-American world, from 1930 to 1980, the dominant theories influencing mathematics education have been associationism and behaviourism. Minor influences have been the progressive education of Dewey (1916), and humanist education as espoused by writers like Neill (1968) and Rogers (1969). During this time, in other countries, theories such as Gestalt education (Wertheimer, 1980) have been influential.

Since the eighties we have seen a growth of interest in constructivist theories—radical and social constructivism (von Glasersfeld, 1995; Cobb, 1994) and variations of these. These have been valuable in questioning the assumptions made about what is knowledge, how people learn, and how we teach, assess, and present curriculum.

Starting from constructivism, enactivism seems to be emerging as a result of five interrelated influences which will be discussed in turn. These five are: the criticisms of constructivism, the Cartesian dichotomies, phenomenology, non-cognitive knowing, and issues from biology.

DISCUSSION OF CRITICISMS OF CONSTRUCTIVISM

While constructivism has been valuable, it has also been criticised. Five criticisms follow.

- There is a lack of models for constructivist teaching. This is hardly a criticism when the theory has always purported to be about learning rather than teaching, but related to this lack of a teaching model is a lack of curriculum and assessment models, and this has meant that teachers are subjected to mixed messages when they talk about constructivist learning but are presented with curricula and assessment in a behaviourist format.

- There is a lack of a critical dimension which means that there is no mechanism to avoid the construction of undesirable outcomes (Taylor, 1996). This seems no different than with traditional learning where the critical dimension depends mainly on the teacher.

- There is an undue influence in education and in what constitutes knowledge by the dominant culture which is currently the white middle class (Zevenbergen, 1996; Taylor, 1996). This concern about the hegemony of the dominant culture is relevant to theories other than constructivism could perhaps have been anticipated when the emphasis moved to social construction or enculturation. While the consideration of prior knowledge in constructivism and the emergence of ethnomathematics might suggest that this domination is being reduced, in fact there seems to be little evidence of change.

- Constructivism is concerned only with cognitive knowing. It does not explain unformulated or subconscious knowledge, it does not consider how things might be known intuitively or instinctively, and it does not consider how emotions are constructed or their role in learning.
- There do not seem to be explicit links made between constructivism and the learning theories that brain-science or neural biology offer.

ISSUES FROM PHILOSOPHY—CARTESIAN DICHOTOMIES

Related to these criticisms of constructivism are some philosophical assumptions that have been made about knowing that are now considered problematic. These include the dichotomy of self and non-self, or self and world, which lead to the subject/object, the mind/body, and the knower/known dichotomies (Davis, 1996). These have been referred to as Descartes' error (Damasio, 1994) but Davis (1996) says we should not denigrate Descartes' contributions, we need to see them in an historical context and as providing a way of seeing the world. He says that these Cartesian dichotomies have caused us to put more reliance on reason rather than on the senses and have led us to focus on building better and better representations of reality. These dichotomies of mind/body, self/other and self/world are not only Cartesian notions, their roots belong within the traditions of individualism in Aristotelian philosophy and Judaic and Christian religions. In these the dichotomies start with the body/spirit split which is reinforced by the notion of a vital principle or soul distinct from the body. This has connotations of intelligence not being connected with the material body.

In the phenomenology of Merleau-Ponty (1962) we see a reinterpretation of this Cartesian bipolar way of thinking that dominates western thinking. Merleau-Ponty rejects both the rationalist and the empiricist ways of knowing and claims that "the body renders mind and world inseparable" with the body being "our means of belonging to our world".

Varela, Thompson and Rosch (1991) suggest the same when they say that "cognition depends on the kinds of awareness that come from having a body with various sensorimotor capacities". Their argument contends that the individual and the environment must be considered together, that one can not separate knowledge from doing and from the body, and that knowing is doing which in the end is inseparable from self-identity or being (Varela, Thompson & Rosch, 1991). These notions of embodiment and systems theory (together with enactivism) seem to resonate more with Zen, Bhuddist and Taoist philosophies than with traditional western ones.

Tremmel (1993) is another educator who sees links between learning and Zen, he sees a need for learners to give themselves up to the learning situation and the action of the moment, he writes of the need to balance the external ecology with the internal ecology, and to study the self which is Zen.

Such an embodied view of knowing is not new, according to Boisvert (1998), Dewey had a similar problem with Cartesian philosophy and he also took what is now called a post-Darwinian or embodied view and did not accepted the mind/body split nor the subject/external world split.

PHENOMENOLOGY

Another aspect of philosophy that has influenced Maturana, Varela and Davis who are perhaps the most influential writers on enactivism is phenomenology. Phenomenology is a philosophy that was founded by Husserl and developed by Heidegger, Satre and Merleau-Ponty (Kvale, 1996). The basic notion in phenomenology is that phenomena are the objects of knowledge. It started with a focus on consciousness and awareness and was expanded to include the life-world and human action. It is concerned with both "that which appears and the manner in which it appears", and it "attempts to get beyond immediately experienced meanings in order to articulate the pre-reflective level of lived meanings, to make the invisible visible" (Kvale, 1996). The question seems to be how are these phenomena internalised as knowledge.

Ideas from the biologist Maturana have been shown by Winograd and Flores (1987) to have many similarities to the ideas from the phenomenology of Husserl and Heidegger, although Maturana makes no reference to them. In particular Husserl believes that "the world is experienced (by me) only through the possibilities allowed by my ego" (Mingers, 1995) and Heidegger uses the phrase "being-in-the-world which summarises the notion that it is this process of experiencing the world that is the way of being of humans" (Minger, 1995), and these notions are similar to Maturana's ideas.

Varela, Thompson and Rosch (1991) have elaborated on the ideas of Merleau-Ponty about phenomenology and they claim that the body needs to be understood as both physical-biological, and at the same time, as experiential/phenomenological. They assume that the mind is influenced by both biological interactions and historical interactions.

NON-COGNITIVE KNOWING

Non-cognitive knowledge does not seem to be explained by constructivism. Davis (1996) suggests that this is because traditionally we have been concerned only with what he calls formulated knowledge because the cognising agent is seen as being separate from world. He discriminates between formulated and unformulated knowledge and says that much of what we do is unformulated as we are not conscious of doing it. He suggests that formulated and unformulated knowledge are not a dichotomy but are complementary and inseparable. He talks about what we say (the formulated) and what we do (the unformulated), and suggests that it is through the *play* between these that we find space for learning. Enactivism for Davis explains unformulated knowing because "every act is an act of cognition" and "we are not separate from but coupled to our situation/context". He summarises this with a phrase from Maturana and Varela (1987) "to live is to know".

Non-cognitive knowing is more than what Davis (1996) calls unformulated knowledge. Davis writes that learning involves resolving tensions between tacit and explicit knowledge, between emotional and reasoned actions, and between intuitive and calculated responses. He speaks of understanding implying sympathy, and meaning implying intent. He says that meaning has an affective dimension which is often ignored because of another Cartesian split, the knowing/feeling split. Other writers also emphasise some of these notions. Feminists writers have always considered emotions more than male writers have and Gilligan's (1982) notion of "connected knowledge" and the term "passionate knower" used by Belenkey, Clinchy, Goldberger and Tarule (1986) are examples of this. Hargreaves (1998) recently wrote that "emotions are at the heart of teaching". In saying this he quotes many other educators. However, for some of them emotions seem to be merely an extra dimension to be considered. Noddings (1992) puts emotions in a more central position when she talks of "the challenge to care in schools". Related to this are some comments from Heidegger, (cited in Mingers, 1995), in describing our experiencing of objects he talks of *dasein* which is characterised by a feeling of concern; and when describing our experience of other people, talks of *solicitude* which suggests a caring for and a concern for others. In a more general way Donaldson (1993) spoke of emotions when she wrote "passionate curiosity empowers the intellect", and "achievement of new understanding is normally accompanied by delight".

Whether we consider emotions as unformulated knowledge, as personal constructions, or as actions, there seems to be a need to consider them not apart from other constructions and actions but as part of our being which is intimately interrelated with our knowing.

ISSUES FROM BIOLOGY

The biological notions about learning have been influenced by the work:
- of cognitive scientists with models that compare human brains with computers;
- on linguistics, semiotics and language in thinking and learning (Pinker, 1994, 1997);
- in systems theory, and
- in neurobiology that is concerned about how the human brain works (Plotkin, 1998).

While each of these theories has distinctive features, many of them fit reasonably with the philosophical notions of constructivism. Within the biological studies concerned with how the mind works, the evolutionary metaphors of Darwin rather than the analytic and reductionist models of Descartes are gaining acceptance (Dennett, 1995).

Such evolutionary ideas in education are not new. According to Plotkin (1998), William James embraced the ideas of Darwinism in his early foundations of psychology but these notions were not taken aboard by his colleagues and virtually disappeared. People like Piaget with a background in biology were also sympathetic to notions of evolution, and the idea of *fit* from such educators is evidence of this.

Capra (1996) and Mingers (1995) have provided explanations of some of these ideas about learning that were developed by Varela and Maturana. Capra shows how the general systems theory from Bertlanffy (1968); the work of other biologists and ecologists, Gestalt psychology from the 30s, the studies by cyberneticians, and the development of self-organizing systems, underpin their work. Capra uses the metaphor of a *web* to explain the interrelatedness within and between living systems and says

> In the emerging theory of living systems mind is not a thing, but a process. It is cognition, the process of knowing, and it is identified with the process of life itself. This is the essence of the Santiago theory of cognition, proposed by Humberto Maturana and Francisco Varela (Capra, 1996, p 257).

Both Capra and Mingers describe Maturana's early work which focused on two question: "What is the organization of the living?" and "What takes place in phenomenon of perception?" and identify the breakthrough as seeing these two as intimately connected. He suggests that Maturana sees a living system as self-organizing and self-referring, and sees perception and cognition as not representing an external reality, but specifying one through the nervous system's organizational process.

According to Mingers (1995), Maturana explored these ideas in *The Biology of Cognition* (Maturana, 1970) and this led him to see humans as autopoietic (or self-producing) systems with nervous systems structurally coupled to bodies and through this to the environment. From this perspective "the body and the nervous system are structure-determined systems; the changes they undergo depend on their own prior structure and can only be triggered, not determined, by interactions with other systems" (Mingers, 1995). He goes on to describe how the capacity for language (including that of mathematics) has opened up new domains in which we make descriptions, descriptions of descriptions, descriptions of ourselves, leading to the emergence of the observer. Such descriptions are not determined by the nature of what is described, but by the describer, they do not reflect an objective reality, they remain subjective constructions. Summarizing Maturana (mainly from Maturana and Varela, 1980), Mingers makes the following points:

> - All our perceptions and experiences occur through and are mediated by our bodies and nervous systems. It is therefore impossible for us to generate a description that is a pure description of reality, independent of ourselves.
> - The structure- determined nature of the observer means that interactions are selective rather than instructive, so that the resulting experience always reflects the observer.
> - We are therefore always, unavoidably, existing within a domain of subject-dependent

> descriptions—that is the only reality we can experience—the reality of the distinctions and descriptions that we make. We must accept this and alter our view of our world accordingly.
>
> - Thus the object of our knowledge does not exist except as it is distinguished by the observer. It is not just that we cannot access as existing reality, but that our realities are brought into existence through our activities as observers.
> - This is true not just of the everyday world of ordinary people, but also of the world of science itself, which is equally confined to exploring the results of its own distinctions.

According to Sacks (1995) who has provided a good overview of the work of Edelman (1987), Edelman sees the brain as likely to be very different from a computer and sees the need for a theory of how the brain works that accounts for or is compatible with "the facts of evolution, neural development and neurophysiology; and neurology and psychology". He sees this as needing to "be a theory of self-organisation and emergent order at every level and scale". Sacks claims that Edelman's theory does this. The theory is known as neural Darwinism or a theory of neuronal group selection where the nervous system (like the immune system) is a system of recognition and where evolutionary processes occur within each individual and within lifetimes. Edelman considers two kinds of selection: developmental (pre-birth or genetic instructions which provide a propensity rather than specific direction) and experiential (based on learning/life experiences).

Another scientist working in this area of neural biology is Plotkin. He (1994) has developed an evolutionary biologist's view of cognition, he sees knowledge as adaptation and believes that we think adaptively rather than logically. He suggests that we have a predisposition to pay attention to particular things and suggests that we might be hard wired for these. He disagrees with many biologists in that he sees language as being unique to humans and he sees it as a tool for learning. Plotkin also acknowledges the idea of emotional knowledge and cultural knowledge that suggest he is disagreeing with some of the cartesian notions. In his later work (1998) he pushes beyond the nature versus nurture argument by including the social context in his discussion of *nurture*. This means that nurture is influenced by historical occurrences in a similar way to the notion that nature (genes) are influenced by ancestors. Davis (1996) also does not see nurture and nature as distinct but sees the cognising agent as being an active rather than a passive agent who selects the actions to be taken.

ENACTIVISM

The criticisms of constructivism, the ideas from philosophy about Cartesian dichotomies and phenomenology, ideas of non-cognitive knowing, and the issues from biology have influenced the emerging theory about knowing, learning and teaching called enactivism. This theory is based on the ideas of Maturana and Varela (1987) and has been developed by Davis (1996) who has interpreted it within the context of mathematics education.

In enactivism, instead of seeing learning as "coming to know", one envisages learner and learned, knower and known, self and other, as co-evolving and being co-implicated. In this situation context is neither the setting for a learning activity, nor the place where the student is, but the student is literally part of the context. With enactivism the complexity of learning is emphasised:

> learning should not be understood in terms of a sequence of actions, but in terms of an ongoing structural dance—a complex choreography—of events which, even in retrospect, cannot be fully disentangled and understood, let alone reproduced (Davis, Sumara and Kieren, 1996, p 153)

Enactivism contrasts with constructivism where there is an emphasis on knowing rather than on knowledge. With constructivism knowledge is viewed as a human construction that is evaluated in terms of whether it fits with the experience of the knower. The representation people construct of their realities can be viewed as a process of redefinition which occurs when ideas are transformed in a social setting so that they can be stabilized, inspected, edited, and shared by others. In this situation viability is the criteria and this fits with the neo-Darwinian idea of *natural drift* rather than *natural selection* (Varela, Thompson and Rosch, 1991). Even with the radical constructivism of von Glasersfeld (1995), while the emphasis is on individual interpretation of and abstractions from experience, these are acknowledged as being shaped by the learning context, by interaction with others, and by the social milieu. According to Davis (1996), radical constructivism emphasises the monologic authority of the individual and social constructivism emphasises the dialogic authority of the society and the individual, but both these are based on the modernist separation of self from other and self from world. Both, Davis claims, have difficulty because they see knowledge as something, and therefore need to assign it a location. An alternative to this need to locate knowledge is provided by Bateson (1972) who says that there is no such thing as information, it is not knowledge-as-object but knowledge-as-action.

This emphasis on action ties in with the way Davis (1996) sees it, he says that in enactivism collective action is not for individual sense-making but as a location for shared meanings and understanding. In radical constructivism understanding is seen as a continuous process of organizing one's knowledge structures, but Davis (1996) describes cognition as being not in minds and brains, but in the possibility for shared action. He talks of cognition/knowledge and knowers as being co-implicated and co-emerging. Knowledge is not apart from world but embedded in it in a series of increasingly complex systems (groups, schools, communities, cultures, humanity, biosphere), and embodied knowledge extends to these bodies that are larger than human. Varela (1992) reinforces this, he sees enactivism as providing an alternative to the constructivists' notion of representation by focussing on self-organizing systems. From this viewpoint he questions the existence of a world independent of the knower and sees the knower, the knowing and the known as emerging together. Clark (1998) describes emergence as complementing reductionist analysis as a way of making sense of this 'embodied' cognition. From his perspective an explanation can be said to emerge in a similar way to the way that the result emerges from a dynamic system (in chaos theory)—from a small start, gathering momentum and direction as more inputs 'fit' with the initial starting point.

Within enactivism Davis (1996) talks about 'self'. He says that self is a 'transformative' process. Piaget saw self as continually changing but enactivists see change as not happening to self but as being self. Davis sees self as a network of relationships and says that one does not don different masks, but one enacts different selves.

In summary, while the rationalist says 'I think', the empiricist says 'I observe', and the enactivist says 'I act'. Whether we accept the constructivist knowledge schema, the web notion of Capra, or the enactivist theory of Davis as our way of thinking about coming to know, we see how we 'come to know' as being more complex than was implied by a simple list of objectives. A similar perspective of complexity is needed when we look at curriculum and assessment.

CONCLUSION

While this paper was written to provide some background to enactivism, being concerned with practice, many educators will say "so what". Davis (1996) has taken the next step and in his book he teases out some of the main requirements for teaching that he sees as being congruent with enactivism.

His most important requirement is listening—not at the shallow level but at a deep level. He discriminates between evaluative listening (the traditional evaluative role taken by teachers), interpretive listening (which cuts through the noise of 'play' and leads to more flexibility in the classroom), and hermeneutic listening (which involves more negotiation and co-implicated activity within the classroom).

This listening is often situated in a 'play' situation which on the one hand reminds one of the emphasis on play in the work of Dewey and of Piaget But Davis also asserts that play is a situation where 'subjectivity loses itself' and play is, in Gadamer's words, 'not the opposite of seriousness' but rather "seriousness in playing is necessary to make play wholly play' (Gadamer, 1990). Davis's (1996) notion of play recognizes that play only exists in the playing, he sees that when playing around we use 'body time' and get totally involved. He sees play as an essential human quality that is evident in all we do and is something that can be realized in stillness, and in solitude.

In terms of curriculum Davis (1996) writes about curriculum anticipating. This means that the teacher works from good learning activities but must anticipate different ways that the lesson might move in response to the students' interactions while still linking with the major ideas that underpin the particular curriculum.

It would be presumptuous for me to try to summarise Davis's book, or to assume that what he calls enactivism is the only meaning that can be ascribed to it. However I would recommend his book as a critique of and a source book for teaching based in the context of mathematics; it provides some ways 'towards a sound alternative'; and is a reference that develops the notions of enactivism further.

Just as I started with some of the critiques of constructivism, so this explanation of enactivism will no doubt be critiqued. This paper was only intended as an introduction to the theory and it is hoped that readers will continue to suspend their judgement until they have delved deeper into the literature and considered other aspects of the theory.

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THE WRITING OF EXPLANATIONS AND JUSTIFICATIONS IN MATHEMATICS: DIFFERENCES AND DILEMMAS

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This paper reports on aspects of a study which investigated the writing of explanations and justifications in mathematics with Year 11 students. Six teachers and 36 students from the same school responded to problem solving tasks; 14 of these students were interviewed. Student and teacher views about the writing process are reported. The study suggests that students and teachers need to jointly negotiate an understanding of what is meant by an explanation, a justification, and what makes a quality response.

INTRODUCTION

In recent years there has been a growing interest in the importance of language factors in mathematics learning. Language is used in mathematics to make and clarify meaning. It provides a medium for creating, preserving, and communicating mathematical thinking (Brown, 1997). Communication is an integral feature of current curriculum reforms in mathematics. 'Mathematics in the New Zealand Curriculum' (Ministry of Education, 1992, p9) states that:

> The mathematics curriculum intended by this statement will provide opportunities for students to:
>
> develop the skills of presentation and critical appraisal of a mathematical argument or calculation, use mathematics to explore and conjecture, and learn from mistakes as well as successes;
>
> develop the characteristics of logical and systematic thinking, and apply these in mathematical and other contexts, including other subjects of the curriculum;
>
> develop the skills and confidence to use their own language, and the language of mathematics, to express mathematical ideas.

Communication in mathematics can be oral, written, formal, and informal. Writing in mathematics involves processes that are fundamental to learning and can be viewed as a reflective process, one which helps students clarify their thinking as they try to explain processes and demonstrate understandings in their own words (Clarke, Waywood, & Stephens, 1993; Pimm, 1987). Students' writings can be informative; providing access to how they think, illustrating misconceptions, patterns of thoughts and beliefs and indicating to the teacher students' variant conceptions of some notion (Burns, 1995; Pimm, 1987). Writing and speaking involve different levels or forms of cognitive activity (MacGregor, 1990). The focused articulation of one's thoughts is a higher order ability and one of the goals of mathematics education. Students may be used to operating at this higher cognitive level in verbal interactions; making judgements, justifying, and evaluating but not in written form, especially in mathematics (Ernest, 1989).

Writing varies according to the purpose and function of the text (genre). This study focused specifically on the writing genres of explanation and justification in mathematics. There are differences between these two writing genres: An *explanation* can be defined as making clear or telling why a state of affairs or an occurrence exists or happens whereas a *justification* provides grounds, evidence, or reasons to convince others (or to persuade ourselves) that a claim or assertion is true (Thomas, 1973). The explanation genre is seen to be predominantly the domain of the teacher. Only occasionally are opportunities given to students to perform similar linguistic demands (Marks & Mousley, 1990).

> Students should always be encouraged to think things through carefully, to understand, and to be able to explain. As students' arguments grow more sophisticated, the explanations should increasingly be conveyed in the formal language of mathematics (National Council of Teachers of Mathematics, 1998, p84).

Writing is viewed as a reflective process, one which helps students clarify their thinking as they try to explain processes and demonstrate understandings in their own words. It is from students' mathematical explanations, both oral and written that teachers endeavour to gain a perception of students' mathematical understandings. As one teacher wrote in a student's journal: "Unless you can explain it to me, you don't really understand" (Clarke, Waywood, & Stephens,1993, p249).

Justification is viewed as being most important to mathematicians.

> Students who cannot effectively explain the meaning of, and justify the use of, mathematical symbols, concepts, and operations are not yet fully fledged members of the community of discourse (Niemi, 1996, p361).

Mousley and Marks (1991) found in the classrooms they observed that students were asked to make mathematical judgements but rarely asked for an oral or written justification for their answers. They believe that much might be gained by asking students for expository answers to mathematical tasks.

In the mathematical community communication is in the form of discussion, argument, justification, and proof. The writing of explanations and justifications can lead students to the next fundamentally important step in mathematics; developing and evaluating mathematical arguments and proof. According to Silver, Kilpatrick, and Schlesinger (1990) much can be learned by providing justification and constructing a proof. Proving an assertion can lead to insights into further statements, to refinement of ideas, and modifications to improve the clarity and precision of justification.

The primary aim of this study was to find out how students respond, in writing, to questions requiring explanations and justifications, and secondly, to ascertain student and teacher views about the process. A detailed analysis of student responses to problems requiring written explanations and justifications is documented elsewhere (Bicknell, 1998). This paper reports on the student and teacher views about the writing of explanations and justifications in mathematics.

THE STUDY

The research was conducted at a large provincial co-educational secondary school with students coming from a full range of socio-economic backgrounds. There is an urban-rural mix and about 14% of the students are Maori. The school has a decile rating of 6[1]. The students who participated in the study came from the two highest ability grouped mathematics classes. The 14 interviewees were students aged 15 and 16 years in their third year at secondary school; seven came from each class. They were selected according to the following criteria: gender, and the type of responses given to the problem solving tasks.

Data collection instruments used in this study included a problem solving task sheet, interview, and questionnaire. The problem solving task sheet consisted of a set of word problems designed to generate students' written explanations and justifications. There were five problem solving tasks; one from each of the content strands of the curriculum: number, measurement, geometry, algebra, and statistics. The problems were based on those used in national School Certificate examinations, incorporated a range of prompts, and were designed to elicit a range of responses. An example of one problem is given in Figure 1.

Figure1
Example of a Problem

> Derryn investigated the packaging of snack bars. She measured the packet with a ruler and found that it was 16.5 cm long, 9.2cm high, and 4.6 cm wide. She calculated the volume to be 698 cm^3 (3sf). The packet had 6 muesli bars in it. Each muesli bar was 7.5 cm long, 4 cm wide but the thickness of the bars varied between 2.6cm and 2.8cm.
>
> Find the volume of the 6 muesli bars as a percentage of the volume of the packet.
>
> Explain what you are calculating at each step and show your working.
>
> Round your answer appropriately, stating the degree of accuracy.
>
> Justify why you have chosen this degree of accuracy.

Semi-structured interviews, conducted with 14 of the students, were used to obtain information about students' attitudes, perceptions, and beliefs, and also to question students about responses to the problems posed in the problem solving task sheet. A questionnaire was completed by six teachers in the mathematics department in order to gain an understanding of the teachers' views and perceptions of the writing of explanations and justifications in mathematics.

RESULTS AND DISCUSSION

Teacher Views

The teachers all agreed that the writing of explanations and justifications is an important aspect of the mathematical processes and recognized advantages to students' learning in mathematics. It was articulated that the writing process helped students to sort out and clarify understanding of concepts in their own minds and thereby understand the work that they were doing. The teachers believed that the process of writing explanations and justifications helped students "develop their reasoning and analytical ability" and their "communication skills". More specifically, it was viewed as helping students "crystallize the task" and a way to "encourage alternative ideas and methods". Teachers felt that it will "often show up other areas for developing and exploring while they are working through it". Having students write justifications can reduce the "guess and leave it at that" approach. Written explanations were seen as particularly useful for helping diagnose students' difficulties or misconceptions.

The teachers all believed that the writing of explanations and justifications should be specifically taught. The teacher's role was seen as important in helping students develop skills in efficient and systematic recording methods, to help improve the clarity and specificity of explanations. To assist students in developing these skills the teachers felt that regular practice should be provided, especially through investigations and assignments. The teachers all acknowledged giving their students opportunities in class to develop skills in the writing of explanations and justifications. Students are regularly encouraged to explain, although for most of the teachers this is in oral situations.

> I insist that they should explain what they did to get the answer. (Teacher A)
>
> I get students to explain at every opportunity - it helps students to clarify their understanding of a concept. (Teacher C)

One teacher uses a strategy of having students write out their explanations first and then picks volunteers to read out their answers to the rest of the class. Another of the teachers uses group work as an opportunity to encourage students to provide oral justifications.

The teachers all expressed the view that students acquire an understanding of what constitutes a good explanation from being given opportunities to write explanations to problems posed. They felt that this process should begin to be addressed in the primary school. In order to encourage students' awareness and development of quality responses, the teachers felt that students should be encouraged to attempt explanations, share responses, and consider the range of answers to a problem. It was considered important that students be given examples of good explanations so that they can evaluate them and make comparisons with their own responses. Modelling by the teacher, both orally and in writing, was also considered a useful strategy for helping students develop an understanding and awareness of what makes a quality response.

Teacher Concerns

Despite both the curriculum and external assessments emphasising the importance of students being able to communicate mathematical findings teachers were concerned about whether this is happening in practice.

> It's in the too hard basket and is promised but shelved. A sudden burst of 'have to do Mathematical Processes' and so questions are put into the test. However, it's good to see thinking not rote learning more in balance. (Teacher D)

There was a common concern about the time factor involved when incorporating this approach to writing in the mathematics class. While time is a problem, Teacher C contends that "the value in the learning process far outweighs this disadvantage".

> The curriculum needs changing with some content eliminated - this sort of work requires hands-on, investigative work which takes time. As maths teachers some decisions need to be made as to what is worthwhile and then worked on with greater time for reflective thinking, writing, and analysing. (Teacher F)

The teachers expressed a common concern about the lack of support given to them to develop expertise in the teaching of the mathematical processes. The limited support that they had received had come from both internal and external sources. Internally, some support had come from departmental meetings and discussions. This was viewed as one way of providing collegial support in learning how to teach the writing of explanations and justifications. However, as one teacher commented: "Basically there's been trial and lots of error. The students themselves have probably had the most input into my work." The teachers had also addressed the issue by trying to find exemplars. Concerns were also expressed that there are distinct disadvantages for students weak in language skills.

Student Views

Students supported the teachers in believing that writing explanations is an important process in mathematics. In examining students' comments it became apparent that they believed that the writing of explanations and justifications helped develop their mathematical understanding and metacognition.

> It's a little unexpected having to write it down but it is important because it personally helps me understand it more. I know what I'm doing if I write it down. It's easier to look back and say: 'I know why I put that number there' . (Paula)

> It makes you have to think about what you're actually doing. That, sometimes helps you understand and puts you back on track. (Anne)

This viewpoint was supported by other students who also acknowledged that it helps them identify possible errors and track down specifically where they went wrong in the process of explaining an answer.

> If you get things wrong you can go back through and catch on to where you've gone wrong. It helps you to go through the steps. If you just write down an answer you don't know where you've gone wrong. It's better than just writing down an answer. (Kim)

> You know that you understand it if you're able to write out an explanation. (Jessica)

Many students acknowledged that writing out an explanation also helped them remember what they had learned.

> It's revision, too, it jogs my memory about how to do it rather than just writing down the answer. (Paula)

Some of the students clearly identified benefits, outside of the mathematics classroom, from being able to write explanations and justifications. Interestingly, the teachers had not articulated this link to the 'real world'.

> It's pretty important as it shows that you understand the question and in the workforce they're wouldn't give it to you as a question just with the numbers. They'd want you to show some understanding and put all the pieces together. (Mark)

> If you are putting it to work in daily life you can't just write out answers. (Jessica)

> You need to be able to write explanations and justifications for jobs and work these days, they don't just want the straight answer. They try to find out if you know what you've got to do with the answer. It makes you think about it more. (John)

Student Concerns

Despite most of the students identifying positive benefits of the writing process, two of the students expressed a dislike for writing explanations and justifications in mathematics. Being expected to write explanations and justifications was perceived as a nuisance, something that helped fill in examination time, and really was quite a tedious exercise.

> It helps you a bit in the beginning when you're becoming familiar with something but after you've been doing it for a while it becomes boring and in the exams you spend too much time writing. (Anthony)

> I find it a real pain. We know what we're doing and you can look and see where we got the numbers from or you can see it on a diagram, but writing it down, 'sux!' (Charles)

Charles finds writing explanations a frustrating process. First, he carries out the numerical calculation and then he returns to write out the explanation. He explains;

> I find it a pain doing the question twice. You work it all out and then you have to read through and think 'what did I do there and what do I do here?' You have to write out all that you did, yet what you did is all in the answer. It's just putting it into words which is a pain! I find it heaps easier just doing the maths; I get stuck sometimes writing it out. You end up writing half a page. I find it easier just doing the numerical side.

Some students found writing explanations difficult, not essentially because of the writing process, but the need to spend time on reading and comprehending the problem.

> I don't like some problems which are complicated ones with heaps of writing and problems where they put in heaps of other stuff. It's really confusing, where you have to sort out what they actually want you to do, what formula they want you to use. (Jan)

> I don't like big long word problems. I don't like the proof reading, reading, and trying to work everything out. (Fay)

> You read the question and then you end up having to read them over and over again. Then you try and start working things out. You get worried about time, you're just sitting there and everyone else is writing down pages. (Anne)

For some students the real-life context actually got in the way of the problem solving process causing concern and confusion. The problems in this case did not contain superfluous or insufficient information but one problem, for example, required students to understand the procedure of 'trading in', another used the building of a shed as a way of 'dressing up' trigonometry in a realistic situation. The context of some problems isolated students from being able to interpret the problem as intended by the writer.

> I get put off by heaps of writing and problem solving especially if you put in real-life. If you put in a diagram it's easier. I hate having to sort everything out. (Merryn)

Not only the context but the language used in the problems also appeared to effect students' success in solving the problems:

> A lot of subjects and setting gets in the way. The way that they word things is stupid. They just word it really difficult to make it harder for you to understand. That's the worst part of it. (Paula)

Several students commented on the 'wordiness' of problems and having to spend time reading, interpreting, and often re-reading a problem posed. Time consequently became an issue for the students who could not read and comprehend problems as quickly as others.

In mathematics it is not how much or how neatly students write but what they write that is important. However, students expressed concerns about not knowing how much they should write. For the problems presented in this study, no lines were provided for the students to write on but a reasonable space was allocated. For some students the amount of space influenced how much they thought it was intended that they should write.

> I sometimes think about the space and if I do just the calculations I know it won't take up the space. That's why I write but then it goes over. (Jessica)[2]

> You feel like you need to fill up the whole space. The space provided makes you think that you have to write heaps. (Paula)

The common misconception was that students are expected to 'fill' the allocated space when a 'write-on' script is provided.

Despite the use of comprehensive prompts such as 'explain what you are calculating at each step' and 'clearly explain your working so that someone else can understand it' students still expressed concern about not knowing how to write an explanation and what was expected. Students used varying modes of representations for an explanation from purely symbolic to a combination of symbols, diagrams, and words. Regardless of whether they wrote detailed answers or only recorded calculations, many of these students expressed doubt about whether they had responded to the problems as intended.

The students experienced difficulties writing justifications. They did not understand what it meant to justify an answer and the majority of the students in the study avoided writing justifications as prompted by the questions.

> I really don't know what they mean. Justify is not really my sort of word, like, it's a **big** word. Too big, usually you're asked to explain, not justify. (Kim)

Richard makes a comparison between the prompts 'explain' and 'justify', and articulates how he feels when he meets these two instructions in a problem:

> It's quite hard to understand what it is that they want you to write. You don't meet the word justify that often.... Explain and justify, they're sort of the same thing. If it was explain I could probably do it a lot easier. When you come across justify you think that they want a really long answer or why you have chosen this. If it was explain you could do it in a sentence or so, but justify you have to think more about what it is they want you to write down.

Common Concern

A key concern for the students was not knowing what makes a quality response. This issue was also raised by some of the teachers who recognized the need to develop an understanding of what is meant by a quality response. Students realize that their intended audience is usually the teacher or examiner and are therefore usually motivated to do their best. However, it is knowing what students should be aiming to achieve that is problematic when there are few suitable exemplars available to students and teachers. A report of brief comments from an examiner's report is insufficient feedback to develop teacher knowledge and expertise of what makes a quality response. As schools incorporate more internal assessment and greater use of open-response items, informed reliable professional judgement becomes paramount.

CONCLUSION

Tasks requiring students to write explanations and justifications are consistent with the aims of the curriculum focusing on communication in mathematics. The teachers recognized the importance of writing explanations and justifications and believe that the process should be specifically taught. They felt that the writing process helps students develop skills in thinking and communicating. Examining students' written explanations and justifications gives teachers an opportunity to diagnose students' difficulties or misconceptions. The teachers suggested that they could help students by modelling good examples, encouraging class discussion, and by involving students in the evaluation process. They viewed it as important that both students and teachers acquire an understanding of what makes a quality response.

The teachers identified a number of barriers when endeavouring to incorporate this aspect of communication into their classes. Concern was also expressed about the time factor and the lack of teacher resource material. They expressed concerns that the writing process could cause difficulties for some students who are not strong linguistically and it could therefore be a threatening process.

The majority of students felt positive about the writing of explanations and believed that it helped develop their thinking skills. They found that it helped them remember and confirm new mathematical understandings. Students need to be presented with a range of mathematical problems to be solved and encouraged to write explanations and justifications as part of the problem solving process. Concerns were expressed about the reading and comprehending of problems, knowing how much to write, and interpreting the context. The students also were not confident about knowing what makes a quality explanation and did not know what it meant to justify.

A learning environment should be established in which writing is a regular and natural part of classroom discourse so that students become more confident at writing explanations and justifications. This study has shown that students believe that writing explanations is an important process in mathematics. However, the students viewed the writing of explanations differently from the writing of justifications. They showed a reluctance to write justifications and had a poor understanding of what it means to justify. Together, both students and teachers face the dilemma of not really knowing what makes a quality response.

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Endnotes

[1] Each state and integrated school is ranked into deciles, low to high, on the basis of an indicator. The indicator used measures the extent to which schools draw from low socio-economic communities.

[2] Jessica writes prolifically in all her responses and frequently uses the blank back of a page to continue her response.

THE MATHEMATICAL ACHIEVEMENT OF CHILDREN IN THE COUNT ME IN TOO PROGRAM

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This paper reports findings from a study designed to investigate the impact of an early numeracy program on the mathematical achievement of young children. Two groups of Year 1 students were assessed using the Schedule for Early Number Assessment—once in May, prior to the experimental group participating in the numeracy program, and once in November. Results indicate that the experimental group performed significantly better than the control group at the post-test phase.

The first three years of school have been acknowledged as having a profound effect on the rest of a child's mathematical education (Wright, 1994a). While numeracy levels continue to receive attention internationally, it is not until children in NSW government schools are well into their fourth year of schooling that any form of systematic or formal assessment is made (for example, the Basic Skills Test is given to Year 3 and Year 5 students in NSW). By this time, vast differences are already evident in children's mathematical abilities (Fuson, 1988; Wright, 1994a; Young-Loveridge, 1989).

Numeracy, and its identification as a fundamental aspect of education, has been recognised for some time. However, Commonwealth government attention to numeracy and its relation to school education, has only been quite recent. It has now taken on a leading role in ensuring that numeracy, along with literacy, is made a national goal. In 1997, the Commonwealth, State and Territory Ministers agreed to a national literacy and numeracy goal:

> *That every child leaving primary school should be numerate, and be able to read, write and spell at an appropriate level.*

Furthermore, they agreed to the sub goal:

> *That every child commencing school from 1998 will achieve a minimum acceptable literacy and numeracy standard within four years.*

The National plan for literacy and numeracy places "importance on the early years because research indicates that if children have not met appropriate literacy and numeracy standards by the end of primary school, they are unlikely to make up the gap through the rest of schooling" (AAMT, 1997, p.49). The plan therefore calls for early intervention to address the needs of all students and for comprehensive assessment of students by teachers in the first years of schooling.

The NSW Department of Education and Training's early numeracy program, Count Me In Too, addresses these two aspects of the National Literacy and Numeracy plan. While it's focus is on the professional development of teachers so as to enhance their teaching of numeracy to all students, comprehensive assessment is an integral component of the program. The purpose of the investigation reported in this paper was to evaluate the impact of Count Me In Too on the mathematical achievement of young children. While the original investigation included children from both Kindergarten and Year 1, only the results of Year 1 are presented in this paper.

BACKGROUND TO THE STUDY

Count Me In Too (CMIT) is a professional development initiative of the NSW Department of Education and Training (DET) focusing on the early years of mathematics. Its main purpose is "for teachers to better understand children's mathematical strategies and their development from less sophisticated to more sophisticated strategies" (Stewart, Wright & Gould, 1998, p.557). It is researched based, originating from the theory and methods of the Maths Recovery (Wright, Stanger, Cowper & Dyson, 1996) and Reading Recovery Programs (Clay, 1993) incorporating aspects such as the Learning Framework in Number (Wright, 1998) and a clinical interview based assessment instrument—the Schedule for Early Number Assessment or SENA (DET, 1998). Rather than being a packaged program, CMIT is a continually evolving school-based initiative that involves a close liaison between the district consultant and a group of teachers at each school.

The work-based model of professional development operating in CMIT schools varies from school to school, but generally there is much more focus on children's solution strategies, on reasoning, reflection, problem solving and conceptual understanding rather than on the rote memorisation of algorithmic procedures. A video-taped clinical interview referred to as the Schedule for Early Number Assessment (SENA) is an integral component of the program. It is used to diagnose children's strengths and weakness, providing teachers with a 'blueprint' of each child's arithmetical development. Guided by results on the SENA and the Learning Framework in Number, teachers are able to map future lines of development for children on an individual basis. Currently, implementation is focussed on number, however a research base in Space and Measurement is being developed that can be implemented in a similar fashion.

In 1998 the NSW Department of Education and Training (DET) extended CMIT to include seventy-eight DET funded schools, two hundred and fourteen non-funded schools, over one thousand teachers and approximately twenty thousand K-2 students. While the focus of the project is on developing the knowledge of K-2 teachers in early number, the ultimate aim is to improve young children's mathematical abilities.

CMIT employs a work-based model of professional development, with mathematics consultants working in classrooms alongside teachers. Exactly how consultants become involved varies from school to school, but basically their role is to assist tachers with the implementation of the learning framework espoused by the CMIT project. Generally, this is achieved by consultants helping teachers assess the mathematical development of children in their class, and by helping them plan and implement developmentally appropriate learning and teaching experiences.

RESEARCH PLAN

Participants

Participants were largely from middle class families and attended one of two suburban metropolitan Sydney schools. The experimental group consisted of a Year 1 (n = 21) class involved in the CMIT project and the control group consisted of a Year 1 (n = 23) class from a nearby school not involved in CMIT. All teachers whose students were involved in the investigation had three or more years experience teaching in the Kindergarten to Year 2 range. While an effort was made to match the experimental and control groups as closely as possible in regard to socio-economic status, experience of classroom teachers and the like, the ultimate selection rested on each of the schools' willingness to be included in the study.

Materials and Procedure

The SENA was developed over a period of approximately five years and has been used extensively by teachers and researchers to assess the early arithmetical development of young children (Wright, 1996). It involves the presentation of 58 'tasks' or problems to a child in a one-on-one interview. The SENA assesses each child on five aspects of number development—Early Arithmetical Strategies (EAS), Forward Number Word Sequences (FNWS), Backward Number Word Sequences (BNWS), Numeral Identification (NID) and Base 10. It is the role of the interviewer (the classroom teacher) to elicit a child's most sophisticated strategy (or EAS) and then determine where each response might be categorised within a framework of predetermined stages and levels of development (Wright, 1994b). For a more thorough description of the Learning Framework see Wright, (1994b).

Children from both the experimental and control groups were interviewed individually on two occasions—once prior to the CMIT project beginning (May/June 1998) and once at its conclusion (November 1998). Research assistants, who conducted the interviewers, were four experienced teachers who had been trained to administer the SENA.

The SENA took approximately 15 to 30 minutes to conduct, depending on the ability of the child being interviewed. Children with greater mathematical ability were asked to perform more difficult tasks and therefore took longer to complete the interview. Each assessment session began with a brief introduction to the nature of the tasks the child would be asked to perform. If a child became noticeably frustrated by their inability to complete any task, they were stopped and asked to complete a task from a different section of the SENA. For example, if a child could not count five counters from a small pile of counters, they were not asked to count fourteen counters. Once it became obvious that subsequent items on the SENA were beyond the capability of a child, the interview was stopped. Research assistants attempted to elicit each child's most sophisticated strategy. If a strategy was not obvious, the assistant asked a child to explain how they found an answer. All SENA interviews were video-taped and later analysed by the chief investigator so as to determine where each response might be categorised within a framework of predetermined stages and levels of development.

RESULTS AND DISCUSSION

Results on the five early number aspect of the SENA for the Year 1 experimental and control groups are summarised in Table 1. Separate two tailed t-tests on the each of the five aspects of the SENA indicated no significant differences between the performances of the two groups at the pre-test phase. However, at the post-test phase, there were significant differences between the performances of the two groups on each aspect (EAS, $t = 4.7$, $p < 0.001$; FNWS, $t = 3.2$, $p < 0.05$; BNWS, $t = 3.6$, $p < 0.05$; NID, $t = 4.9$, $p<0.0001$; Base 10, $t = 3.7$; $p < 0.05$) indicating that the experimental group performed significantly better than the control group.

Graph 1 presents box plots for the experimental and control groups' performances on each aspect of the SENA at the pre-test phase. Graph 2 presents the results for each group at the post-test phase. Box plots allow more comparisons to be made between the performances of the experimental and control groups. In particular, they provide detailed information regarding the distribution of students' performances at the pre-test and post-test phases of the study. Generally, box plots are composed of five horizontal lines that display the 10th, 25th, 50th, 75th and 90th percentiles of student performances. All scores above the 90th and below the 10th percentiles are plotted separately. The two horizontal lines at each end of the plot (called 'whiskers') indicate where the majority of scores lie. The further apart the whiskers, the more 'spread out' are the scores. The middle 50% of scores are between the upper and lower borders of the box. The longer the box, the more 'spread out' are the scores. The horizontal line inside the box shows the middle score (or median) if all scores were arranged in order from smallest to largest. For instance, in Graph 1, a comparison of the box plots representing performances of students in the experimental and control groups on the Early Arithmetical Strategies aspect of the SENA (EAS E1 and EAS C1, respectively), indicate that at the pre-test phase the range of scores was the same (between Stages 0 and 4). However, the taller box for the control group means that student performances were more spread out—the majority of students performing at either Stages 1, 2 or 3. The absence of a horizontal line in the middle of graph EAS E1 indicates that the majority of students (the middle 50%) in the experimental group performed at Stages 1 and 2.

Table 1
Experimental and control group performances (means and standard deviations) on the SENA and each aspect of the SENA pre-test and post-test

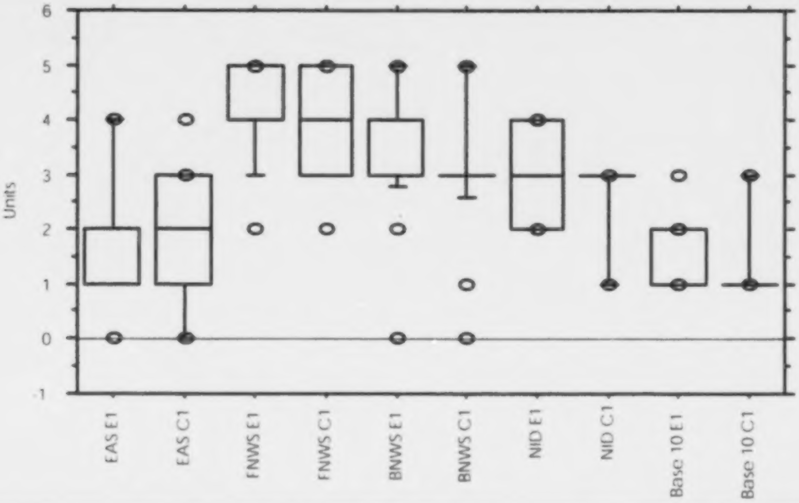
		Experimental Group		**Control Group**	
Aspect		Pre-test n=23	Post-test n=23	Pre-test n=23	Post-test n=23
EAS	Mean	2.0	3.5	1.7	2.4
Max=4	SD	1.1	0.9	1.2	1.0
FNWS	Mean	4.1	4.9	4.0	4.3
Max=5	SD	0.8	0.3	0.9	0.8
BNWS	Mean	3.4	4.5	3.1	3.7
Max=5	SD	1.2	0.8	1.1	1.2
NID	Mean	3.1	3.9	2.7	2.9
Max=4	SD	0.9	0.7	0.7	0.7
Base10	Mean	1.4	2.1	1.3	1.4
Max=3	SD	0.6	0.7	0.7	0.7

EAS	Early Arithmetical Strategies	BNWS	Backward Number Word Sequence
FNWS	Forward Number Word Sequence	NID	Numeral Identification

An advantage of using box plots to represent results is the ability to obtain information 'at a glance' about the full range of academic abilities and to make comparisons. For instance, it is evident from Graph 1 that the two groups were performing at similar levels for each of the five aspects at the pre-test phase. It is particularly interesting to compare the plots EAS E1 and EAS C1 on Graph 1 with EAS E2 and EAS C2 on Graph 2. The box plot EAS C2 (see Graph 2) indicates that while the majority of the lower achieving students (those at Stages 1 and 0 at the pre-test phase) had advanced to Stages 2 and 3, very few high performing students (those at Stage 3 at the pre-test phase) had progressed beyond their initial performances. In contrast, EAS E2 shows that the majority of students from the experimental group were performing at Stages 3 and 4 and that there were no students performing at Stages 0 or 1 at the post-test phase. Furthermore, a comparison of box plots EAS E2 and EAS C2 on Graph 2 reveals that the bottom 25 per cent of students from the experimental group were performing at the same level as the more able students from the control group on the SENA post-test. A possible explanation for the lack of improvement on this aspect for the control group lies in the fact that EAS, or early arithmetical strategies, such as counting-on and counting-back, are not closely related to Syllabus content (they are thinking strategies or processes which children use to solve computational problems) as other aspects, such as Forward and Backward Number Word Sequences, Numeral Identification and Base 10. While most students develop arithmetical strategies without the explicit instruction that occurs in CMIT, many continue to use inefficient strategies (e.g. counting on fingers) well into upper primary and even adulthood. Results from this study indicate that, with explicit teaching, *all* students can learn to use more efficient arithmetical strategies and that they can do so from as early as Year 1.

It is also apparent from Graph 2 that the performances of students from both groups were less 'spread out' at the post-test phase. This is particularly evident for the experimental group. While box plots reveal how children of various academic abilities performed at each phase of the study, caution needs to be exercised when interpreting the plots for the more able students in Year 1. For example, box plot BNWS E2 (Graph 2) looks as if the majority of the Year 1 students from the experimental group were all performing at the

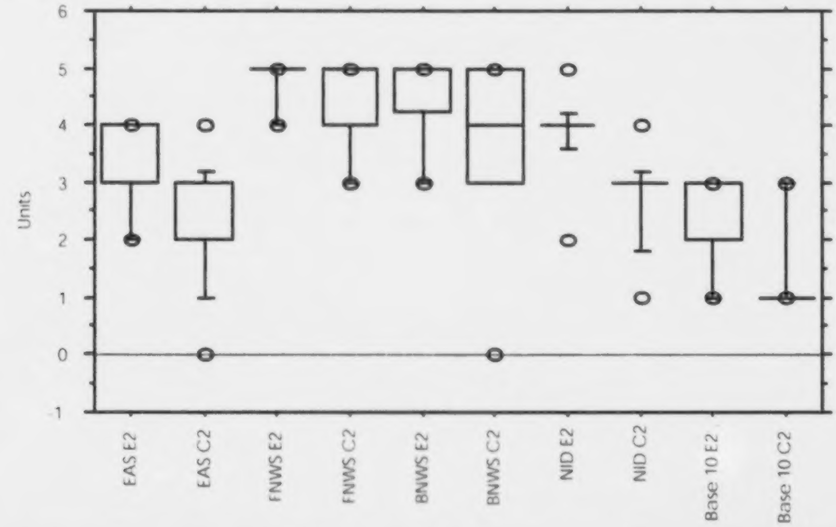
Graph 1
Box plots of experimental (E) and control group (C) performances on each aspect of the SENA for the pre-test



* Units correspond to Stages of development for EAS and Levels of development according to the Learning Framework in Early Number for all other aspects

same level (Level 5). In reality, this may not be the case, since Level 5 is the upper limit described on the Learning Framework and many able students could probably perform at even more advanced levels if the framework were extended. In other cases, such as for the box plot EAS C2 (Graph 2), where students have not reached the upper level of the framework, it is clear as to how students of various abilities performed in relation to each other.

Graph 2
Box plots of experimental (E) and control group (C) performances on each aspect of the SENA for the post-test



* Units correspond to Stages of development for EAS and Levels of development according to the Learning Framework in Early Number for all other aspects

While the clustering of performance scores may be a result of students from both groups reaching the upper limit of levels on the Learning Framework for aspects such as Forward and Backward Number Word Sequences, it cannot explain the smaller variation in performances on Numeral Identification and Base 10 aspects where there were still higher levels to which students could advance. For example, a comparison of box plots NID E1 on Graph 1 and NID E2 on Graph 2 illustrates how performances of students from the experimental group were distributed between Levels 2, 3 and 4 at the pre-test phase, but at the post-test phase almost all the students performed at Level 4.

SUMMARY AND CONCLUSION

The fact that there were no significant differences on the SENA performances overall between the experimental and control groups at the pre-test phase indicates that they were well matched in mathematical ability. The significant advances made by the experimental groups on all aspects of the SENA are clear evidence of the positive impact Count Me In Too can have on the mathematical ability of children involved in the program.

The suggestion that the experimental group may have been unfairly advantaged because the SENA contained tasks requiring skills emphasised by the CMIT program and thus were able to 'practise' them before the post-test phase, is not a viable explanation for the results. Aspects, such as Forward and Backward Number Word Sequences, Numeral Identification and Base 10, are included in the NSW Mathematics Syllabus (Department of Education, 1989) and should therefore be emphasised in all K-2 classrooms and not just ones in which CMIT is operating. Also implausible, is the suggestion that CMIT classrooms devote more time towards the teaching of number and forgo work on the measurement and space strands. Teachers involved in CMIT are required to fulfil Syllabus requirements across all the strands.

An obvious limitation of the study is the small sample size. Future investigations that incorporate larger samples should endeavour to make comparisons between schools with populations drawn from a range of socio-economic and ethnic backgrounds, and from rural and metropolitan schools. While the current study included children mostly from middle-class backgrounds, the findings cannot be extrapolated to the general population without further investigation. It would also be beneficial to monitor the progress of children on a longitudinal basis, say over a period of one or two years. In this way, it might be possible to determine if CMIT gives children a 'head start' in mathematics that continues on in to their upper primary years.

Another limitation of the present study was the fact that many students, particularly in the experimental group, reached the upper limits of many early number aspects being assessed by the SENA. Thus, their true mathematical abilities were not reflected in this study.

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MATHEMATICS, ICT AND EFFECTIVE TEACHING

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This paper reports some initial findings from a substantial project funded by a UK government agency. The project aimed to investigate effective pedagogy in numeracy and literacy using ICT in primary schools. Some preliminary findings relating to pupils' attainment and a series of lesson observations are reported. Some of the broader findings from the development work in mathematics and ICT are also discussed in the context of two particular case studies which focused on developing effective pedagogy in mathematics.

INTRODUCTION

This paper describes some aspects of a two year project investigating effective pedagogy in numeracy and literacy in primary schools (for 4-11 years) in England. The research was funded by the Teacher Training Agency (TTA), a government body responsible for the initial teacher training curriculum, standards and continuing professional development of teachers. Although the final report of the project will not be published until the middle of 1999 some of the implications from specific parts are discussed here, particularly in relation to mathematics teaching and the effective use of Information and Communications Technology (ICT). In England there is considerable change taking place in the teaching of mathematics in the primary age range. At the same time the government is funding, from the UK National Lottery, training for all serving teachers to develop their skills in the use of ICT in their teaching. This paper discusses some of the observations of mathematics teaching from the project, some of the work of the project team concerning teachers' thinking about ICT and some relationships with their practices as well as presenting two of the development projects as brief case studies. We then draw out what we see as some of the overall implications for effective development work in mathematics and ICT.

THE RESEARCH PROJECT

As part of its remit to improve the quality of teaching in England, the TTA commissioned the University of Newcastle to undertake research into effective classroom pedagogy using ICT in primary schools. The TTA intended that this research should:

- test and develop a generic framework highlighting the potential benefits and pitfalls of using ICT in classrooms, particularly in the areas of numeracy and literacy;
- help teachers raise pupil achievements in these areas through supporting informed choice about the use of ICT in the classroom;
- refine and illustrate specific aspects of ICT use through "rich pictures" - detailed classroom case studies of effective teacher practice and development.

The research was carried out in three main stages. During the first stage, a range of schools was identified and the extent of ICT provision determined. Some of the schools were identified using value-added (VA) data from the Performance Indicators in Primary Schools Project (PIPS) based at Durham University, initially through a brief postcard survey (2053 schools: 66% return rate). The postcard survey yielded data for 740 classes for which 1996-7 value-added scores were available. Then, to ensure the aims of investigating effective pedagogy were met, a sample of only those teachers who were identified as having high or average VA was selected. This enabled correlational links to be examined between a more detailed follow up questionnaire and the VA performance data from the schools to identify any factors which were distinguishing characteristics of the high VA teachers. At this stage, 250 questionnaire returns were received (a response rate of 34%). In the second stage of the project specific teachers from these schools and other 'ICT-rich' schools were investigated to identify pedagogical factors which correlated with increased pupil attainment. Structured observations of four lessons in each of 32 classrooms were compared with data on pupils' performance. In the final stage of the project, data from the questionnaire, the observations and existing research information were used to plan specific development projects in numeracy and literacy with 20 teachers in their own classrooms. Evidence for the effectiveness or otherwise of this development was focused upon pupil attainment using standardised tests. In addition further observations and interviews with teachers were completed. Then a construct elicitation exercise was undertaken with teachers to investigate their thinking and beliefs about teaching and learning.

Value Added (VA) and PIPS

PIPS is one of a series of performance indicator (PI) systems used in monitoring education. They were developed originally by Prof. C.T. Fitz-Gibbon (see Fitz-Gibbon 1996 for details) and all use VA scores as their basis. VA data are measures of relative performance calculated annually at the individual pupil level. They are based upon a model which uses a least squares regression analysis technique resulting in relative residuals. The model requires very large numbers of pupils in each cohort - PIPS currently has approximately a third of a million pupils involved each year. Individuals' scores in a variety of tests are measured using material which has been shown in independent research to be the best available predictors of future performance. For each child, a predicted score is calculated using their individual input variable scores. Their actual achievement score is then compared to this. For example, a child in maths may score 50 and the model predicted 45. S/he would have a relative residual of +5. Clearly, any child achieving the same score as their predicted score will have a VA of zero. For the purposes of this project, these individual VAs were aggregated up to teacher level. A teacher whose pupils' aggregated score is zero is taken to be 'averagely effective' whereas one whose pupils' scores are significantly positive is said to be 'highly effectively'.

SOME ASSOCIATIONS BETWEEN QUESTIONNAIRE DATA, THE OBSERVATION DATA AND PUPILS' ATTAINMENT

Analyses of the initial postcard survey data were carried out to see if high computer use across a school year group was linked with high levels of achievement, high VA scores or very positive attitudes. Generally there was little evidence for this. There was almost no association between the reported amount of computer use and either achievement level or attitude. However there was a relationship with VA in that the small number of year groups where pupils were said to use computers "more than once a day" had positive VA scores on average. Year groups where pupils used computers "less than once a month" had negative VA scores on average. This suggested that, at the extremes, highly effective teachers tended to have pupils using computers a lot and less effective teachers have pupils using them rather infrequently. The effect size differences between the high and low groups were 0.6 for mathematics and 0.4 for reading, with the most significant result applying to mathematics VA in Reception classes (4-5 year olds). This clearly implies that Reception teachers who are particularly effective in mathematics tend to use computers more often in the classroom. This finding was validated to some extent by a connection between pupil VA scores in mathematics for Reception teachers taking part in the development work who also reported greater use of computers for number work (a correlation of 0.53, but non-significant, due in part at least to the small number of Reception teachers (8) involved in development work).

Some significant relationships between the first set of lesson observations (on four lessons with each of 32 teachers) conducted by the project team and the VA scores of the pupils were also found.

Teacher Behaviours

(Please note: * indicates significance at the 0.05 level and ** at the 0.01 level)

- Highly effective teachers used more examples and counter examples (0.61**) in numeracy lessons, particularly in the plenary phase of the lesson (0.41*). For example a teacher might give an example of an incorrect solution as well as a correct solution and explain why one was correct and the other incorrect. *(This was a particularly strong relationship and suggests that effective teachers use examples as part of their teaching but also exemplify what they are teaching with counter examples for clarification.)*
- Highly effective teachers of Reception classes (4-5 years) tended to provide different ICT activities for pupils with special educational needs (SEN) from those for the rest of the class (a correlation of 0.75* with VA scores for the eight teachers who reported providing such activities). The opposite trend appears for teachers of older pupils (classes of 6-7 years and 8-9 years), though the correlation was not significant.

Pupil Behaviours

- Pupils tended to model work as examples to their group or the rest of the class in the highly effective teachers' classes (a correlation of 0.51** with VA scores for the 32 teachers observed).
- Pupils tended provide examples more in the highly effective teachers' classes (a correlation of 0.45*): these last two findings are clearly linked, though some caution in interpreting this result is called for. The frequency of pupils' examples in the final plenary phase of the lesson does not seem to be so clearly beneficial. Such pupil behaviours in the final phase of the lesson were negatively associated with value-added scores, a correlation of -0.42 (ns) was found for the 18 teachers for whom data was available.

We interpret these findings as indicating that effective teachers make good use of clear explanations, perhaps an unsurprising result and one which reinforces the work of other researchers (e.g. Dunkin and Biddle's (1974) review of classroom research from the 1960s and early '70s or Rosenshine's work (1983)

IDENTIFYING AND OVERCOMING BARRIERS IN USING ICT TO ENHANCE TEACHING AND LEARNING IN CLASSROOMS

Part of the challenge set by the TTA was to summarise existing work on identifying and overcoming barriers to using ICT effectively in the classroom; to exemplify some of these barriers and the way the project team tackled them through the development work in schools; and finally, to suggest ways of understanding these barriers as a means to overcoming them.

The task was approached in several different ways. Many different barriers were identified and these fell into various categories. *Time* related issues were consistently identified as the biggest group. The next set appeared to be *equipment* related and this was followed by *pupil* related ones. It was possible to make a clear distinction between issues relating to people and those relating to equipment: e.g. the number of pupils in my class; the amount of adult support; and the number of machines available for the pupils to use. Perhaps the most important pedagogical issue identified was knowing which software was appropriate in a given situation. However, in general, these pedagogical barriers which teachers identified were seen by them as relatively less important than the other barriers indicated above. These findings match closely the 'problems and challenges' recently reported in Scotland (William's et al, 1998)

Later, in a group discussion with some of the teachers, several significant issues were again raised: it is difficult for teachers to devote time to only two or three pupils who are working on computers; the number of computers per classroom is still low (usually one, occasionally two); the time needed for teachers to learn new software appears to be increasing with the complexity and sophistication of the software; the time for some to learn how to use basic software for word-processing or data handling well; the lack of immediately available technical support; past frustrations, especially with printers, leading to de-motivation and the ever growing cost of consumables.

A different sub-set of teachers from the original questionnaire was also surveyed (120 teachers, 53% return) and asked to self-rate how confident they felt to tackle particular tasks using ICT. Their responses (n=64) suggest that confidence in using ICT is still a challenging issue for primary teachers. Only *using a spell checker* was an area where the average score indicated that teachers were confident to tackle the task.

The team also tried to explore positive examples of overcoming barriers and several factors were identified as being related to this: supporting the development of pupils' skills to enable a clear focus on objectives in numeracy or literacy; providing information about software, but again with specific subject-related objectives in mind; finding ways to increase access for pupils and providing copies of software for teachers to use at home. In subsequent interviews, teachers reported that offering technical support and supporting their personal skills had a positive impact on their attitudes and confidence.

Further study of our data revealed that teachers with high VA scores: (a) tended to report that computers made them feel good about their teaching; (b) thought that computers did not require too much technical knowledge and (c) said that computer software was not too complicated for children. Since these data were strongly associated with highly effective teachers, it suggests to us that such teachers see the potential of the technology and try to use it effectively by integrating it as part of their repertoire of teaching strategies and approaches. In the same way the team feel that the barriers are seen by some teachers as something which is to be overcome and by others as more of a barricade which prevents them realising the potential of ICT. To use Leat's (1999) metaphor, for these teachers it is a case of rolling the stone *uphill*. For the former grouping, those who may have needed help at first, it is more a case of rolling a stone *downhill*. Once barriers have been overcome the development work gains momentum and overcoming barriers becomes less of an issue as the teacher's skills and confidence increase. On the other hand, for a few teachers in the project, once one barrier is addressed and overcome, another appears. Rolling the stone uphill in this case becomes more like the task of Sisyphus, as once your back is turned the stone seems to return to the bottom of the hill.

To summarise this section, teachers identified ICT-specific barriers as a major difficulty in their attempts to become more effective in using ICT in the classroom. They did not identify subject-specific factors in numeracy or literacy or general pedagogical issues. In some cases these ICT-specific barriers were overcome relatively quickly and easily. In others, these barriers were not overcome easily and required a determined effort on the part of the development team and the teachers involved. All of the information discussed above was used by the team as background to planning specific interventions with 20 teachers in numeracy and literacy, two of which are now described in more detail.

CASE STUDY 1: DEVELOPING COUNTING SKILLS WITH RECEPTION PUPILS USING ICT

This case study focused on developing counting skills with 4-5 year old pupils. The pupils created counting pictures using a children's drawing and painting program[1]. The activities were designed to complement other numeracy activities and to act as a focus for discussion between the researcher and the teacher. The development work took place over two terms with two different classes of pupils. The results from the standardised testing[2] suggest that teacher was able to use ICT effectively as part of her teaching to support the development of her pupils' counting skills. Using detailed knowledge of how young children learn to count and the errors that they commonly make was an important feature of the development work in ICT and mathematics.

The teacher has been at the school for six years and usually teaches the Reception Year. When planning activities she likes to make links between different activities and to draw out connections for the children. "*.... they might need work on number recognition, or using number lines, and it isn't always linked. But usually we do something that is to do with counting, a counting rhyme or acting out a song which is linked to other activities.*" This approach is consistent with the new National Numeracy Framework's (Department for Education and Employment, 1998) recommendations for the Reception year currently being introduced in England and with recent research on effective teachers of mathematics (Askew et al, 1997).

At the beginning of the Summer term, some pupils were making errors in reciting the number names accurately (particularly numbers from 13-19 and at the decade transitions such as 29 to 30), some had inaccurate one:one touch counting skills and strategies which led them to give an incorrect total at the end of a count, and some were also unable to identify the correct numeral to go with a number name with numbers up to 20. These matched previous findings of other researchers such as Denvir and Brown (1986) and Maclellan (1997). The teacher was therefore keen to address these issues in a broader way. She saw ICT as enabling her to do this more effectively than she had been able to do with previously available software which focused only on numeral recognition. "*(It) was quite different from anything (that) I've used before, (which) had been very closed.*"

The teacher found the diagnostic information from the baseline test very useful. It gave her detailed information about the particular counting errors that specific pupils were making. In addition to the work using ICT she planned other numeracy activities to address pupils' particular difficulties. The teacher wanted pupils to be able to practise specific aspects of counting and therefore emphasised different counting skills with different children. The software has the facility to let children stamp a variety of pictures onto the screen thus allowing them to work on the different aspects of counting which had been identified in the initial assessment.

The teacher thought that having pupils create mathematical pictures, and count out loud as they did so, enabled them to use the ICT more independently. They were able to create a variety of pictures over a period of several weeks. This matched a number of the teacher's mathematical objectives for the term such as using their counting vocabulary, improving the reliability of their counting, teaching them more systematic counting strategies and helping them to recognise and use numerals. "*We didn't have much in the way of numeracy materials, whereas using the (ICT) has really helped, and the children have used a lot more mathematical language.*"

A target group of five pupils was identified whom the teacher thought would benefit from the ICT work, together with a control group of 3 others. A standardised test was used to assess children before and after intervention. The target pupils all showed that their counting and number ability in particular had improved. They had also improved relative to the other three pupils (an average of 8 point gain in age-standardised score compared with an average gain of 1 for the control group). This development work was repeated in the Autumn term with the teacher's next class, but without a control group. The class made impressive gains. Their age-standardised score improved by an average of 15 points in just over two months.

It is clearly not possible to claim that the pupil score improvements were directly due on either occasion to the ICT activities, especially as the teacher used the information from the standardised test to inform her planning more broadly. However the results do suggest that carefully planned and structured ICT activities with clear mathematical objectives can play an effective role in improving pupils' counting skills.

CASE STUDY 2: DEVELOPING UNDERSTANDING OF DECIMALS WITH Y4 PUPILS (8-9 YEARS) USING PORTABLE ICT EQUIPMENT

In this case study, the teacher wanted to introduce her class to decimal notation. The school's mathematics policy, like many others, follows very closely the National Curriculum and so she felt she was almost directed to teach this through the medium of money notation. However, in the UK there is a culturally related problem which makes this quite complex. £3.24 is not seen as a single number of pounds but rather as two closely related numbers: three pounds and, separately, twenty four pennies. This leads to the common error of writing the amount as £3.24p. It is almost as if we have an Imperial notation imposed on a metric system. The result is that for teachers of 8-9 year olds, this becomes an unnecessarily difficult hurdle when teaching decimal number, so one of the teacher's main concerns was the issue of how meaningful the pupils would find such decimal numbers. Given that she eventually wanted her pupils to be able to order and even calculate with decimal numbers, she felt that it was important to use numbers where the pupils could draw some understanding from the context.

After discussing matters with the Project Team member, she agreed to try using *time* as the medium rather than money - time measured through the use of ICT. The activity used an e-Mate (a robust portable computer designed for children) connected to a pressure-mat sensor. This allowed the children to generate times in seconds to two places of decimals. The equipment enabled two different types of activities to be timed, each of which needed different interpretations as to which was the 'best'. The first activity required the children to stand on the pressure mat and jump in the air before landing on the mat again. The times for these standing jumps were typically 0.43 or 0.37 seconds. Here, the 'best' time was the biggest number - the longest time in the air. The second activity exploited the portability of the equipment as the children used the ICT equipment as a timing gate. They took it out into the playground and ran from one side to the other and back again. As they set off, they trod on the mat starting the timer. It stopped when they returned and trod on it again. Here the 'best' time was the smallest number - the shortest time taken.

The teacher chose to use the ICT equipment for two reasons. The first was to generate 'real' numbers which the children could understand from the context in which they were produced. The second was as a stimulus to get the children thinking about what 'best' meant and about using decimals in a specific context to achieve this. She also wanted to use aspects of the published scheme which the school uses as part of its approach to mathematics throughout the school but also to make links to understanding decimals in another way through the ICT activity. Using the ICT also had an effect upon the choices related to her expectations of the children. She had not done a lot work on decimals with previous Y4 classes. This, she thought, might have been related to her own needs as well as those of the children.

> "I found decimal fractions quite difficult for a lot of children. It may be the fact that I didn't expect them to take it much further that was part of my problem as well. The e-Mate stimulated their interest to learn and understand beyond my expectations. I would never in my wildest dreams have thought they would cope with this."

Not only was there a global shift in her expectations of what the whole class could achieve, but there was also a few significantly different changes for individuals.

The pupils completed a standardised mathematics test[3] before and after the intervention. The results indicate that the mean number-age rose from 8 years and 4 months to 9 years and 5 months. This gain was achieved in four months. (The difference is statistically significant at <.001 level.) The test did not specifically look at using decimals, although that skill was a component of the test. Whilst it is important not to attribute this gain solely to the ICT activities, it is thought reasonable to suggest that they will have played a part in developing specific mathematical skills and understanding. In addition, it also seems reasonable to assume that they will also have played a part in the more general improvement in the pupils' confidence and attitude. The idea that working on developing one area of mathematics has a non-planned but significantly beneficial effect upon others is consistent with the work of Denvir & Brown (1986) and is therefore not surprising.

SOME CONCLUSIONS AND IMPLICATIONS

Obviously we have to be somewhat cautious about over-stating any claim based solely upon this work, but there are some interesting outcomes. It appears that effective teachers use examples and counter examples as part of their normal teaching. The pupils of such teachers tend give examples during lessons and to model their work both when talking to the teacher and to their peers.

The teachers' and the pupils' levels of personal confidence appear to have conspicuous effects upon the outcomes when measured through pupil achievement. What is more, the degree of their confidence in using ICT appears to be rooted quite firmly in their personal skill levels. Our data would support the view that, on the whole, effective teachers who use ICT are teachers who are confident with ICT and that they are much more comfortable with ICT as an enabling addition to their pedagogical armoury. Where ICT is not adopted as a normal part of their teaching, certain barriers play important parts in determining whether and how teachers in primary schools adopt ICT in their teaching. In trying to explore this, the research team found that if the specific ICT-related issues were addressed head-on, this would not necessarily lead to future success for the teacher. What we felt was important was to try to work around the problem rather than to confront it. This was done by helping the teachers to keep a very clear focus upon desired mathematical content targets for the lesson(s) and not to get side-tracked into the ICT issues. Principally, this was achieved by having someone in their classroom helping to get their pupils over an initial threshold of personal ICT skills which allowed the teachers to concentrate upon the mathematical purpose of the lesson. Using this approach, it was as if the original hurdles were by-passed rather than confronted. The result was that the importance of solving ICT issues diminished considerably. Once past this hurdle, it was as if the changes became self-fuelling and the teachers began to use the ICT to achieve improvements in their teaching without further help. ICT was seen as insufficient in itself to stimulate changes in their pedagogy. However given assistance to get their pupils sufficient skills to then concentrate upon the mathematical task in hand (rather than the technicalities of using the equipment), they could achieve some surprising results - surprising often to themselves as well as to outside observers. We believe this may well have implications for a UK government which seems to believe that the large scale and very expensive adoption of ICT could be an important factor in solving a pressing teacher shortage.

The data and case studies presented here are only a small selection of the initial findings from a substantial, very data-rich project and do not attempt to be a complete report of all that has been measured and analysed. They are seen as being principally illustrative and not conclusive. Possibly the most important thing to say is that whilst many of results reported are correlational, the team are crucially aware that there should be no attempt to suggest that there is necessarily any causal link implied or intended, only that the findings may well indicate areas where more detailed and carefully structured investigation may be necessary to establish firmer links. However, we believe there are some very interesting and possibly important findings that may well prove to be significantly influential in the future.

At this stage of our research however, and somewhat fundamentally, we conclude with two points: first, we cannot see machines replacing the essential element of the personal relationship that exists between teacher and pupil in the primary years of schooling, though the addition of new technologies may enhance this teaching and learning relationship, and, second, we would not want to underestimate the effect of the personal relationship which develops between teachers and those supporting their professional development as an important factor in successful development.

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Endnotes

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MATHEMATICS TEACHERS AND THE VCE: BROADENING THE EDUCATIONAL LANDSCAPE

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The Victorian Certificate of Education (VCE) has been subject to ongoing modifications since its full introduction in 1992. These changes largely focused on reducing curriculum choice and the value of the school-based assessments. The recent government initiated Review has maintained this direction. We draw on the experiences of close to 500 senior mathematics teachers -interview and survey data - to consider the impact on them of the evolution of the VCE, particularly on their teaching practices at the junior secondary levels.

INTRODUCTION

Education is a large budget item and governments seek greater accountability for the increased economic outlays manifesting as greater government intervention in the outcomes of schooling (Horwood, 1998). In the early 1990s educational reform at both the National and State levels was aimed at meeting the requirements of a complex, rapidly changing and increasingly technologically dependent society (Hiebert *et al*, 1996). The Victorian Certificate of Education (VCE) is one example of this reform movement which aimed to change classroom practice through government intervention.

Prior to the introduction of the VCE in 1990 there were alternate forms of high school certificates. These had evolved to cater for the different types of students who wished to remain at school and complete some form of certificate. The VCE aimed to bring all of these courses and certificates together under the one umbrella while still catering for the diversity of student aspirations. The structure and requirements of the VCE anticipated modified classroom practices throughout all levels of secondary schooling through changes to both the standard curricula and the assessment tasks set. In its original format, the VCE provided a reduced emphasis on statewide examinations with equal weighting given to alternate and school-based assessment.

Opposition to this educational reform soon emerged, with considerable criticism of the assessment format from within the university sector. In one way the original VCE could be considered to have signalled an effort to disrupt the powerful influence of the university professoriate on the school curriculum. In science and mathematics in particular, students and teachers have traditionally been forced to cope with an academic curriculum primarily designed as a preparation for university studies (Horwood, 1998). As a consequence, the VCE was controversial, frequently capturing the attention of the press, which eventually led to two waves of major modifications in the curriculum and assessment format during the mid-1990s. These changes resulted in a much reduced and more hierarchical subject selection, and a return to a greater emphasis on examinations.

The terms of reference of the recent government Review of the VCE (1997) largely focused on finding ways to reduce students' workload and maximising authenticity of students' school-based assessment tasks. School assessed CATs have been replaced by school assessed coursework which are to be now completed as part of the regular teaching and learning program. The examinations remain unchanged and will be used to moderate schools' assessment of coursework to ensure comparability across the State. The outcomes of the

VCE Review appear to have continued the relentless direction towards a reduced emphasis on time spent engaging in alternate forms of assessment. The most significant components of the curriculum in mathematics, the problem solving and investigative tasks, have been eroded to a small shadow of their original status over the decade.

Historically, the successful implementation of educational innovations has been thwarted largely when teachers do not have a clear understanding of what is expected of them, and when they lacked the skills and knowledge required to perform their new role (Gross, 1971). When describing the tendency for educational reform to fail, Fullan (1982) argued this was due to a greater emphasis on policy-making at the expense of developing an understanding of how to work through a process of change. Whether this is the way to now view the current situation is open to debate. The final status of school-based assessment is yet to emerge fully. Evaluation of educational change is daunting because the process is complex and dependent on many interacting factors. Despite this, one of the crucial factors that affects the adoption and continuation of innovations is the quality of the relationship between policy-makers and local practitioners (Fullan & Stiegelbauer, 1991). Quality education is also time consuming and requires resources, up-to-date technology and qualified teachers.

The current situation in Victoria certainly reflects the influence of bureaucratic control and is consistent with treating the content as non-problematic with the focus on assessment and moderation (Horwood, 1998). While the use of external moderation procedures are designed to guarantee comparability across the State for school-based assessed coursework, the equitability of the assessment procedures have "the added effect of providing a means for constructing teaching in terms of quantifiable outcomes and external objectives" (Horwood, 1998, p.269). If greater equity was ever the real intention, moderation procedures might also take into account different assessment outcomes in terms of socio-economic status and gender. In mathematics, for example, research continues to highlight enormous gaps in assessment outcomes between students from different socio-economic circumstances (Teese, Davies, Charlton & Polesel, 1995). In terms of gender, the 1992 VCE mathematics assessment data demonstrate uniquely that boys do better overall than girls in examinations while girls tend to perform better than boys on the more intensive school-based assessment (Leder, Rowley & Brew, 1999). Subsequent modifications made to the VCE study design led to this situation becoming clouded. With statistical modelling to control for the tendency for girls to select themselves out of higher level mathematics, the clear gender pattern evident in the 1992 data remerged for 1994-1996 (Rowley, Brew & Leder, 1997). It is conceivable that the proposed scheme to use the examinations to moderate the school-based coursework may well exacerbate the advantage to males in the overall assessment.

It is the role of this paper to broaden the educational landscape by presenting the views of mathematics teachers on these broader issues with respect to the VCE. It is hoped that this paper will serve as an historical marker from which to consider the impact of the latest round of VCE modifications. It is our perception that the voices of the secondary school teachers have been largely absent in public discussion and in the policy directives that have been instigated. In this paper we present evidence that indicates there is a high level of support among mathematics teachers for alternate assessment at the VCE level. There is a widespread perception that it encourages the development of different mathematical skills in students, in particular, to be able to apply their mathematical knowledge rather than just follow rules and procedures, and that it also encourages more innovative teaching practice in the junior levels.

THE STUDY

The Sample

In Oct-Dec 1996, interviews with 40 mathematics coordinators from across all school sectors and regions were conducted to explore the impact of the evolution of the VCE on schools, teachers, and teaching practice (Brew, Rowley & Leder, 1997). These interviews formed a pilot study that led to the development of a survey to explore the views of a representative sample of teachers across the State. Issues that emerged in the interviews structured the survey into a range of themes and a selection of these are drawn upon here to report the findings.

In April 1997, surveys were mailed to 250 randomly-chosen secondary schools within the three school sectors. For each of the sectors, 100 State schools were approached, 50 Catholic and 50 Non-Catholic Independent schools (hereafter referred to as Independent). Close to 500 VCE mathematics teachers responded from 162 schools (83 State, 44 Catholic and 35 Independent). The school response rate was 65%. A more detailed summary of the survey findings is reported by Brew, Rowley and Leder (1998).

Format of the Survey

The survey contained 140 items. Teachers were asked to identify whether their school had obtained funding under the disbanded Disadvantaged Schools Program [DSP](pre 1993) or the Victorian Equity Program (1994). And to provide biographical data to allow comparisons between teachers' views and experiences by gender, years of teaching, qualifications and studies taught. A standard Likert five-point scale was adopted for most items with teachers being asked to indicate their strength of feeling from "strongly agree" to "strongly disagree." A "yes/no" response was requested in some cases; further comments in others.

RESULTS

The Impact of the VCE upon Junior Level Mathematics

One aim of the VCE was to encourage innovative teaching practice across all levels of secondary schooling through a diversified assessment program at the highest level. Seventy-five percent of teachers said that they had modified their mathematics teaching at both senior and junior levels since the inception of the VCE. Support for the major innovations with their greater emphasis on problem solving and project based studies was high (83%) across all school sectors and regions. These innovations were reported by 68% of teachers to have led to an enhanced focus on mathematical communication and justification of work. Just on half of the teachers also wished they could include more project work and problem solving in their junior classes but the time required for marking prevented them from doing so. From the interviews with the coordinators it also emerged that it was more difficult to be innovative at the higher levels due to the severe time limits and high stakes nature of the assessment. Despite the restrictions, junior level teaching provided the greatest opportunity to be inventive and flexible in their teaching practice. The following quotes from provide examples of this perspective:

> There is a lot less chalk and talk than there was, especially in the junior levels. A lot more activity based learning. (At) VCE level, there is still a lot of chalk and talk as you have to get through the course.
>
> Problem solving ... was the best innovation which filtered its way down the school so we now have specific problem solving exercises from Year 7 up. (The VCE) certainly has changed teachers' attitudes to the importance of problem solving and projects. At the top end it hasn't worked as well where you have a four week project. That is too demanding.

Female teachers were also more likely to say they had modified their junior mathematics teaching practice than were male teachers (females: 87%; males 71%). This is consistent with the notion that the VCE had to some extent feminised the curriculum through the introduction of assessment that required greater language skills. Such skills were more likely to be possessed by girls than by boys and hence provided a greater opportunity for girls generally to participate in mathematics.

The attempt to introduce and broaden innovative practice in the junior level teaching was not without its problems. The rationalisation process that was sweeping through State schools in the mid 1990s, epitomised by reduced budgets and staffing, would appear to be a major factor, not just a backlash against top down change (Fullan & Stiegelbauer, 1991).

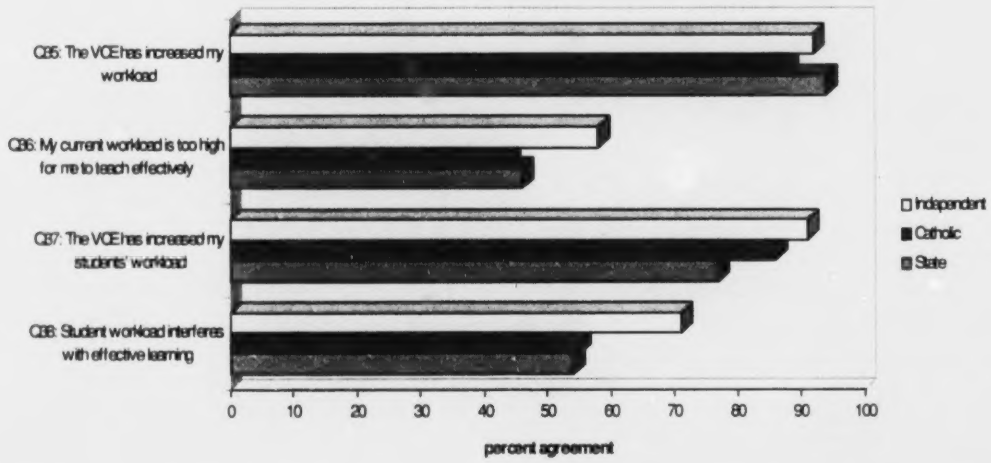
> The VCE has changed mathematics right through down probably as far as year 8. There was certainly an enormous change two years ago. I think a lot of that has gone by the board now because people have cut back a lot of the project work because of the time features. I know I was doing a lot more project work three years ago compared to now.
>
> One big problem is the VCE is enormously time consuming to do it correctly, (because of) the workload, a lot of people are paying lip service. ... realistically, if you do a Year 10 class and do the projects correctly, you have created 10 hours of work. To put in 10 hours at one class and then you have to do similar things at Year 7, 8 and 9, you are looking at a phenomenal work load, you can't run a high quality program and cut the guts out of it which is what is happening.

High Workloads

A huge increase in teacher workload was a major factor associated with the introduction of the VCE that emerged from both the interviews and survey findings. Nearly one in every two teachers reported that it was interfering with effective teaching (Figure 1) and this was more evident in Independent schools where the pressure is more likely to be higher on teachers to maximise their students' academic achievements.

What the teachers were effectively saying was that the introduction of problem solving and project work was very time consuming, both in the classroom and in the assessment of generated student work. It was increasingly becoming impossible for State school teachers

Figure 1
Teachers' Views on the VCE Workload by School Sector



to maintain the momentum and commitment to an innovative curriculum right across all levels during a rationalisation process and casualisation of the workforce.

> We have eighteen staff here involved in maths, and I would estimate that there are five who are full time maths. Considering we have 930 students, I would have hoped for more. There are six teachers that have got one class and that doesn't represent the building of strength.
>
> One of the major problems is that the government has now brought in this rule about absorbing your own excess. We were in a position of having seven teachers in excess ... and this has caused us to be pulling in a number of people from other faculties ... with little maths experience to take junior classes. I have always had a rule at the school you put your most experienced people with the year 7s and the VCE. ... But when you are in a situation where you have someone who is the careers teacher and who has taught junior maths ten years ago, you are forced into putting that person in year 7 or 8, not in the middle school.

Roberts (1994) also noted a related problem. Despite teachers in Years 7-10 attempting to introduce more VCE-like exercises into their courses, they were frequenttly not well qualified in mathematics and therefore evaluation and analysis, the two key areas that allow students to obtain higher grades, were not generally a focus.

Teachers' Qualifications

The large majority of mathematics coordinators (86%) did, however, indicate they had enough qualified mathematics teachers for the VCE. A breakdown down by years of teaching senior mathematics found that nearly 80% of teachers with more than 20 years experience had studied three years of university mathematics while fewer than 60% of the teachers with less than five years experience had done so. The figures also revealed that a few schools have always been using unqualified mathematics teachers to fill shortfalls, but there was also a trend to suggest that the shortfall is growing.

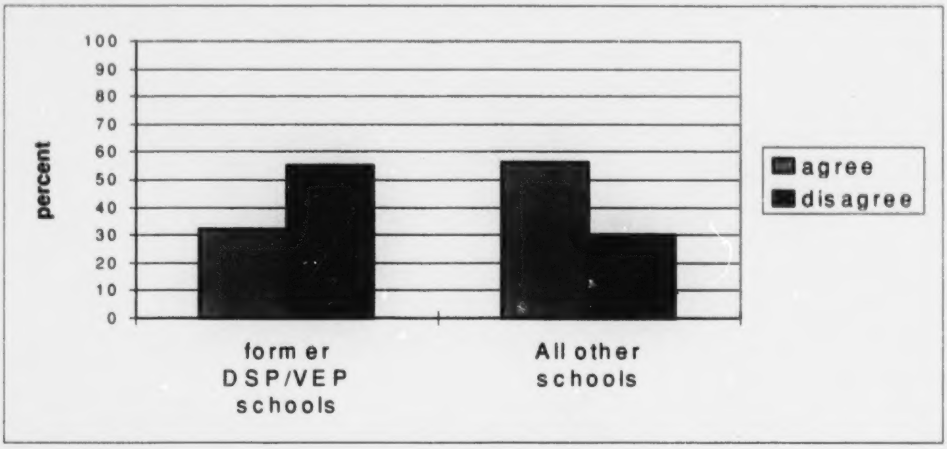
Restrictions on Innovative Teaching Practice

While the VCE was aimed at encouraging innovative practice, schools that had developed their own initiatives for catering for students individually felt thwarted in their efforts.

> (We have not changed the way we teach) at VCE, we have changed the way the we teach from 7 to 10, but VCE is getting in the way. ... We adopted a system where everybody does their own thing at their own speed all continuously assessed ... each one of them is a case management system. (So) we would like to teach individually and VCE as well, but because of the administration ... it might prove difficult. For example, suppose I have a student in Year 8 who is ready for VCE and is quite capable of teaching themselves the whole lot and they still have to hand in their projects and they can teach themselves all the theory very quickly. I don't see why it should take them 12 months. I don't know what is wrong with 3 months. ... So I think that ties you down to teaching in old fashioned ways.

There was further evidence that the VCE was experienced as restrictive for teachers rather than providing a means toward being more creative with course structure. This was evident after the second round of modifications in 1994 when three hierarchical mathematical studies replaced the existing six study choices that incorporated broad choice through a cluster system. While there was widespread support for the three study structure (83%), there was not a similar level of support for the associated reduction in study choice (61%). While the content and flexibility of both Mathematical Methods and Specialist Mathematics were perceived as appropriate for their intended clientele (84% and 89% respectively), Further Mathematics, the mathematics perceived as the least academic, was the least likely to be experienced as meeting the needs of the students (63%). This outcome is consistent with only half of teachers (51%) expressing the view that the VCE curriculum was allowing them to cater well for the lower ability mathematics students compared to 90% for the middle and higher achievers. This situation was even more evident for teachers at former DSP/VEP schools (Figure 2).

Figure 2
The VCE Structure Allows our School to Cater Well for the Lower Ability Mathematics Students



Gender and Assessment

The views of teachers on any bias in the assessment format by gender was consistent with the statistical evidence. More teachers agreed that the school-based assessment provided a better opportunity for girls to succeed compared to boys (38% vs 27%) and more teachers agreed that the examinations provided a better opportunity for boys to succeed compared to girls (44% vs 12%). That many teachers were undecided may well reflect the now clouded situation because of the greater tendency for girls compared to boys to self-select out of mathematics altogether and out of higher level mathematics in particular.

The Relationship Between the Policy Makers and Teachers

The VCE Review has maintained the status of the General Achievement Test (GAT) in assisting the Victorian Board of Studies (VBOS) to moderate school assessment. In the submission we made to the Review we highlighted that while the large majority of teachers were confident in following the marking criteria set by VBOS they had far less confidence in the moderation process, particularly the use of the GAT scores (Table 1).

Table 1
Teachers' opinions about Verification and Authentication Procedures

Survey statement	% agreeing	% disagreeing
I am confident of my ability to assess CAT 1 in accordance with VBOS marking criteria	83	8
I have confidence that if any students' work is recalled it will be marked fairly by VBOS in their review process	36	32
The use of the GAT scores to support the review of school results is appropriate	16	56

CONCLUSIONS

Providing for the diversity of student aspirations, the requirements of tertiary institutions, the economic needs of the State, and a fair and equitable education system for all, is clearly a daunting balancing act. The VCE in its original format clearly stands as a genuine attempt to meet all of these needs. The subsequent modifications made to it in response to powerful political forces have made an accurate assessment of its impact more difficult. It would appear that the recent outcomes of the VCE Review have continued to undermine some of the major innovations. The high stakes nature of the assessment has clearly inhibited the achievement of the aims documented above, and it is perhaps likely that some of the pre-VCE mathematics courses may well have catered better for the diversity of student aspirations, particularly those who did not wish to pursue a purely academic stream.

What does remain crucial to the on-going educational outcomes of students is the quality of the relationship between the policy-makers and the local practitioners. There would appear to be strong evidence of a low level of trust between VBOS and teachers. While shifting to an even greater emphasis on centrally set examinations is one way to reduce the conflict that teachers may feel in their dual role as educators and assessors, the longer term implications for quality education remains clouded. If there is a relationship between mathematics examinations and reliance on rules and procedures, this is clearly maladaptive for meeting the requirements of a complex, rapidly changing and egalitarian society and for meeting the requirements of education institutions which prepare students for such a world.

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Acknowledgments

This study was made possible by an Australian Research Council Grant. We would like to thank the Department of Education and the Catholic Education Office for giving permission to conduct this study and to all the teachers who generously contributed within their limited time. We would also like to thank Anne South and Julie Ryan who assisted with the data collection and the initial interview analysis.

BEST COPY AVAILABLE

AN EXPLORATION OF GENDER DIFFERENCES IN SUBJECT CHOICE AND BELIEFS AMONG YEAR 12 STUDENTS

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This paper explores gender differences within data collected in a study of the factors influencing Year 12 students to study or not to study mathematics. In particular, differences in subject choice, beliefs about mathematics, mathematicians and users of mathematics, and reasons for studying or not studying mathematics are examined. While the results of analysis generally confirm the trend towards a reduction or elimination of gender differences at this level, some differences persist. Possible reasons for this are conjectured.

INTRODUCTION

Decreasing participation rates in mathematics as a subject of study at Year 12 have been of considerable concern to mathematics educators because they signal the very real possibility that too few people are becoming mathematically and technologically literate. As a result of such concern, a number of studies have been carried out to investigate the reasons why more and more students are opting out of mathematics e.g. Jones (1988), Ainley, Jones & Navaratnam (1990), Dick & Rallis (1991), Hamlett (1991), Brinkworth & Truran (1998).

Participation rates in Year 12 subjects in South Australia are fairly indicative of the trend. An examination of the enrolments in Year 12 South Australian Certificate of Education (SACE) mathematics subjects between 1992 and 1996 shows that the total number of students enrolled in mathematics dropped from 70% of the total SACE enrolments in 1992 to 50% of total enrolments in 1996 (SSABSA, 1992; SSABSA, 1996). Furthermore, of those students enrolled in mathematics subjects, the percentage of students enrolled in 'advanced' mathematics, which leads to mathematics, 'hard' science and engineering courses at university level, dropped from 16% to 13·5%. Thus not only are less students overall doing mathematics, but less are undertaking it at an 'advanced' level.

Enrolments according to gender present a mixed picture. Over the same period, the percentage of females enrolling in Year 12 SACE mathematics actually increased marginally from 44% to 46%, but there was a small but steady drift of female students away from advanced mathematics. Of particular concern is that the number of males doing advanced mathematics continues to be more than double that of females. This situation has persisted for more than a decade both in other states of Australia (Dekkers, de Laeter, & Malone, 1986; Watt, 1996) and overseas (Pallas & Alexander, 1983; Revak, 1995). Inevitably, since less females are taking advanced mathematics courses, they continue to be under-represented in the mathematics and science professions. More generally, by opting out of mathematics, they are limiting their educational opportunities, occupational choices and career paths.

Why has the pattern of reduced participation rates for female students continued? In order to investigate this, gender differences were explored within data collected in a survey of the influences on Year 12 students' reasons for studying or not studying mathematics (Brinkworth & Truran, 1998). A review of research was undertaken about relevant variables associated with gender differentiated participation in mathematics as a basis for interpreting the results of the survey data. The following summary of representative findings is based on the categorisation of variables adopted by Leder (1992) and Forgasz (1994).

Basis for career choice: Males attach significantly greater importance to the *money to be earned* in making a career choice than females (Dick & Rallis, 1991) and are more influenced to choose mathematics because of its importance as a *credential* for a future career (Russell, 1984). For females, it is more likely to be *interest and enjoyment* of the subject which is the basis for choosing it (Quality Assurance Directorate, 1994), although females are more likely to see mathematics as irrelevant to their interests and of little use to them in their future careers (Barnes & Coupland, 1990). Often this is because they are not given information about the possibility of careers in mathematics (Schwartz & Hansen, 1992).

Gender and learning style: The learning style preferred by female students is based on *cooperation* rather than competition, which is favoured by males (Schwartz & Hansen, 1992; Forgasz, 1994). Females more greatly value the importance of mutual support and the sharing of information which is a feature of cooperative learning.

Attributions of success and failure: According to the review of literature by Kloosterman (1993), males are more likely than females to attribute success to ability and failure to either bad luck or lack of effort, while females are more likely to attribute success to effort and failure to lack of ability. Thus females feel that they have to work harder to have their efforts and accomplishments recognised.

Mathematics as a subject of study: It is commonly believed by students that mathematics is a fragmented set of facts, formulas and procedures to be memorised (Southwell & Khamis, 1994). While Barnes (1995) reported that females tend to view the subject as cold, impersonal and lacking in imagination and creativity, leading many to dislike it and opt out early, Crawford et al (1994) found no gender differences among students' beliefs about mathematics: all tend to see it as rigid, barren and meaningless, and the learning of it mechanical and boring.

Risk-taking and assessment: Leder & Taylor (1995) found that females tend to be more cautious and less prepared to take risks in tests than males, while Forgasz & Leder (1991) observed that they make greater use of the "I don't know" option when this is available on multiple choice items: this may be a measure of their confidence in doing mathematics.

Mathematics as a stereotyped domain: The belief that "mathematics is a male domain" has a number of negative effects on females (Forgasz, 1994), resulting in lowered expectations and performance, and opting out of mathematics, although it has been argued by some researchers that as well as opting out of mathematics, females self-select into other subjects for positive reasons (Tobias, 1990), particularly English, which is seen as a female domain (Forgasz & Leder, 1996).

Socialisers and subject choice: Teachers can inspire or encourage students to choose to study mathematics or not, and in the process tend to favour male students (Rodgers & Mahan, 1989; Walkerdine, 1989). Parents too can influence students' subject choices in mathematics (Eccles & Jacobs, 1986), tending to favour male over female offspring (Schwartz & Hansen, 1992. Pressure from friends and peers can also be influential. Being seen as good at mathematics can carry a social stigma, so that some students, especially females, prefer underachievement to ostracism from the peer group (Forgasz, 1994).

METHOD

A survey was undertaken of students from Year 12 classes in 10 South Australian secondary schools and provided data from a sample broadly representative of the whole Year 12 cohort, including schools which were large/small, urban/rural, state/independent /Catholic, and boys only/girls only/mixed. Of the 392 students surveyed, 70 (10 male, 60 female) were no longer studying mathematics, and of the students continuing with mathematics (169 male, 222 female), 78 were enrolled in 'advanced' mathematics courses.

Students were asked about their current study of mathematics, their level of achievement in mathematics and their expectations for future work and study, and then to indicate their level of agreement/disagreement with a number of sets of statements of "opinion" about mathematics, mathematicians and users of mathematics. In the case of mathematics, students responded to 33 statements on a four-point Likert-type scale (Strongly Agree = 1; Agree = 2; Disagree = 3; Strongly Disagree = 4), while in the other two cases, which used the same set of 13 statements, students also had the option of "Don't Know". Those who were currently studying some form of mathematics were asked to judge the level of influence on a three-point Likert-type scale (Strong influence = 1; Some influence = 2; Little or no influence = 3) of 16 possible reasons for choosing to study mathematics, while those who were not doing so judged influences on 16 loosely matched possible reasons for choosing *not* to study mathematics. Finally students were asked to identify some of the characteristics of their most recent mathematics classes and their familiarity with a number of mathematical techniques.

Gender differences were investigated with the aid of an SPSS package by means of cross-tabulations, using χ^2 to test for significance.

RESULTS & DISCUSSION

Student Destination

In a result which is consistent with typical Year 12 enrolment patterns, 41% of males but only 20% of females indicated that they expected to study any mathematics course (beyond Year 12) apart from statistics. Otherwise there were no gender differences apparent among students regarding their future study plans, despite 70% of both males and females agreeing that "careers are available for people who graduate from a university with a major in mathematics".

Beliefs about Mathematics

There was surprising unanimity of response from males and females to the 33 statements of 'opinions' about mathematics: only 6 yielded significant differences (see Table 1 below).

Male and female students appear to have largely similar beliefs about mathematics as a subject of study (Table 1). For example, no gender differences were found in students' shared perceptions that to succeed in mathematics requires working through exercises but not memorising textbooks and notes, and this was true of most other statements. In fact where there were differences, they were mainly focussed on beliefs about people who study mathematics and jobs which require mathematics. In all cases, females were more likely to disagree with the given statements.

There was no strong evidence that males attach greater significance to *money to be earned* than females when considering mathematics-related careers. Both males and females differed widely in their beliefs about whether jobs for those with mathematics qualifications earned higher incomes and whether mathematicians or users of mathematics earn large salaries (Table 1), although it was noted that they tend to believe that mathematicians are better paid than users of mathematics. Apart from males being more strongly agreed than females that users of mathematics earn a large salary (Table 2), there were no other gender differences on this issue.

There was some evidence that males doing mathematics are more influenced by the importance of mathematics as a *credential* for further study. They rated needing mathematics for further study as a reason for studying mathematics more highly than females (Table 3). While males were also more convinced that there are more jobs for people with mathematics qualifications (Table 1), suggesting that their choice is career-related, they did not rate mathematical job opportunities (Table 3) any differently from females as a reason for choosing to study mathematics. This may indicate that some students are studying

Table 1
Students' Opinions about Mathematics (excerpt)

Statements (in survey order)	Level of agreement	sig. diffs	mean
M2 It is important to discuss mathematics with others			2.05
M5. To succeed in mathematics requires hard work studying at home.			1.95
M6. There are more jobs for people with mathematics qualifications.	m>f	*	2.63
M8. People who study mathematics are uncultured.	m>f	**	3.28
M10. Jobs for those with mathematics qualifications produce higher incomes.			2.71
M11. People who study mathematics are conventional.	m>f	**	2.81
M13. To succeed in mathematics requires memorising textbooks and notes.			2.95
M16. There's not enough room for creativity in Year 12 Mathematics.			2.33
M19. Mathematics is useful for understanding literature.	m>f	*	3.35
M21. People who study mathematics are dull.	m>f	*	3.15
M22. To succeed in mathematics requires good luck.			3.25
M24. Jobs for those with mathematics qualifications are more interesting.	m>f	**	3.10
M26. You can either do mathematics or you can't.			2.62
M29. To succeed in mathematics requires lots of natural ability.			2.45

Significant gender differences: * $p < 0.01$, ** $p < 0.001$

mathematics for other reasons such as gaining a high TER score. That is, the "further study" for which Year 12 is a credential may not involve mathematics at all.

That females consider jobs with mathematics qualifications *less interesting* than other jobs is confirmed by the data. While both male and female students generally felt that mathematical jobs were less interesting (Table 1), females were significantly more convinced of this than males. Furthermore, both groups gave a low rating to mathematical jobs being more interesting as a reason for choosing to study mathematics (Table 3), with females rating this reason even lower than the males.

There were no significant gender differences found in students' high level of agreement that it is important to discuss mathematics with others to clarify thinking in the classroom (Table 1). This could be considered evidence that mathematics teachers are encouraging more discussion in class or simply that students feel the need for it. Either way, it could lend support to the view that it is not just female students who prefer a *cooperative, discursive style of learning*, although other data collected in the survey indicates that despite some positive features most students are still being exposed to traditional teaching methods in Year 12 which encourage individual and competitive learning.

Contrary to the findings of previous research, no significant gender differences were apparent in students' *attributions of success and failure*. Both male and female students agreed that success in mathematics requires both hard work *and not* good luck, although they were divided in their belief that natural ability is required (Table 1). They were also divided over the statement "you can either do mathematics or you can't". Thus it appears that for these students neither luck, hard work nor ability are gender-linked to success.

Students appear to see Mathematics and English (or other creative, language-rich subjects) as separate kinds of activities, but there is no strong evidence here that they are *gender-stereotyped*. Both males and females disagreed that mathematics is useful for understanding literature (Table 1), with females being more emphatic about this than males. It might be argued that this result arises because females prefer literature over mathematics, are better at it and thus do not associate as strongly with mathematics. Yet there were no gender differences (or unanimity) regarding the appeal of language-rich subjects as an influence on choosing not to study mathematics (Table 4). Nor were there gender differences in choosing not to study mathematics being influenced by the perception that "mathematics doesn't provide enough creativity/self-expression" (Table 4): both males and females ranked this as the second most influential reason to give up mathematics. Furthermore, there were no gender differences on the issue of whether students believe that there is enough creativity in Year 12 mathematics (Table 1), an issue about which all students were divided in their opinion. Choosing to study creative, language-rich subjects rather than mathematics does not appear to be strongly related to gender according to these students.

There appears to be no strong evidence that *peer pressure* has an influential role in subject choice through labelling or stigmatising those who study mathematics (Table 1). Both males and females disagreed that those who study mathematics are uncultured and dull, although females disagreed more strongly. It seems that females are more tolerant than males of students doing mathematics and thus probably less likely to shun them because of it. However, females are less likely than males to agree that mathematics students are conventional. In other words doing mathematics may be considered a slightly offbeat thing for females, but not enough for them to be labelled in a negative way.

Beliefs about Mathematicians and Users of Mathematics

There were no significant gender differences in students' beliefs about mathematicians: all students seem to have a similar image of what mathematicians are like, possibly the result of their lack of direct contact with any of these people. However, gender differences did surface among students' opinions of users of mathematics, of which teachers are the closest role models for students (see Table 2 below). While students' views were divided, females were more inclined than males to agree that users communicate results to the community, but less inclined to accept that users assess the work of other professionals. Why this is so is not clear, but it is interesting to note these are among the four statements about users of mathematics with the highest "Don't Know" responses, with varying proportions of male and female students expressing an opinion. The gender differences may be more apparent than real.

While the frequency of "Don't Know" responses to opinions about mathematicians and users of mathematics indicates that many students have a very limited view of such people, the finding that significantly more females than males gave "Don't Know" responses to most statements in these categories may suggest that females are less willing to offer an opinion when they are unsure. This mirrors the differential test-taking behaviour of males and females found in other studies, and may be an effect of using a multiple choice format in the questionnaire, where "Don't Know" is given as an option.

Table 2
Students' Opinions about Users of Mathematics (excerpt)

Users of mathematics spend a significant amount of their working time				Don't know
DIVIDED OPINIONS	U8 assessing the work of other professionals	*	2·32	28·4%
	U1 communicating results the community	‡	2·50	34·1%
	U10 earning a large salary	*	2·74	29.7%

Significant gender differences: * = m > f, $p < 0{\cdot}01$; ‡ = m < f, $p < 0{\cdot}001$

Reasons for Studying or Not Studying Mathematics

As noted earlier, while both males and females rate the requirement to have mathematics for later study quite highly, males appear to view it more strongly as a *credential* for further study (Table 3).

Table 3
Students' Reasons for Choosing to Study Mathematics (excerpt)

Statements (in decreasing order of influence)	mean	mode	sig diff
S16 Mathematics is required for the course I want to pursue	1.97	1	p<.01;m>f
S4 I took notice of advice from my parents	2.16	2	
S10 I took notice of advice from my mathematics teacher	2.17	2	
S8 There are more job opportunities for those with maths qualifications	2.20	3	
S7 I have had inspiring mathematics teachers	2.37	3	
S6 Jobs for those with maths qualifications are more interesting	2.53	3	p<.001;m>f
S14 I wanted to understand the nature of the universe	2.72	3	p<.001;m>f
S1 My friends are studying mathematics	2.77	3	p<.001;m>f

Males and females were not however significantly different in their ratings of the influence of taking advice from mathematics or class teachers, parents, careers advisers and school counsellors to choose to study or not to study mathematics. For those who were studying mathematics (Table 3), advice from *parents* and *mathematics teachers* was more highly valued than advice from other sources, while for those who were not studying mathematics (Table 4 below), the influence of parental advice was ranked even lower. Thus overall, these *socialisers* provide only moderate to low influence on students' subject choice, and generally have less influence on the decision to opt out than to continue with the subject.

There is little evidence that *friends* have much influence on students' subject choice (Tables 3 and 4). Both males and females ranked the influence of friends on their choice to study or not to study mathematics last behind all listed options, with males more influenced than females. 81% of students doing mathematics indicated that having friends doing mathematics had little or no influence on their decision to study it at Year 12, while 90% of those not doing mathematics stated that friends had little or no influence on them giving it up.

Wanting to understand the nature of the universe as a reason to choose to study mathematics (Table 3) and blaming mathematics for the problems of the world as a reason not to (Table 4) both produced significant gender differences, but in each case since at least 80% of both male and female students rated these reasons as having little or no influence on their decision, it is not possible to draw conclusions with any confidence.

Table 4
Students' Reasons for Choosing to Not Study Mathematics (excerpt)

Statements (in decreasing order of influence)	mean	mode	sig diff
D15 Maths doesn't provide enough room forcreativity/self-expression	1.83	1	
D11 The teaching in language-rich subjects is much better	2.23	3	
D13 I took notice of advice from my mathematics teacher	2.54	3	
D3 I took notice of advice from my parents	2.67	3	
D6 Mathematics is to blame for many of the world's problems	2.81	3	p<.001;m>f
D16 My friends are not studying mathematics	2.89	3	

CONCLUSION

Analysis of survey data has yielded comparatively few gender differences among paricipants' beliefs and reasons to study or not study mathematics at Year 12. The apparent lack of differences in these data among variables such as learning style, attributions of success and failure, mathematics as a male domain and socialisers such as teachers, parents and the peer group, contrary to previous research findings, suggests that there may be greater gender equity in mathematics classrooms than used to be the case. This is to be applauded.

However, gender differences still exist. Where they have been identified through this analysis, they tend to relate to beliefs about jobs for those with mathematics qualifications, the qualities of people who study or use mathematics, self-confidence and mathematics as a credential for further study. The relevant issue is whether these beliefs are instrumental in influencing gender differential participation in mathematics: is the under-representation of females in advanced mathematics at Year 12 (and beyond) the result of such beliefs? If so, then there is need for a greater effort to promote mathematics to those who have the potential to succeed in and benefit from it, for example, by building confidence, providing positive role models and making available better information about careers and career paths. Given the limitations of the study, though, the causal factors may lie elsewhere as well.

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SPEAKING WITH AUTHORITY IN EPISODES OF MATHEMATICAL DISCOURSE

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In this paper we employ three interpretations of the ZPD to frame an analysis of episodes of teaching and learning in a year 7 mathematics classroom. The analysis is concerned with the way authority is constituted locally by the teacher and students within the norms of Collective Argumentation. In traditional classrooms, authority is assumed to reside in the teacher and texts whereas in Collective Argumentation, authority is attained through discourse practices that privilege socio-mathematical norms such as meaningfulness, communicability, and testability. We show how students in the classroom speak with unusual authority as they communicate and negotiate their claims.

INTRODUCTION

The zone of proximal development (ZPD) can be interpreted from three different but interrelated viewpoints. First, the ZPD can be characterised as the distance between a problem-solver's actual ability exhibited when working alone, and the problem-solver's potential ability shown during collaboration with a more-experienced other. With regard to the notion of authority, scaffolding tends to highlight a one-way transfer of expertise and status from the expert to the novice. Novices are assumed to gain authority as their practices match those of the established experts. Recent reformulations of scaffolding, however, have foregrounded the contribution of the novice alongside that of the expert in creating the ZPD, so that it is seen now as a co-constructed space where issues such as independence/dependence, leading/following, authority/compliance are negotiated by partners, and where the learning outcomes are neither totally predictable nor constrained by the predetermined goals of the expert (see Renshaw,1998).

Second, a "cultural knowledge" interpretation of the ZPD is based on Vygotsky's argument that mature concepts develop in educational institutions and communities through interweaving 'scientific' (mathematical) understandings with students' everyday understandings and experiences. This interpretation formulates the ZPD as being the distance between the cultural knowledge provided by an instructional context and the everyday knowledge and experience of a learner. With regard to the issue of authority, the institutional context of schools and classrooms where such interweaving of concepts typically occurs, provides powerful messages to both teachers and students about their respective roles. The ideal student is constructed most often as a receptive, attentive and respectful learner who is willing to follow teacher directions and complete assigned tasks to the teacher's satisfaction. The ideal teacher is constructed most often as a benign classroom leader who can transmit cultural concepts using efficient and effective methods. Here authority resides in the teacher on two levels, first in his/her role as classroom leader, and in the disciplinary knowledge that he/she is assumed to possess. To move beyond these perceived ideals can be difficult because teachers and students need to learn new ways to distribute authority more symmetrically in the classroom and school.

A third interpretation of the ZPD takes a "societal" or "communal" perspective. Following the work of authors such as Lave and Wenger (1991) this interpretation of the ZPD focuses on processes of social transformation rather than social transmission, and extends the notion of learning beyond the context of scaffolding and the local classroom, to include a consideration of the nature of pedagogy in relation to broader social, cultural, and economic factors. In this interpretation of the ZPD, authority is revealed in the social patterns of participation and influence that different individuals and groups achieve within an institutional setting such as a classroom. These local patterns of influence and privilege are not regarded as merely random and incidental but as revealing social and cultural distributions of power. A recent example of this type of analysis can be found in Gee's critique of collaborative classrooms in relation to "new fast capitalist" discourses (Gee, Hull & Lankshear,1998).

In this paper, we are concerned primarily with the second interpretation of the ZPD – the cultural knowledge viewpoint. In our analysis of recent classroom episodes of collaborative mathematical inquiry, we captured students speaking with unusual authority as they challenged the teacher to listen to and respect their ideas. We argue that the students' confidence to resist the teacher's traditional voice of authority was supported by the culture of the classroom which privileged socio-mathematical norms such as meaningfulness, communicability, and testability. Clarifying the way that authority is achieved in the classroom has practical implications in terms of designing more collaborative learning environments. It also has implications for students' understanding of epistemology. In a classroom where authority is negotiated rather than imposed, open discussion of the appropriate criteria for accepting or rejecting claims is foregrounded. In such a classroom students are challenged to reflect on the basis for their claims and to appraise the claims of their peers in terms of agreed norms and practices.

METHODOLOGICAL APPROACH

To begin to address the issue of *authority* in mathematical inquiry we've employed Bakhtin's notion of voice. Voice provides a means of identifying processes of appropriation and resistance as the teacher and the students ventriloquate, revoice or challenge each other during collaborative activities. Voice also directs attention to the multiple stances or speaking positions that teachers and students adopt during their interactions. The teacher at times speaks as if he were a student, while students adopt the voice of the teacher as they attempt to influence the direction of the collaborative activity. The multiple and shifting voices of the teacher and students reveal also the way the institutional setting of the school and the culture of a specific classroom privilege interact to certain voices over others. The teacher and students seek to influence each other by speaking with authority, that is, by adopting what are situated and privileged voices in a specific classroom.

The specific classroom we investigated is a year seven classroom where students have been using "Collective Argumentation" to guide their mathematical activity. Collective Argumentation is organised around a key word format involving a number of processes - represent the task or problem alone, compare representations within a small group of peers, explain and justify the various representations to each other in the small group, reach agreement within the group, and finally present the group's ideas and representations to the class to test their acceptance by the wider community of peers and the teacher (see Brown, 1994).

COPYING, REMEMBERING, REVOICING: WHEN CAN I SPEAK WITH AUTHORITY?

Some of the dilemmas that we have noted with regard to authority are illuminated by considering the similarities and differences between processes of copying, remembering and revoicing. In a collaborative classroom ideas are often "in the air", so to speak, as students represent, compare, explain and justify particular points of view. Collaborative activities are premised on the assumption that students will work with each others' ideas. However, as we have noted(see Brown,1998), students can be sensitive to the notion of copying when working with each other's ideas. During one episode of Collective Argumentation, a small group of students was presenting their solution to a task when one of their classmates asked them if they had copied the idea from someone else. They replied that they had not copied it, but had remembered the idea from a previous session. The students' distinction between copying and remembering provides insights into the nature of authority in this classroom. Copying implies that the group was neither the original author of the ideas, nor had the members of the group appropriated the ideas in a meaningful fashion. To retort that they had *remembered* it, is actually a subtle variation of the notion to copy, with the important difference, however, that the idea had been stored over time, retrieved, and seen as appropriate for use in this particular context. To remember in this way, therefore, implies meaningful appropriation and the students clearly felt that they could speak confidently and with authority about "their" ideas. This brief exchange between these students reveals an emerging awareness of epistemology, and an ability to legitimate their practices by drawing quite sophisticated distinctions.

Learning may occurs when students or the teacher use each others' ideas to advance the mathematical discourse. Summarising, paraphrasing and rephrasing ideas have been referred to by O'Connor and Michaels (1996) as "revoicing", and like us, they have noted the use of "so" at the beginning of an utterance as a salient marker of revoicing (see Renshaw & Brown, 1998). Teachers within collaborative classrooms are likely to employ revoicing quite often as they attempt to incorporate students' contributions into the ongoing discourse. A student whose contribution has been revoiced is positioned to make a judgment regarding its accuracy (Is this really what I said and meant?), relevance (Is it really appropriate in this context?) and acceptability (Now that I've heard what I said, do I still agree?). Students thereby acquire a particular kind of authority within the classroom discourse, because they are able to either assent to or challenge ideas accredited to themselves.

So, to conclude this section – the public and collaborative nature of knowledge construction during Collective Argumentation provides many opportunities for students to reflect on the basis of their claims, the origin and authorship of ideas, and to become aware of the socio-mathematical norms of meaningfulness, communicability, and testability as the basis of knowledge claims.

Authority in Collective Argumentation

Collective argumentation is based around social practices and shared norms that challenge the traditional authority framework of the classroom where the teacher is assumed to be in control and where students are expected to comply with teacher directions. These practices and shared norms take time to establish and require the teacher to constantly embody a different stance to issues of authority. There are times, however, when both the teacher and students may revert to more traditional voices or stances. To illustrate how Collective Argumentation can create a space for students to assume a dialogic position where views and ideas are retained through appeals to shared evidence and reasonable argument, a short extract from one episode of small group work is presented below. This episode is particularly interesting because it is the student who remains within the norms of collective argumentation and who resists the teachers momentary adoption of the traditional authority space.

We enter the dialogue where the students are attempting to find the area of an eight-pointed star enclosed within a square, and the teacher has joined the group to review their progress. Annie has employed a conventional representation to successfully solve the problem. However, her partner, Allan, has adopted an imaginative, but inadequate representation, which requires viewing the figure as two equivalent rectangles. Annie is exploring Allan's idea to see if it can be successfully adapted to solve the problem. Annie's attempt to work with Allan's idea demonstrates that revoicing can have an explicit instructional purpose. O'Connor and Michaels (1996) make the point that the 'revoicer' of a speaker's contribution often sees more significance in the ideas than the speaker was aware of. This clearly occurs in the following episode where Annie sees potential in Allan's idea where neither the teacher nor Allan sees any. In fact the teacher demands that Annie cease her attempts to co-construct a response to the problem, and comply with his directions. Annie maintains her viewpoint in the final line of the transcript even after the teacher's insistence that she follow his directions – "I didn't ..."

(In the transcripts below the Teacher's words are in *italics*, Annie's are **bold**, and Allan's are normal text)

Teacher	*You've turned the eight-pointed star into two rectangles, but you're no longer measuring the eight-pointed star.*
Annie	**So what we did. I found the area of these little triangles and it was twenty-four centimetres squared. So I got that idea off mine and took it away from Allan's answer.**
Teacher	*No, that's not going to work. You just can't make things fit together. Okay? You can't get two different ideas and make them fit together.*
Annie	**No, I just knew that . . .**
Teacher	*Stop arguing and listen to me for a moment. You can't take his ideas and take your answer away from his answer. He's coming at the problem from a completely different perspective to what you are. You have to work with your ideas and convince him that your ideas are accurate.*
Annie	**I didn't take my answer away from his.**

In the dialogue that follows, the teacher and Annie re-visit the calculations evoked by her representation of the problem space, confirming Annie's answer that the area of the star is 40 square centimetres. The teacher then re-visits the calculations evoked by Allan's representation and compares the two results. Upon comparing the results the teacher concludes that even though Allan's idea is "beautiful" it "just doesn't work". The teacher then directs Allan to move onto the next problem - "Can you work the next (problem) out?" - and tells him to use Annie's idea to assist him.

In the above sequence the teacher had engaged a voice characteristic of the "expert" operating within the framework of pedagogical scaffolding. However, in response to the teacher's statements Annie maintains the argument that she is not simply subtracting her answer to the problem from Allan's answer ("I didn't take my answer away from his"), but combining his ideas with her ideas to solve the problem in a novel way ("So I got that idea off mine and took it away from Allan's answer"). This contradiction of the teacher's voice is not an act of defiance by Annie, but an example of a sociomathematical norm which Polya refers to as 'wise restraint' - where a mathematical point of view is not changed wantonly, without serious examination. Annie's mathematical voice resonates with the confidence of the knower, struggling to represent what she knows and to connect that to the knowledge of others - an emotionally risky resonance, but necessary to developing authority of voice (Kutz, 1990). Annie's desire to have Allan and the teacher appreciate her insight regarding Allan's representation persists in the following sequence.

Annie	**Okay, let's go with your idea.**
Allan	No.
Annie	**Yes.**
Allan	We don't have time.
Annie	**No, we're going to fix up your idea. We're going to find out where you went wrong.**
Allan	But it (the work-sheet) is wrecked.

Annie **Allan, we'll do your idea. Can you draw that shape (the figure) please on the back (of the sheet)? On the back of this and we'll fix up your idea.**

(Allan commences to draw the problem figure on the back of the work-sheet. Teacher approaches the group.)

Teacher *How are we going?*

Annie **I know where he went wrong.**

Teacher *It doesn't work!*

Annie **I know, but I think it can.**

Teacher *I'll get you another sheet.*

(Teacher gives the children a new problem sheet and leaves the group.)

In the above sequence, Annie first recruits Allan's participation in the co-construction of a solution by expressing confidence that his idea can be 'fixed' and by organising their work-space so that time can be used efficiently (working on the back of the work-sheet). Annie then recruits the teacher's tacit participation by her confidence ("I know where he went wrong"), affirming the teacher's argument that the idea does not initially work ("I know . . .), and expressing faith in the status of the idea as being an important element of a co-constructed response ("I think it can").

Annie's stance continues to give direction to the discourse as the other participants (Allan and the teacher) take up reciprocal positions relevant to the norms of Collective Argumentation. The teacher's attempt to enforce his authority by adopting what may be referred to as the traditional teacher voice, has been successfully resisted by Annie's maintenance of the voices of Collective Argumentation. In the sequence below it is the teacher who now tries to follow Annie's definition of the task, grants her the status of the 'knower', and begins to work with her as a co-participant.

Annie **Okay. Talk to me. Talk me through what you did.**

Allan I went Oh! and got this here (a triangle) and put it here (points). Then I went Oh! and saw there was another one there and another one (both children reconstitute Allan's original representation).

Teacher *But if you keep putting the parts which are not part of the star onto your rectangles, are you finding the area of the (indicates the area of the whole figure)? You've got a big problem with the middle anyway, because it overlaps (shades in the middle square that overlaps both big rectangles).*

Annie **Ah ha!**

Allan What are you doing?

Teacher *But that can be solved by finding out the area of that square and taking it away once. That problem can be solved.*

Annie **So you've got to find the area of the square . . .**

Teacher *Yeah, we're not interested in that at the moment. But the question you have to ask yourselves is by putting these bits (the triangles) of the square which are not part of the star onto the end of this rectangle, are you finding the area of the star?*

Allan No.

Teacher *Or are you finding the area of the whole square (the figure)?*

Allan We're kinda finding the area of the whole square.

Annie **I'm not particularly worried . . .**

Teacher	*You are, you're finding the area of the whole square. Aren't you?*
Allan	Because we've got all these parts (the triangles) which were the square, on the end.
Teacher	*In other words you've turned the square into two rectangles. You've got the other problem here . . . that these two rectangles overlap here.*
Annie	**So we've got to take this square here (the overlapping square) away from our answer.**
Teacher	*So if you take that square there away from your answer you should get the area of the square which is sixty-four square centimetres. See if it works.*

In the above sequence, it is the teacher who is learning to master the children's definition of the situation, and it is Annie who speaks authoritatively in the teacher's voice as she asks Allan to "Talk me through what you did". It is Annie who revoices the teacher's ideas as a way of enlisting his appreciation of her original insight - "So you've got to find the area of the square . . .", and "So we've got to take this square here (the overlapping square) away from our answer." It is the teacher who signals his appropriation of Annie's mode of thinking by prefacing his final contribution with the word "so" - "So if you take that square there away from your answer you should get the area of the square which is sixty-four square centimetres. See if it works". In this sense, Annie may be said to be 'scaffolding' both Allan's and the teacher's engagement in the task, and to be sharing a symmetry of authority with the teacher which is rarely seen in primary classrooms (Edwards & Mercer, 1987).

CONCLUSION

In this paper we have explored the issue of authority from a sociocultural perspective, and related it to three interpretations of the ZPD. The *cultural knowledge* notion of the ZPD focuses on the classroom as the site of knowledge transmission and construction. A clear hierarchy of authority has been maintained in most classrooms, with the teacher assigned the institutional status both of "knower" and "leader", and students expected to enact the more compliant roles of "novice" and "follower". A sociocultural perspective challenges this classroom hierarchy by proposing more symmetrical, participatory and collaborative forms of interaction between teachers and students.

Collective Argumentation, which has been devised and implemented in an upper primary classroom over a number of years, provides a generative space for students and teachers to begin to adopt these new ways of interacting. In this space, students are expected to co-construct mathematical ideas relevant to various problems, and to communicate their ideas effectively not only within their small groups but to the teacher and the class as a whole. In these various communicative contexts, students have the opportunity to adopt different speaking positions, to speak as the "the knower", or "the explainer" and to lead the teacher and the other students to appreciate insights that they have developed.

In our analysis of a number of episodes of Collective Argumentation, we suggested that students were beginning to speak with authority in this classroom, and were beginning to adopt an epistemology consistent with sociomathematical norms – namely that ideas had to be meaningful (personally understood rather than just copied) communicable (able to be explained and represented for other members of the community to appreciate), and testable (able to be supported by logical argument and/or empirical demonstration). The insightful exchange between the students regarding the issue of *copying* versus *remembering* provided evidence that students were beginning to reflect on the basis of their arguments, and were becoming aware of the relevance of the sociomathematical norms for their small group practices. They regarded copying as problematic because it implied reproduction without personal understanding, whereas remembering seemed legitimate because it implied transformation of the original idea and effective application in a novel situation.

The analysis of the interaction between Annie, Allan and the teacher addressed a number of important issues regarding the dynamics of authority in the classroom. A key episode shows Annie refusing to accept the direct command of the teacher "to stop arguing and listen for a moment". Annie is able to counter the teacher's authoritarian stance by drawing on a voice that has a privileged status in this classroom, namely, that speakers have the right to be heard if they have good reasons to support their views. If the teacher had insisted on compliance in the face of Annie's firm resistance, an authoritarian framework would have been reasserted, with the implication that the students simply had to listen and obey the commands of 'experts'. So, this episode suggests that both teachers and students find it difficult at times to remain within the collaborative classroom norms of symmetrical authority. It also reveals the complex interplay between institutional and local classroom cultures when collaborative forms of learning are introduced to replace traditional teaching methods. Institutionally, the teacher and students were more familiar with hierarchical roles and expectations, so the creation of the new culture required a joint effort to resist old practices and to regulate each other's adoption of the more symmetrical roles.

New practices cannot exist in a vacuum or be insulated either from the histories of the participants, or the influence of the surrounding institutional mileau. We have glimpsed in the episodes analysed above, the interplay of the personal histories of the participants and the institutional mileau of a typical school, as the teacher and the students struggled to maintain symmetrical authority relations, rather than return to the hierarchical authority relations so familiar to them.

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DO GAMES HELP THE LEARNING OF PROBABILITY?

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This paper reports on a research project which investigated the value of using games to assist Year 7 and 8 students' learning of probability concepts. Games, although generally useful in mathematics for helping children learn, may not automatically be as useful in helping students develop normative probability concepts. The study found that the type of game impacted on the value of using games for learning probability, and that there are implications for the teacher's role when using games for learning.

INTRODUCTION

In the world as it currently is, and will be in the future, concepts involving chance are pervasive. Thus an understanding of chance and randomness is desirable so that we can make the best of the probabilistic decisions that face us in everyday situations.

Although probability concepts are an integral part of life, they have not always been included in the school curriculum at the primary school level. New Zealand's national curriculum statement *Mathematics in the New Zealand Curriculum* has raised the profile of probability in the school curriculum, compared with previous curriculum statements. Because probability learning is introduced at earlier levels than has happened in the past, primary teachers need ways to help ensure that effective learning of probability occurs.

Learning Probability Compared with 'Other' Mathematics

It is readily acknowledged that many of the ideas behind probability are difficult to learn and therefore hard to teach (Ahlgren & Garfield, 1991). Freudenthal (1973), when discussing the types of problems that arise in probability, indicates that pure mathematics does not know of similar analogies; it appears that probability learning may be different from learning in other areas of mathematics. Borovcnik and Bentz (1991) suggest that our desire for deterministic explanations works against the development of an adequate understanding of randomness, which is inextricably linked to an understanding of probability.

In most areas of primary school mathematics, the use of equipment, teaching experiments, and practical activities can be used to demonstrate various concepts. From the specific examples and results obtained, students are encouraged to look for patterns and generalize, thereby developing their mathematical understanding. However, this process of learning through induction may not be as relevant and useful in probability, because of the nature of randomness and random events - the specific examples and results from chance events may not be the expected ones. These 'unusual' results conflict with what is expected, and students are therefore faced with experimental evidence that does not clearly illustrate the concept, and furthermore, it is unlikely that they would realize this. If the student ascribes the result to an overall pattern and views the outcome deterministically, then this unusual result may interfere with the understanding of probability that the teacher is hoping to encourage.

Additionally, the intuitive knowledge that students bring to the classroom can advantage the learning process, but unfortunately, it can also impede the learning (Borovcnik & Bentz, 1991). This appears to be especially pronounced in probability learning, because of the often strong probability intuitions that students develop prior to any formal learning in the classroom (Fischbein, 1987). These intuitive ideas of probability are particularly resistant to change (Konold, 1995). This is problematic for learning because normative probability concepts are often counter-intuitive.

Konold (1991) advocates a multifaceted approach to instruction designed to confront the students' misconceptions. In this approach, students are encouraged to evaluate their current beliefs and how well they fit against: 1) the beliefs of others; 2) their other, related beliefs; and 3) their own observations. The first aspect involves discussion and Konold describes the ways in which teachers can facilitate this by helping to keep the conversation going, and focusing the discussion. However, according to Konold, the most important role of the teacher is to create a classroom atmosphere in which the students can discuss, argue, and convince one another of their views, while at the same time exploring their own views and understandings. The teacher, through the use of probing questions and prompts, encourages the students to analyze the consistency and completeness of their own beliefs – both related to the misconception being considered as well as other related beliefs.

Games for Learning

Games are commonly-used mathematics activities in the primary school. Numerous resources for teachers give details of games which can be used in the classroom for a variety of purposes. Many games are for practising and consolidating skills, but of interest in this study were those which help students learn new concepts. Since games involving chance are a common part of many cultures and therefore part of the real-life experiences of many children, it would seem sensible to use games as a context for learning mathematics, and in particular probability, in the classroom. Also because games involving chance historically brought about the development of probability theory, it has been suggested that games of chance could profitably be used in the classroom to help children learn probability concepts (Biehler, 1991). However, there is a limited amount of research available on the use of games as they relate to the learning of probability concepts for younger students.

Bright (1980) expressed caution about the use of games for probability learning because of the possibility that, as the students become more familiar with a game, they may become more skilful in the strategy required in the game without necessarily increasing their understanding of probabilities related to the dice outcomes. He indicated the importance of carefully examining the players' moves during the game in order to determine how much is attributable to practice at playing the game compared with real understanding of probability. Falk, Falk and Levin (1980) also referred to the possibly passive role of children during the playing of games involving only chance. They suggested that to overcome this, games involving some decision-making from the players should be used to ensure that probability concepts are more likely to be explicitly thought about during the game. Also, the game should be played a number of times. Because of the nature of random events, less likely outcomes could dominate small trials and would not necessarily help develop an appropriate understanding of probability.

Eade (1988) recommended games that may induce 'cognitive conflict', where the players' intuitions about the game and its outcomes conflict with the empirical results, thereby challenging the students' understanding. Through this, it is hoped that the conflict would be resolved with an improved (normative) understanding.

Burnett (1993) argues that the discussion which arises from the game playing as well as the social context in which it is played facilitates learning. This view is supported by others (eg., Ellingham, Gordon & Fowlie, 1998) who contend that games encourage students to listen to the viewpoints of others and make sense of those interpretations. These group interactions provide the opportunity for the individuals in the group to develop understanding more sophisticated than the original, individual ideas (Wood, Cobb, & Yackel, 1995). Games have the advantage of providing a natural situation for the teacher to be able to question children about their understanding (Ainley, 1990).

Sometimes when students are using games for learning, the players may be able to complete a game without being challenged or required to consider the concepts for which the teacher has chosen the game (Ellingham et al, 1998). They suggest that when the educational aspects and the gaming components are not be fully integrated, students treat it merely as a game rather than a learning experience.

RESEARCH QUESTIONS AND DESIGN

The research project focused on the use of games for helping students learn probability concepts (Burgess, 1999), and this paper reports on part of that research, specifically the following questions:

1. Does the use of games encourage students' thinking in relation to specific probability concepts as well as any misconceptions that the students may have?
2. Is there a difference between the use of a game involving only chance compared with a game involving chance and strategy on students' thinking about probability, and on the interaction between students while they are playing the games?

Eight Year 7 and 8 students from each of two classes were selected by their classroom teachers to be involved in the project, ensuring a range of mathematical abilities within each group. The only information given to the researcher about the students was a grading of mathematical ability, based on a 3-point scale (1 for below average, to 3 for above average).

Since this study was mainly focused on the students' understanding of probability concepts, qualitative methods were seen as the most appropriate for exploring that understanding. A questionnaire was administered, with items based on the objectives from *Mathematics in the NZ Curriculum* – some of which had been used by other researchers in the area of probability misconceptions. All the questions were of multi-choice or true-false type, but most also required written justifications for the answers. Observations and group interviews were used during the subsequent game-playing sessions, which involved four students at a time, playing in pairs. The two game sessions, which each lasted for about 45 minutes, were audio-taped as well as video-taped. The stimulus for the interview questions was both the students' activity during the game, and the discussion which arose from playing the game. The mixture of data collection and analysis methods helped reduce any possible bias, and contributed to the verification and validation of the qualitative analysis.

Two simple games were chosen for the students to play. They needed to be relatively short in duration, so that within the allocated time for the games sessions, a number of repetitions of the games could occur. The repetitions of a short game allowed random outcomes to be observed as well as having the potential for showing the 'longer term' trends of the random outcomes.

The selected games were adapted to enable a type of 'record' to be kept of the outcomes through the use of the counters on the game board, instead of relying on the students' memories of the outcomes to gain some understanding of the (lack of) fairness in the game. Reliance on memory is known to be unsatisfactory in relation to probability-based events; the availability heuristic, as described by Kahnemann and Tversky (1972), leads to 'incorrect' reasoning and decision-making.

The first game involved chance only, where the difference between the numbers on two dice determined which team was able to move their counter. If the difference was 0, 1, or 2, one team moved their counter one space; for a difference of 3, 4, or 5, the other team moved their counter one space. The game board consisted of 12 spaces; the first team to have their counter reach the end was the winner. This game was not 'fair' – the lower differences are more likely to occur than the larger differences. The second game consisted of strategy as well as chance, and the game session was split into two parts: the first part used a game based on the sums of two dice (with the board having the numbers 1 to 12, against which the team could choose where to place their ten counters), while the second involved the differences between the numbers on the two dice (and this board was similar to the first, but had the numbers 0 to 6). The strategy part of the game required both teams (each with two students) to place their counters on the game board, in order to optimize their chances of winning, by being the first team to remove their counters from the board. At each throw of the dice, if a team had a counter beside that corresponding sum (or difference for the second part), they could remove that (one) counter from the board.

RESULTS AND DISCUSSION

The questionnaire revealed that most of the common probability misconceptions, which have been described by various researchers, were held by the 16 students involved in the research, with most of the students demonstrating more than one type of misconception. For 12 of the sixteen students, a dominant type of reasoning emerged for each of those students: four displayed normative (correct) reasoning, while the others mainly used one misconception, such as the outcome approach (Konold, 1995), the representativeness heuristic (Kahnemann & Tversky, 1972), or the equiprobability bias (Amir & Williams, 1994), as their dominant type of reasoning. Overall, the predominant type of reasoning used by the students was the outcome approach – all students used it in some form, even those whose main type of reasoning was normative. Often there were inconsistencies between answers from a student, or inconsistencies between an answer and the justification given for that answer. These results provided a base against which to compare their actions and comments in the subsequent game playing sessions.

Students' thinking about probability was quite clearly stimulated by playing the games. Prior to playing game one the first time, three of the four groups of students indicated that they believed (incorrectly) that the game was fair, and in the other group, there was disagreement as to which team would have the advantage. After playing this game a number of times, there was only one group that still maintained their viewpoint that the game was fair; the other two groups had been convinced by the empirical evidence, that the game was not fair. The discussion within these groups ensured that they moved towards a shared understanding about the lack of fairness in the game. The only group to have predicted that the game was unfair, came to an agreement quite quickly as to which team had the greater chance of winning. The reasoning used within this group was normative, and relied to some extent on the empirical results from the games.

The one group that did not change in their belief about the fairness of the game, were obviously affected by unusual results in two of the games: both games were very close, with the first having a probability of only 0.05 of occurring, and the second game had a probability of 0.03. In between these two games was one where the result was much more conclusive, as was the result from the fourth game. Through all this, the group maintained their belief that the game was fair. The empirical evidence, particularly with two close results out of the four being quite unlikely, was not useful in helping the students develop their understanding of probability. At times like this, the teacher's role is crucial to help students work through the results and clarify their misconceptions, which may otherwise be consolidated.

Comments from students, unprompted by the researcher, during the games also indicated that they were thinking about probability concepts. For example, in relation to the possible differences when throwing two dice, one student suggested after a few throws that it was not fair because her team was losing:

> Because there's more chance of getting a one or a two than having a higher number like a three or a four or a five.

Another student (from a different group), who had demonstrated various misconceptions in the questionnaire, referred to a belief in luck and a causal approach to probability, through the following comment:

> I think yellow is very unlucky for them, because the board is yellow, one dice is yellow, and their counter is yellow. And they're not winning at all.

The negotiation of meaning within the groups was not straightforward however. For instance, the influence of some rather flippant comments by particular individuals could not be underestimated, as these tended to cause some of the less assertive students in the group to contribute less to the discussions. For instance, one student stated (jokingly):

> The dice must be rigged....there are probably magnets inside the dice that make them fall that way.

Also, some comments could be considered influential, in that they altered the 'path' of the discussion, leading the group away from developing the normative understanding that had been occurring. For example, one group had been using a reasonable amount of normative reasoning, but then a suggestion came from a student that the colour of the counters may have something to do with the results - a causal approach to probability. This altered the discussion for some time, with the students 'exploring' other possible influences on the outcomes, before finally reverting to the reasoning which was used previously.

Games are claimed to be useful for encouraging students to ask questions and reflect on responses, hence to make new deductions and inductions. Although the students were encouraged to talk and discuss their ideas throughout the game sessions, the level of involvement of the students varied, as measured by the number of times each spoke. For example, although the mean number of contributions per student over the two game sessions was 27, the minimum was 10, and the maximum was 41. There was no signficant difference between the mean number of contributions for each of the three ability groups.

There was significant variation between groups in the amount of interaction within the group: some groups played the games with minimal verbal interaction, while there was a large amount of discussion in other groups. This was measured by the researcher identifying "spontaneous interactions" – students' comments relevant to the game and unprompted by the researcher. The spontaneous interaction was identified as having ended, when there was a signficant pause, a change in topic of the conversation, or intervention through a question from the researcher. The number of comments for each spontaneous interaction was determined, as well as the number of spontaneous interactions per game session, and this is summarised in Table 1.

Table 1
Number of Comments per Spontaneous Interaction during Games

	Game 1				Game 2			
	Group 1	Group 2	Group 3	Group 4	Group 1	Group 2	Group 3	Group 4 *
Average number of comments per spontaneous interaction	2.4	2.0	1.8	2.2	2.5	1.7	1.7	2.1
Number of 'spontaneous interactions' during game session	32	24	25	24	21	21	13	29

** Note: Group 4 consisted of only three students for game session 2, which may have affected these results.*

Game one involving chance generated 105 interactions overall, whereas game two, involving strategy and chance, generated only 84 such interactions. However, it can also be determined that the average number of comments per interaction is almost identical for the two types of games - 2.1 compared with 2.0. This indicates that the greater number of interactions during the first game must have been shorter in duration than those in the second game. Three groups reduced the number of interactions from game one to game two. For one group (number three) this can probably be attributed to one student who was concerned that he was going to miss some of his class's physical education lesson, and consequently he contributed almost nothing to the second session. For the other two groups, there is no obvious reason for the reduction.

It was necessary, when comparing the games, to consider the quality of the interactions, and this was done by examining the transcripts for evidence of changes in the students' understanding as a result of playing the games. One group particularly showed an 'improvement' in their understanding from game one, where they did not move at all towards a normative understanding, compared with game two, where some normative reasoning was used. As a group, they accepted that some numbers occurred more often than others, and they identified some of the numbers that did occur more often. But against this, one student, with a strong tendency towards the equiprobability bias, argued using the outcome approach. She also indicated that some of it was due to luck. Neither the rest of the group nor the empirical evidence from the games was able to 'shift' this student's understanding. Some students' comments indicated that they were more involved in the second, strategy-based game:

> In these games you are thinking of what move you are going to make, or what numbers you need.
>
> In the other game [game one], we didn't get to choose the numbers so it didn't make you think too much about it.
>
> It has made us think more than in a game like Monopoly, where you just think about how much money you have to hand out or something.

However, an opposing view was put forward by one student:

> I think it made me think harder because it was more of a competition, sort of. And you wanted to win, and you realized it was an unfair game.

Games of strategy and chance, which have been suggested as possibly being more useful to the learning of probability than games of chance only, showed some interesting aspects in regard to the strategies used by the students. The students tended to improve their chances of winning from one trial to the next through the choices they made, but rarely did a team take notice of what the opposition had chosen, in order to further optimize their chances. They tended to focus only their own choices. The nature of random events meant that a well-chosen strategy did not necessarily guarantee success. The random outcomes did not always favour the team with the better choice of numbers. This contrasts with the claim that games give feedback on the "consequence of actions" (Inbar & Stoll, 1970). There were games played in which a 'poorer' choice of numbers resulted in a win for a team over the opposition who had a 'better' choice, in relation to the relative probabilities.

CONCLUSIONS

There was evidence of students holding contradictory beliefs of which they were not aware. This may occur because the beliefs were expressed in different contexts, in which case the students may consider the problems to be different. For effective learning to take place, the students should be made explicitly aware by the teacher of these contradictions, to have cognitive conflict induced. This would enable the students to consider their conflicting ideas and have an opportunity to resolve them.

Students do not necessarily transfer their learning and understanding from one context to another. Most groups concluded (correctly), as a result of playing game one, that the dice differences were not equiprobable. But they did not transfer this knowledge into the second game that used dice differences, as their choices for placement of the counters in the second game did not reflect the understanding that they had developed previously. A teacher would need to make the students explicitly aware of these, so that the students can address and, hopefully, resolve them.

It is important that students are not left on their own to play such probability games, as it may happen that without the influence and input of the teacher, a group does not become as actively involved as is desirable. Also, the importance of the teacher listening to the students to gain an understanding of their concepts is obvious when comparing the results of the questionnaire with the views expressed by the students during the discussions. The questionnaire results were limited by not being able to follow up on what the student meant with some of the responses, but through being involved in the games' discussions, the teacher could easily seek clarification if necessary.

The design of the games was intended to provide a type of record of the results, as these are known to be important so that students can see long-term trends, rather than relying on memory (which is when students are prone to use the availability heuristic). The design of the games did not meet this requirement sufficiently, as there were instances of inadequate recall of the results by the students, indicating use of the availability heuristic. The design of each game needs further refinement to overcome this problem, or some other strategy involving recording needs to be incorporated into or alongside the game.

Active involvement of the students along with the intention of learning is required to ensure maximum benefit. So, although games are strongly advocated because of the way that they assist with the mathematical development of children, they may be inadequate on their own: it is the teacher's planning of the educational process which is essential.

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INVENTED ALGORITHMS: TEACHERS FACE THE CHALLENGE

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This paper explores teachers' responses when implementing a teaching approach that encouraged children to develop and use invented algorithms for computation. The children demonstrated number sense and used methods that predominantly reflected left-to-right computational processes. While acknowledging the children's success and their ability to explain and justify the methods used, the teachers faced a number of challenges. They confronted previously held beliefs, perceived curriculum requirements and social expectations regarding the place of standard algorithms for computation.

Background to the Study

The debate surrounding the use of standard algorithms for numerical computation is not new. Ginsburg (1977) suggested that children "fail to understand the necessity or rationale for such methods ... The result is not only a bizarre written arithmetic, but a gap between it and children's informal knowledge" (p125). In contrast to standard formats children's invented algorithms often reflect a left-to-right mental process. It is argued that children resolve the apparent conflict between their own mental or written strategies and the standard algorithm by giving up their own thinking (Kamii & Dominick, 1998). More recently greater emphasis has been placed on children's development of number sense and mental computation skills rather than mere fluency in the use of algorithms. It has been well documented that children develop a meaningful understanding of numbers if given the opportunity to use their own procedures (e.g. Carpenter, Franke, Jacobs, Fennema & Empsom, 1998; Kamii, 1989). It is believed children's number sense allows their understandings to be used in flexible ways (McIntosh, Reys, Reys, Bana & Farrell, 1997) and is a factor in allowing them to depart from traditional or taught algorithmic methods (Bobis, 1996).

Mathematics in the New Zealand Curriculum (Ministry of Education, 1992) suggests children develop flexibility and creativity in applying mathematical ideas and makes no reference to teaching the traditional algorithm. Nevertheless the teaching of traditional algorithmic methods for multidigit computation remains common practice in New Zealand classrooms. Rather than debating the place of children's invented algorithms as opposed to traditional taught algorithms discussion in teaching circles has tended to focus on which methods to teach (eg decomposition or renaming as opposed to equal additions or "borrow and pay back" methods of subtraction). The use of place value blocks to model the algorithmic processes has also been a major focus in New Zealand classrooms. Teaching approaches advocating the use of place value blocks were tightly structured. The four operations were taught in sequence with expected mastery of multidigit addition before the introduction of multidigit subtraction. For each operation children would be presented with examples requiring no regrouping of place value blocks before being introduced to examples requiring regrouping (eg 10 ones renamed as 1 ten). The use of structured equipment was aimed to assist children to develop a mental image of the computational process however, rather than reflecting the mental strategies used by the children the approaches were teacher directed and designed to lead children to the standard algorithms for computation.

Literature shows that while children have the ability to develop effective strategies for solving computational problems, most teachers continue to expect the use of standard algorithms (Peters, 1997). The work of authors recommending emphasis on number sense and invented algorithms is well known by mathematics educators in New Zealand. However, while many teachers have introduced into their mathematics programmes activities that encourage mental computation and estimation, few have extended this approach to include children using their own invented algorithms. If more emphasis is to be placed on identifying and developing children's legitimate strategies (Cooper, Heirdsfield & Irons, 1996) what challenges do teachers need to be prepared to face?

The Research Study

This study set out to explore the implications for teachers when implementing a change in their mathematics instruction for multidigit computation. It follows the experiences of two teachers in an Auckland primary school as they encouraged their children, aged 7-9 years, to develop and implement their own mental strategies and algorithms. It views not only the classroom as a site of social construction of concepts, but also the teachers' wider community as a site in which beliefs and pedagogical practice are constructed.

The school was approached with the offer of professional development in which invented algorithms would be demonstrated as a way of encouraging number sense. Identifying this as in keeping with the school's general policy of developing children as independent thinkers, the principal accepted the offer and two teachers volunteered to participate in subsequent implementation of the approaches in their classrooms. The professional development took place over six months, with three initial meetings followed by a period in which the two teachers were observed in their classrooms.

The methodology of the study was a case study of the two classrooms with the author as participant observer. During a five-week period I joined each class on three or four occasions each week when children were working with a unit of work that focused on multidigit computation. Classes were videotaped and audiotaped and samples of children's work were collected every 3 or 4 school days. While my major focus was to observe the classroom interactions, it was agreed that I would offer support for the children and teachers as required during the ongoing classroom activity. Following each observation session the teacher and I discussed the day's activities and considered any issues that arose. At the conclusion of the five-week observation period each teacher participated in an interview during which they discussed their reactions to the implementation of the teaching approach.

The teachers, Ms N and Mr C, were both experienced primary school teachers. Their approaches to the teaching of multidigit computation had previously focused on work with place value blocks, the development of the standard algorithm and exploration of computational word problems. Neither teacher had experience with children implementing their own invented algorithms. Ms N, who taught a Year 3 class (aged 7-8), had 14 years teaching experience but expressed a lack of confidence in her own abilities in mathematics and a perceived need for support in the teaching of mathematics. Mr C, who taught a Year 3&4 class (aged 7-9) had 20 years teaching experience and was confident in both his own mathematics and his teaching of mathematics.

Results

Ms N introduced her "number unit" by revising addition and subtraction basic facts with the class. She extended this activity by subsequently asking children to record multiple names for numbers (eg 19=10+9, 19=1000-981). She then focused on two-digit place value asking children to identify the tens and ones values (eg 36 is 30 and 6 or 3 tens and 6 ones). Multidigit computation was formally introduced by providing children with addition problems involving values that required no renaming of ones as tens (ie 54+38 would not have been included during this introduction as the ones values when added exceed 10). Children were encouraged to use any methods that resulted in an accurate answer and to share their methods with others in the class. She suggested that children look for ways they could use known information to solve more difficult problems. She focused on how basic facts knowledge could be used to solve problems with larger numbers (eg if 3+4=7, then 30+40=70 and 300+400=700). As children displayed confidence adding two-digit numbers with no renaming Ms N introduced examples where renaming was required (ie the ones values when added equalled more than 10). Toward the end of the five-week unit Ms N introduced subtraction, although she expressed a concern that some children "may not be ready for it".

Although her planning appeared tightly structured and influenced by her earlier experiences using place value blocks, throughout the five-week unit Ms N encouraged children to express their own ideas and to share these with others. Opportunities for discussion were introduced into each day's activity and all children were encouraged to participate in an environment where all contributions were openly valued and received with interest. Recordings were presented in formats that reflected the spoken descriptions given by the children and although two children had home experience with standard algorithms these were not formally introduced in this unit.

Acknowledging the age and ability range within his class, Mr C decided to allow flexibility for some children to extend beyond addition and subtraction during the five-week unit. He began the unit by assessing children's skills in multidigit addition and subtraction. Children were set two word problems to work on as homework. It was emphasised that the problems, requiring addition of 53 and 39 and subtraction of 27 from 62, were to be solved using any method and children were asked to use words or numbers to write down how they "got the answer". Within classroom activity Mr C focused on similar introductory activities as used by Ms N however these activities included greater emphasis on multiplication and division as well as addition and subtraction. He focused on the inverse nature of multiplication and division and explored strategies for mental computation such as the use of doubles and derived facts. Multidigit computation problems were posed in contexts (as word problems) by the teacher or selected by the children as a Lucky Dip from which they chose two or more counters each with 1-digit, 2-digit or 3-digit numbers that could be added, subtracted, multiplied or divided. This latter activity allowed children to work at their own level but Mr C also monitored children's choices.

Children were encouraged to use mental methods as well as record the way in which they worked. As in Ms N's classroom, children were encouraged to share their ideas and extensive opportunities were provided for discussion each day. The phrase, "Does anyone have a different way of working this out?" was used in most discussions and children were openly encouraged to look for alternative methods to try. While Mr C was openly committed to children devising and using their own strategies he also perceived greater pressure from parents and future teachers to introduce the standard algorithm. At the midpoint of the unit he introduced the children to the algorithm by demonstrating on the whiteboard the way he would record the problem. Although he emphasised that this was only one way of recording there was a marked change in the recording used by the children after this demonstration.

Children in both classrooms demonstrated a range of strategies for computation, for example: counting on, counting back, addition or subtraction by place values, use of known facts and use of compensation. Children's written work samples were collected at seven regular intervals throughout the unit and were analysed in terms of presentation (whether the children recorded their working using horizontal or vertical working formats) and process direction (whether the children started with the larger values or the units when calculating). Samples from the first five collection points in Ms N's classroom (Class 1) reflected addition and the final two sets of samples showed subtraction examples. The first two sets of samples collected from Mr C's classroom (Class 2) were the addition and subtraction homework examples completed by the children. Subsequent sample sets included addition, subtraction and multiplication examples. Table 1 indicates the way in which children presented their work at each sample collection point and Table 2 shows the process direction used by the children.

Most children in Ms N's class used horizontal working formats throughout the five-week unit. Those children using vertical formats identified they had received instruction from the home in the use of these methods however it was noted that all of these children used horizontal formats on other occasions. Collection point 6 reflected samples collected when subtraction was first introduced in this classroom and several children showed no working.

Table 1
Percentage of children at each sample collection point using horizontal and vertical formats for calculating. Class 1: n=22, Class 2: n=23

Data Collection Point	1	2	3	4	5	6	7
Class 1 Horizontal	91%	100%	100%	95%	95%	77%	95%
Class 1 Vertical	9%	0%	0%	0%	0%	0%	4%
Class 2 Horizontal	39%	70%	78%	43%	43%	56%	56%
Class 2 Vertical	57%	30%	17%	57%	57%	43%	43%

The children in Mr C's class demonstrated greater use of vertical formats. Some children had had prior experience with the standard algorithm in their previous year of schooling. It was also noted on some work samples that children had received assistance from parents or siblings when completing their homework task. As the children explored new ideas and were encouraged to use their own formats the number of vertical representations decreased however there was a marked change following Mr C's demonstration of his vertical format, the standard algorithm, prior to collection point 4.

Table 2
Percentage of children at each sample collection point calculating larger place values before units (left-to-right) and units before larger place values (right-to-left). Class 1: n=22, Class 2: n=23

Data Collection Point	1	2	3	4	5	6	7
Class 1 Left-to-right	59%	82%	82%	73%	77%	27%	41%
Class 1 Right-to-left	14%	4%	4%	0%	0%	0%	4%
Class 2 Left-to-right	39%	22%	35%	4%	0%	0%	0%
Class 2 Right-to-left	26%	22%	0%	4%	13%	17%	26%

The process direction used by children in their recordings was often difficult to establish from the written work samples. This was most evident in Mr C's class with children more frequently used the vertical formats. Where problems required no renaming (eg 10 ones as 1 ten) it was unlikely that there was any indication whether the children had started with the units, tens or larger values when calculating. The predominance of a left-to-right process in Ms N's class reflected the frequent use of horizontal recordings by the children. Problems were approached as they would be read (ie left-to-right) and recording clearly indicated the children calculating tens values before units (eg 30 + 50 = 80, 5 + 7 = 12 so 35 + 57 = 92).

Discussion

While acknowledging the children's success and their ability to explain and justify the methods used, the teachers faced a number of challenges within the context of their own beliefs, the classroom programme and school and community expectations.

The teachers identified that children completed fewer examples but spent more time discussing their work with the teacher and others in the class, reflecting greater emphasis on the verbalisation of the processes used. They believed that the frequent opportunities for discussion increased children's confidence in presenting their ideas to others. Mr C commented that children were "*thinking deeper*" and that they were "*freed to think for themselves, to challenge themselves, to build independence*" and to no longer be "*bounded by the task*". Mr C and Ms N both acknowledged that many children exhibited skill levels beyond their expectations and noted that in a traditional classroom programme some of these children would have been limited to practising skills they had already mastered.

Both the teachers identified their role as encouraging children to put forward their ideas and valuing these contributions. Ms N openly expressed to the children the dilemma she faced when introducing something new to the class. She said, "*Part of me wants to get out there and show you how to work it out but I'm going to let you have a think about it first. Then you (emphasised on tape) can decide how best to work it out to suit your way of thinking.*" She recognised such statements as reflecting her perceived lack of knowledge in mathematics. When reflecting on her development of the number unit with the class Ms N identified that she had imposed limits on the range of problems to be explored in order that she was better able to predict possible difficulties and prepare for these. She also acknowledged that the structure of her teaching, although encouraging children's own methods, was similar to the format of instruction she had used when working with place value blocks, an approach with which she was familiar and had experienced perceived success. During the final interview Ms N indicated an intention to include a wider range of examples at an earlier stage together with opportunities for children to extend their ideas.

Although there was an acceptance that children could develop their own methods for computation, the teachers expected that these methods should also lead to accurate responses. Throughout the five-week unit high levels of accuracy were evident in the activities observed and samples collected. However in each classroom these levels dropped for a short time. Given Ms N's continued emphasis on addition before the introduction of subtraction, the children in her class had strongly developed successful methods (most commonly 46 + 35 would be solved as 40 + 30 = 70, 6 + 5 = 11 so 46 + 35 = 81) that they then assumed could be applied to subtraction. In attempting to apply these methods, many children made a number of errors (eg 52 – 35 solved as 50-30=20, 2-5=3 so 52-35=23) however during the next lesson successful strategies were being shared and implemented successfully. In Mr C's class the number of errors increased markedly after the introduction of the standard algorithm as children attempted to apply this method. At this time one child was recorded as saying, "*I know the answer should be ... but I can't get the numbers to work*". She had successfully calculated the solution mentally but in applying the standard algorithm was achieving an incorrect result.

As pointed out by Yackel, Cobb and Wood (1992), teachers work in a context in which procedures must be institutionally sanctioned. Both teachers highlighted the expectations they perceived from parents and subsequent class level teachers. Parents consider the acquisition of computational skills, along with the knowledge of the tables, as the most important mathematical task of primary schooling (McIntosh, 1998) and demonstration of these skills is seen to be through paper-and-pencil computation using standard working formats. Mr C commented that his introduction of the standard algorithm was in response to a perceived demand from parents that children know these methods. Although the use of invented algorithms had been accepted by the school as appropriate for implementation it was evident that both teachers saw this approach as an interim step and that standard algorithms would still be expected subsequent to children's exploration of their own methods. They expressed concern that future teachers might wonder what they had been teaching all year if the children had no experiences of standard algorithms. They recommended that any school taking on this approach should view it as a school-wide policy whereby all teachers were familiar with expectations and parents could be kept informed. Mr C pointed out, *"If it is going to happen you need a climate that asks people to take the risk of trying something new from the start."*

While accepting the benefits to children of using this approach the teachers perceived expectation of the wider community acted as a constraint in making changes to their teaching. For change to be implemented in the New Zealand setting it appears that the focus must come from the wider educational community and be backed up with support for teachers and schools.

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DEVELOPING PERFORMANCE ASSESSMENT TASKS IN MATHEMATICS: A CASE STUDY

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Development of assessment tasks that both provide quality insights into students' mathematical understanding and produce rigorous data for measurement purposes is a challenge for mathematics educators. This paper presents a case study of the development of three performance assessment tasks. The tasks addressed several strands of the mathematics curriculum and were designed to match the teaching strategies being used. The tasks were refined following the application of a qualitative developmental model, the SOLO Taxonomy. They were also tested against criteria for good assessment.

Developing assessment tasks that both elicit information about students' deeper understanding of mathematical concepts and provide reliable and valid data for measurement purposes is a challenge for mathematics educators. One way in which this may be achieved is through the use of well designed performance assessment tasks with appropriate scoring rubrics that provide students with the opportunity to draw on a range of mathematics skills and apply these to interesting and motivating situations. Students' approaches to the problems reflect their cognitive processes and mathematical understanding (Harmon *et al* 1997).

Assessment should describe what a student knows and can do in relation to the learning and teaching program within the classroom and should fit within the overall framework of the development of that student (Griffin & Nix 1991). As well, it should "...enhance student motivation ..., provide starting points for further learning..." and "... be fair for all groups of students" (Australian Association of Mathematics Teachers n.d.).

This paper reports the trial of three performance assessment tasks as a case study in preparing suitable tasks that meet the criteria outlined above. These assessments are part of larger study that aims to improve understanding of factors that affect the numeracy development of Indigenous adolescents. The on-going project, Improving Numeracy for Indigenous Students in Secondary Schools (INISSS), targets 17 Tasmanian high schools that had a relatively high enrolment of Indigenous students who were achieving poorly on state-wide measures of numeracy achievement (Callingham 1998). In some of these schools Aboriginal Education Workers worked with Indigenous and non-Indigenous students to provide additional support. Many of the target students were being taught by skilled teachers who had not had training in mathematics teaching; others were taught by highly experienced, well qualified mathematics teachers. The INISSS project provides professional development to teachers and Aboriginal Education Workers involved with the target group of low achieving students in years 7 to 9. Evaluation of the project will focus on year 8 students since most teachers in the project teach students in this grade.

The professional development for teachers is based around the use of Mathematical Problem Solving Task Centres that have been successfully used with Indigenous students (Williams 1997). The tasks come packaged for use with all required materials and manipulatives in reseal plastic bags. Students work on them singly or in small groups. Teachers in the INISSS project schools reported using these tasks in a variety of ways, similar to teachers elsewhere (Pavlou & Clarke 1998). The tasks proved to be motivating to students and, according to the teachers, provided the first real success in mathematics for many students.

RESEARCH QUESTIONS

Because of funding arrangements, the project needed to report on student learning outcomes as part of its evaluation. With this in mind, it was decided to develop a set of assessment tasks that would reflect the nature of the Task Centre material, but which also could be marked rigorously by teachers. The purpose of this initial small-scale trial was to collect qualitative data about the presentation of the tasks, clarity of the instructions and the range of responses elicited from students. The research questions addressed by this trial were then:

- Were the tasks presented in a way that was both understandable and motivating to all students?
- Did the tasks allow for a wide range of responses along a developmental continuum?

Theoretical Model

The theoretical model used as the basis for analysis of the responses was the SOLO Taxonomy (Biggs & Collis, 1982; Collis & Biggs, 1991). This neo-Piagetian model proposes five modes of thinking: sensori-motor, ikonic, concrete symbolic, formal and post-formal. These modes are increasingly abstract and develop as an individual matures from infancy through to adulthood. Although each mode to some extent builds on the previous mode, each mode also continues to develop in its own right. The model can thus provide a description of the development of a student's understanding.

Within these modes, cycles of response to different learning situations can be found. The five categories of response depend on the Structure of the Observed Learning Outcome (SOLO) produced by the student from the use of information available to the individual when giving an answer. The categories are of increasing complexity and are described as follows:

- Pre-structural - no use or irrelevant use of the information available;
- Uni-structural (U) - use of only one piece of the relevant information;
- Multi-structural (M) - a number of pieces of information are strung together, usually sequentially;
- Relational (R) - the various pieces of information are integrated into an understanding of the relationships involved in the situation;
- Extended Abstract - over-arching principles are called upon, and the thinking is thus moved to a new level. This response provides the uni-structural response for the next higher mode of thinking.

In the school context, cycles that occur in the ikonic and concrete symbolic modes are particularly important. These progress through the uni-structural, multi-structural, relational cycle and are termed U-M-R cycles. Several studies have identified at least one and sometimes two U-M-R cycles of response in the concrete symbolic mode in different learning situations (e.g. Watson, Collis & Moritz 1997). From a measurement perspective, the use of U-M-R cycles potentially provides a basis for the development of a scoring rubric, which can then be translated onto a developmental scale addressing the underlying concept. For this reason, it was important to ensure that the tasks allowed for at least one cycle of U-M-R responses.

THE TASKS

Of five tasks trialed, one was discarded and one completely rewritten following comments from teachers and students. The three assessment tasks reported here addressed aspects of several strands of the mathematics curriculum. They were presented in a similar format to the Task Centre materials. Students were provided with a record sheet for each task. Each task had between 8 and 10 individual questions, each of which was provided with a scoring rubric based on anticipated responses. This provided a standard format for collecting the assessment information, but did make the tasks more structured than those of the Task Centres. Overall, however, the tasks were more open than traditional assessment materials and had the 'flavour' of the teaching tasks. An initial scoring rubric was developed for every question on each task, based on experience and information from the initial sources where this was available. It was hoped that this approach would be easier for teachers to use rather than a holistic score on the whole task.

Bean Counter was adapted from a similar assessment task used in New Zealand (Flockton & Crooks 1998). It required students to use number patterns to complete magic squares. The name came from the beans that were provided as part of the task package. Students used the beans to 'juggle' the numbers in the empty squares. The underlying concept was recognition and use of number patterns. *Come in Spinner*, as its name implies, addressed aspects of the students' understanding of chance. It was developed from an idea in the *Mathematics Work Samples* (Curriculum Corporation 1994). Students made spinners and tested them to see if they met the specified criteria.

Street Party was an original task addressing pattern and algebra concepts. Students developed patterns by building small tables into a 'long table' for a street party, and used the patterns to find the numbers of small tables needed to seat a given number of people. Both direct and inverse relationships were targeted.

THE TRIAL

The tasks were trialed in three year 8 classrooms in one school in early December 1998. The school enrolled students from a very wide area and its students came from a range of socio-economic backgrounds. The students in the grade were divided broadly into upper and lower ability classes. The trial classes included all levels of ability and were taught by two very experienced mathematics teachers. Altogether 32 students produced 45 completed record sheets covering these three tasks.

Students worked through the tasks, usually in pairs, but providing an individual response on the record sheet provided. They were also asked what they thought about the tasks they attempted and these written comments were collected separately. No pro-forma was supplied for this—students were free to respond in any way that they wished.

The two teachers had both previously used Task Centre materials in a variety of settings. They marked the tasks using the scoring rubrics provided based on anticipated responses, and provided feedback about these and practical aspects of using the materials. Marked responses and students' and teachers' written comments were collected for analysis.

FINDINGS

Motivation

Teachers reported that the students enjoyed the tasks and were motivated to complete them. In general the students were on task throughout the lesson—a first for some of these students. One teacher reported that he had never seen one student work so well. This student was of Polynesian descent and usually struggled in the classroom, becoming at times a behaviour problem. At the end of the task the student wrote "Thanks to [the teacher] and I really like the Tasks".

The positive comments by teachers were borne out by other students' comments.

I thought it was a good worksheet and fun to do.	Comment on *Bean Counter* RL
I found this activity more interesting. Although it was a bit easy.	Comment on *Street Party* CH

Comments about the apparent lack of difficulty were frequent, even from students who produced a low-level response:

> The spinner unit was pretty good because it made you think a lot and was pretty easy.
> Comment on *Come in Spinner* BC

The comments also produced some interesting insights into students' perceptions of problem solving:

> ...also question 4, 5, and 6 are a bit obious[*sic*] and aren't problem solving just looking problems. It's a good activity but a bit short! Comment on *Bean Counter* CJ

This student had produced one of the lowest level responses to the task.

It was clear from these comments that students enjoyed the tasks, found them understandable and non-threatening and were prepared to work at them—even those who were unable to produce high level responses.

Developmental Progression

By considering the nature and complexity of students' responses, the development of the targeted concept could be evaluated. Findings are summarised for each task.

Bean Counter

This task provided students with packets of beans and some blank grids to help them find their answers. This allowed a range of strategies to be used—some students enjoyed using the beans while others preferred to use blank grids and write the different solutions.

Placing beans on one square and then arranging others to fit was the commonest problem solving strategy. Where this was unsystematic, it was coded as an uni-structural response in SOLO terms—each response was unique and independent of the others. Where a similar but systematic strategy was applied, this was coded as a multi-structural response because the student was using one bit of information sequentially. One student appeared to be visualising her response:

> Well I went through the numbers in my mind to come up to the answer. I had to go all through the numbers. (McC)

Since the final problem had eleven different solutions, not surprisingly this student had been unable to find all of these. This was coded uni-structural since the student was only using one number at a time, and appeared unable to systematically record her findings.

No student explicitly referred to patterns in the solutions although this was implied in some responses. Probably the most sophisticated response came from the Polynesian student referred to earlier, who had clearly recognised the patterns but lacked the English language skills to express this clearly. Having described his process for the easy initial questions he went on to write:

> We get the answer easyier(*sic*) because we just have to get the answer from number 1, 2, 3 and 4. (JR)

The numbers referred to were those used to label his first few solutions of the eleven-solution problem. This response was also coded multi-structural but with better language skills the student might have reached a relational response. Interestingly, his partner, while participating in the same process had not reached the same insights:

> Me and [JR] worked our methods out of jubiling (*juggling*) our numbers around and trying to work them out and add the number up with the correct answer. (HM)

This response was coded uni-structural as an unsystematic strategy. The common criticism of the weakest student's performance being over-inflated when group tasks are used for assessment was not upheld in this instance. A summary of the highest responses to this task achieved by students is presented in Table 1.

Table 1
Summary of Highest Responses to Bean Counter

Uni-structural N=7	Multi-structural N=8	Relational N=0
Use of unsystematic strategies based on random trials of numbers	Use of strategies using progressive, systematic movement of beans from one cell to another.	Use of systematic strategies based on recognising the patterns and constraints of the numbers in the squares.

Come in Spinner

All students' responses demonstrated an understanding of equal likelihood but their understanding of variability was less certain. They were asked to make two spinners—one with an equal likelihood of landing on each of three colours, and one with four colours where one colour would come up most, one least and the other two were about equal. They had to test these and explain their tests and results. The spinners were correctly constructed in all but one case. Students were able to set up a test and record the findings but they were satisfied with one short run. Some students gave no explanation of their test findings other than a comment unsupported by data, or recorded the data without comment. These kinds of response were classified as uni-structural since the students appeared to focus on only one aspect of the test. Others suggested an emerging understanding of variability. For example:

> I found out that you don't get equal amount of colours even though you have equal spaces. (BC)

These responses were classified as multi-structural since the students were attempting to read their data in relation to the design of the spinner.

Some students seemed to think that the number of spins used to test the spinner should be related to the number of colours on the spinner. Many students tested their 3-colour spinner by spinning it 30 times and their 4-colour spinner by spinning it 40 times. This may be a reaction to a common teaching strategy of asking students to undertake short runs and then combining the results to get a larger sample size. Highest level responses to *Come In Spinner* are summarised in Table 2.

Table 2
Summary of Highest Responses to Come In Spinner

Uni-structural N=7	Multi-structural N=8	Relational N=0
Spinners correctly constructed and tested but test limited to one short run and no comment on the variability of the result.	Spinners correctly constructed and tested but test limited to one short run, test results recorded and explained by reference to the spinner construction only.	Spinners correctly constructed and tested. Long runs suggested to improve predictions of outcomes and possibility of spinner bias acknowledged.

Street Party

Responses to this task indicated students' developing understanding of relationships. The first two questions, which all students successfully completed by continuing the pattern, only required an uni-structural response in SOLO terms.

When asked to record their findings most students simply copied and extended the diagrams that had been provided. Some, however, drew up a table showing the number of small tables used to make the big table and the number of people seated, and two went straight to the underlying rule. This appears to suggest a cycle of understanding of the relationship the students were investigating. In continuing the given pattern in the same form, students demonstrated only a uni-structural response, while those who translated this into a table required at least multi-structural understanding since they had to recognise two variables and abstract these into a different symbolic form.

The two students who used the rule appeared to demonstrate a relational understanding of the information. Their work, however, showed some interesting differences.

One of these, a girl, had described the rule in words:

> To get the results you can times the number of tables by 2 and add 2 for the 2 at the end. (HC)

This student, in later parts of the task, clearly indicated that she had a good understanding of this direct relationship, arising from a concrete situation. She correctly solved several more difficult problems, including those where a different pattern was used, to find the number of people that could be seated. She could not, however, solve any of the inverse problems where she was asked to find the number of tables needed to seat a given number of people. For this reason, her response was coded as multi-structural rather than relational.

The other student, a boy, used a more mathematical notation:

> People at tables = 2 x Tables + 2 (RL)

This student completed the whole worksheet correctly, including the inverse problems. He appeared, however, to need some work on refining his mathematical notation. He correctly solved the problem of finding the number of tables needed to seat 200 people but when asked to explain how he got his answer wrote $200 - 2 \div 2$, rather than $(220 - 2) \div 2$. Despite the poor notation, the response was coded as relational with respect to the underlying concept of understanding relationships.

Few other students correctly solved the inverse problems. One who did solve the problem requiring the number of tables for 200 people, used a laborious method of drawing long lines of tables and counting the people seated. Not surprisingly, this student did not complete the rest of the task. His uni-structural method was successful but very inefficient. The most successful students were clearly able to articulate the relationship using mathematical language. For example, this relational response:

> I subtracted 2 from 200 because of the 2 people at the end and then I divided 198 by 2. This gave me 99. (SJ)

Another response that had not quite reached the same level of understanding was the following, coded as multi-structural because of the apparent inability to resolve it completely:

> There is[*sic*] three people at each end of the tables and two for each of the middle table so you work out a number to times to add to the end. (BG)

The task appeared to be successful in drawing out the range of understanding of relationships. Highest level responses to Street Party are summarised in Table 3.

Table 3
Summary of Highest Responses to Street Party

Uni-structural N=4	Multi-structural N=7	Relational N=4
Correctly solved simple relational problems using counting strategies only.	Recognised a straightforward relationship between two variables and used this to solve direct relationship problems only in a practical situation.	Developed a straightforward rule relating two variables in a practical situation and used this to solve direct and inverse problems.

CONCLUSIONS

The trial met its goal of providing utilitarian information about the tasks and data concerning the range of responses in relation to the underlying concepts being addressed. In answer to the first question, the tasks were clearly motivating and accessible to all students. Both teachers and students commented favourably on this aspect, and it was also clear from student work that they had understood what was asked, even if they produced a low-level response. Teachers also reported that the scoring rubrics were provided in an easy-to-use format and they could match their students' performances to the descriptions of anticipated response provided.

In terms of the developmental continuum, SOLO analysis suggested that some tasks did not attract relational level responses. Modifications were subsequently made to improve the range of responses possible. Additional questions were added to the spinner task to elicit responses more clearly about variability and the 4-colour spinner was made more open by asking students to design their own spinner having a different chance of stopping on each colour. Minor changes were made to the wording of *Bean Counter* to encourage students to look for patterns. Changes to *Street Party* asked students to design their own tables and determine relationships based on these designs to allow a greater degree of freedom of response. The U-M-R cycles identified were also used to inform the scoring rubrics of the rewritten tasks before large-scale use, while maintaining the format. Samples of student work were also used to provide exemplars for the teachers' manual to accompany the final tasks.

There were also some unexpected outcomes from the trial. Both of the teachers involved indicated that these tasks had provided them with insights into their students' thinking that they would not otherwise have had, such as the response from the Polynesian student. These insights also applied to teaching strategies. For example, following discussion about the spinner responses, one teacher said that he would only ask students to complete a limited number of trials in the future if he specifically wanted to compare what happened with different small samples. If he wanted large sample sizes he would instead give students five minutes to complete as many trials as possible so that not all students produced the same amount of data. This teacher has also spent some time providing opportunities for his low-ability students to translate patterns from diagrams into other summary forms such as lists and tables. In these ways, the tasks affected the teaching programs in the classrooms more than might have been expected from a small-scale trial at the end of the school year.

Finally, the trial provided some evidence of the use of a theoretical developmental model to underpin assessment tasks to be used in large-scale quantitative studies. The SOLO model applied to student work provided a means of identifying the developmental sequence of the concept being addressed by a task. In addition the underlying U-M-R cycles offered a basis for developing improved scoring rubrics for individual questions in each task. Zero was used for a pre-structural response throughout. Other questions were coded according to the highest level of the U-M-R cycle possible. Thus a question that could get a relational response was coded 0 - 3, where unistructural (1) and multi-structural (2) responses provided partial credit. Further work is needed in this area, however, particularly in relation to large-scale testing.

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CHILDREN'S INFORMAL COMPOSITE AND TRUNCATED PARTITIONING STRATEGIES

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In this paper, we report two subsets of partitioning strategies utilised by a sample of twelve Grade 3 children when they were asked to solve sequences of partitioning problems. These strategies emerged when the children sought to make the process of partitioning more efficient in terms of generating equal shares, reducing the number of steps involved in partitioning and reducing the load on working memory. Implications for instruction are drawn from the findings.

INTRODUCTION

Informal or intuitive knowledge is what learners have prior to instruction, "the applied, real-life, circumstantial knowledge, which can either be correct or incorrect" (Leinhardt, 1988, p. 12). There is growing evidence to suggest that students have well developed informal knowledge of fractions which they are able to use to solve everyday problems (Behr, Harel, Post, & Lesh, 1992; Hiebert, 1988; Leinhardt, 1988; Mack, 1990; Resnick, 1986), including informal knowledge about partitioning and equivalence (Kieren, 1976; Pothier & Sawada, 1983). Research has generally indicated that informal knowledge can serve as a basis for understanding symbols and procedures (Hiebert, 1988). Thus, in recent years, there has been an increased interest in investigating young children's informal strategies for partitioning. These studies have found that young children tend to use a variety of informal strategies when confronted with partitioning problems (Lamon, 1996; Pitkethly & Hunting, 1996; Pothier & Sawada, 1983, Streefland, 1991).

In the study being reported in this paper, we investigated the partitioning strategies utilised by a purposively selected sample (Patton, 1990) of twelve Grade 3 children when they were asked to solve sequences of "realistic" partitive quotient fraction problems. Our purpose for conducting this study was to add to the literature on young children's partitioning strategies by not only setting out to identify new partitioning strategies but also to develop a taxonomy for classifying partitioning strategies in terms of their ability to facilitate the abstraction by young children of the partitive quotient fraction construct. In the partitive quotient fraction construct, the fraction $^x/_y$ refers to the quantity represented by one of the resulting shares when the quantity x is partitioned into y equal parts. For example, if two whole cakes are partitioned into three equal shares, then each share is two thirds of a cake. The major findings from this study are reported in Charles (1998), Charles, Nason, and Cooper (submitted), Charles, Nason, Cooper and McRobbie (submitted), and Charles, Cooper, Nason and McRobbie (in preparation). In this paper, we report on two subsets of partitioning strategies utilised by children in the study: composite and truncated strategies.

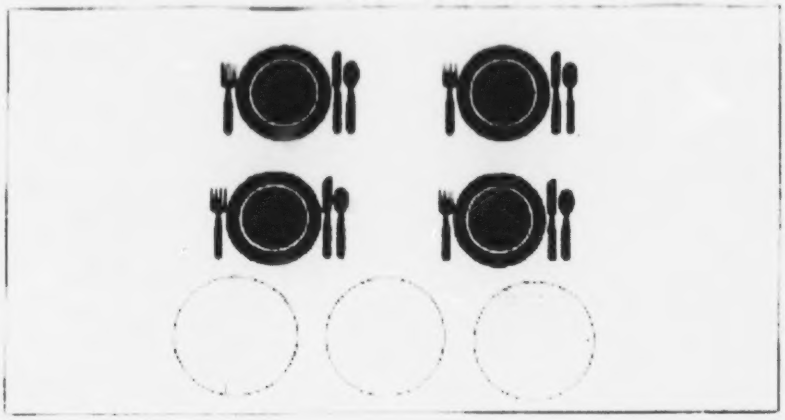
METHOD

The study was couched in the constructivist paradigm. The data gathering technique used was the clinical or mixed method technique of Ginsburg, Kossan, Schwartz, and Swanson (1983) which is a combination of Piaget's clinical interview and the talk aloud procedures of Ericsson and Simon (1984). According to Ginsberg et al., this technique enables researchers to not only elicit complex intellectual activity but also to identify the internal symbolic mechanisms underlying the complex intellectual activity.

Instruments

A set of thirty realistic partitioning problems were developed for the study. In each of the problems, the children were asked to assume the roles of waiters/waitresses serving pizzas, pancakes, pikelets, icecream bars, apple pies, licorice straps to a number of customers sitting at a restaurant table. An example of one of these problems is presented in Figure 1 below.

Figure 1
Partitioning Problem 1(f): 3 pizza among 4 people



Three task variables were taken into consideration during the development of the partitioning problems: (i) types of analog objects, (ii) number of analog objects, and (iii) number of people. The analog objects used were circular region models, rectangular region models and length models (cf., Streefland, 1991). The number of analog objects ranged from one to six and the number of people ranged from two to six. The circular region models for the study included: pizza, pancake, pikelet and apple pie. The rectangular region models included: cake and icecream barcake. The length model was represented by the licorice strap (cf., Streefland, 1991).

PROCEDURE

Each interview began with problems using the "pizza" circular region model (cf. Streefland, 1991), then proceeded onto problems using the rectangular region model and the length model. As the literature noted that children have sound informal knowledge of half (Behr et al., 1992; Ball, 1993) and powerful strategies for halving (Pothier & Sawada, 1983), the interview began with problems which produced shares of half and quarter. The sequence of partitioning problems administered to a child after (s)he had attempted these initial problems which generated shares of half or a quarter was not selected a priori. Instead, the interviewer used the child's responses to each problem to inform the selection of problems administered later on in the interview. Each child thus was administered a unique sequence of partitioning problems.

During the course of the interviews, icecream bar and licorice strap problems were presented to all of the children in order to investigate Ball's (1993) contention that partitioning of length models and narrow rectangular region models is easier than the partitioning of circular region models. It was expected that children who had difficulty partitioning one circular region model among three people may successfully partition one length model or one narrow rectangular region model among three people. Problems were included to explore children's partitioning strategies for four, five and six people. Particular attention also was taken of children's partitioning strategies for the wide rectangular region model as Streefland (1991) noted that it supported both one-directional (i.e., vertical or horizontal) and two-directional (i.e., vertical and horizontal) partitioning.

RESULTS AND DISCUSSION

During the course of this study, twelve partitioning strategies emerged. As is indicated in Table 1 below, six of the partitioning strategies used by the children in this study had hitherto been reported in Streefland (1991), English and Halford (1995), Pothier and Sawada (1983), and Lamon (1996). However, six of the strategies used by the children in this study were hitherto unreported in the research literature. Full descriptions of these twelve strategies are provided in Charles (1998) and Charles, Nason and Cooper (submitted).

Table 1
Partitioning Strategies

Hitherto Reported Strategies	Hitherto Unreported Strategies
Partitive quotient foundational strategy *Streefland (1991)* *English & Halford (1995)*	Partitive quotient part-whole strategy
Proceduralised partitive quotient strategy *Streefland (1991)*	Partitive quotient renaming strategy
Horizontal partitioning strategy *Streefland (1991)* *Pothier & Sawada (1983)*	People by objects strategy
Halving the object then halving again and *Streefland (1991)* *Pothier & Sawada (1983)*	Half the objects between half the people again strategy strategy
Half to each person then quarter to each person strategy *Streefland (1991)*	Repeated sizing strategy
Whole to each person then half remaining objects between half the people strategy *Lamon (1996)*	Repeated halving/repeated sizing strategy

We noticed that as each of the children progressed through his or her sequence of partitioning problems, they all sought to find ways of speeding up the process of partitioning and of reducing the load on working memory. As this occurred, we noted the emergence of two very interesting classes of partitioning strategies which we labelled *composite strategies* and *truncated strategies*.

Composite Strategies

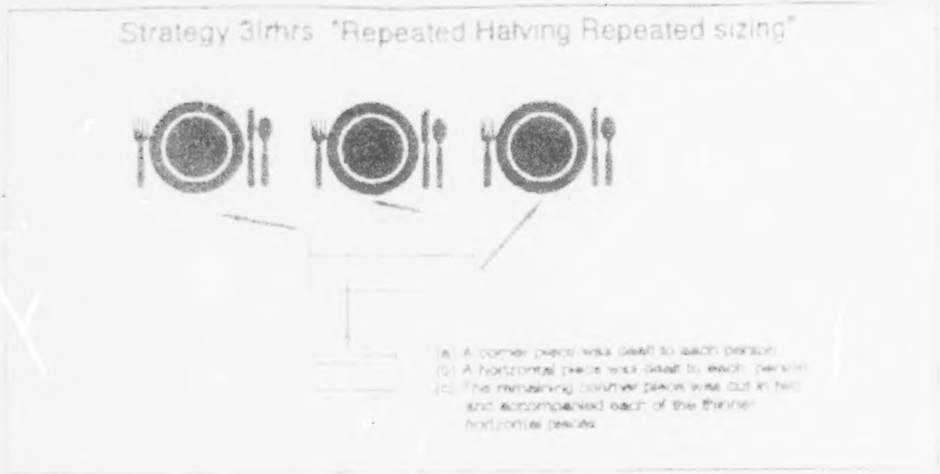
A composite strategy is formed when two or more strategies are combined into a single strategy. In this study, we identified three such strategies: the "whole to each person then half remaining objects between half the people strategy", the "repeated halving/repeated sizing strategy", and the "half to each person then quarter to each person strategy".

The "whole to each person then half remaining objects between half the people strategy" is formed by the combination of the children's strategies for sharing wholes and the "half the objects between half the people strategy" for sharing fractional pieces. Emma employed this strategy to share six pizzas between four people. She shared one whole to each person.

Then she tiered the two "left-over" pizzas, cut them into four halves and shared 1/2 to each person. She quantified each person's share as $1\,{}^{1}/_{2}$.

The "repeated halving/repeated sizing strategy" is a combination of the "halving the object again and again strategy" and the "repeated sizing" strategy. Joshua used this strategy to share a rectangular cake between three people. He cut the cake in half horizontally. Then he cut each half in half and shared 1/4 to each person (i.e., he applied the "halving the object again and again strategy"). He resized the final 1/4 as demonstrated in Figure 2 and shared the pieces as evenly as possible by the attribute of area (i.e., he applied the repeated sizing strategy").

Figure 2
Joshua's Application of "repeated halving/repeated sizing strategy"


Strategy 3/rhrs "Repeated Halving Repeated sizing"
(a) A corner piece was dealt to each person
(b) A horizontal piece was dealt to each person
(c) The remaining corner piece was cut in two and accompanied each of the thinner horizontal pieces

The "half to each person then quarter to each person strategy" is a combination of the "half the objects between half the people strategy" and the "partitive quotient foundational strategy". Two of the children in this study, Joshua and Sally, used this strategy when asked to share three pizzas between four people. Joshua partitioned the first two pizzas into halves and shared them fairly between the four people (i.e., he applied the "half the objects between half the people strategy"). Then he partitioned the third and remaining pizza into quarters and shared these quarters between the four people (i.e., he applied the "partitive quotient foundation strategy"). Finally, he quantified the total amount each person received by adding the 1/2 (generated by the application of the "half the objects between half the people strategy") to the 1/4 (generated by the application of the "partitive quotient foundational strategy") to get 3/4. When attempting the same problem, Sally cut the first and second pizza in halves and the third pizza in quarters. She shared 1/2 to each person and then 1/4 to each person. She quantified each share in a similar manner to that used by Joshua. Each of these three composite strategies shared a number of characteristics. First, they were very efficient time-wise at generating equal shares. Second, each composite strategy could only be applied to a small number of partitioning problems. We also noted that the composite strategies only emerged after the child using them had successfully utilised the component strategies on a number of previous partitioning problems. The emergence of the composite strategies thus was highly dependent on the sequence of problems presented to a child during an interview session. If a problem currently being solved could be broken into a sequence of subproblems which were almost identical to

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problems attempted recently, then the probability of a composite strategy emerging greatly increased.

Truncated Strategies

A truncated strategy is formed by the shortening and/or simplification of an earlier emerging partitioning strategy. We found that the children only truncated strategies after they had successfully applied them a few times. Two types of truncations were identified during the course of this study: *simplification of the physical process of partitioning*, and *cognitive proceduralisation* (cf., Anderson, 1983).

Simplifications of the Physical Process of Partitioning

The "half the objects between half people" strategy is an example of a truncated strategy which greatly simplified the process of partitioning the objects. This strategy involved the following six steps:

Step 1.	recognition of the number of people *(n)*
Step 2.	recognition of the number of objects *(x)*
Step 3.	realisation that halving the objects will generate enough pieces to share between all the people
Step 4.	partition the objects into halves
Step 5.	sharing the halves between all the people
Step 6.	quantification of each share

Joshua, Claudia, Sophie, Thomas, Emma, Sally, Da and Belinda utilised this strategy to share two objects between four people. Thus, instead of partitioning each of the objects into quarters as would have happened if they had applied the "partitive quotient foundational strategy", the children instead partitioned each object into two halves and then distributed the four halves generated between the four people.

Although truncations of this type simplified the physical process of partitioning, they often tended to lead to two types of problems. First, the strategies created by this type of truncation only could be successfully applied to a limited number of partitive quotient fraction problems. Second, the strategies created by truncations of this type, whilst simplifying the physical process of partitioning, more often than not complicated the cognitive processes of accurately quantifying each share and making a conceptual mapping between the concrete activity and the partitive quotient fraction construct. For example, after Claudia had applied the "half the objects between half people strategy" and partitioned the two wholes into halves and distributed the four half objects between four people, she incorrectly assumed that each person had received one part out of four equal parts. It appeared that she had converted the problem from one being represented by region analog models to one being represented by a set analog model consisting of four separate elements. Therefore, she incorrectly quantified each person's share as 1/4 (i.e., one part out of four equal parts) instead of either 1/2 or 2/4 of a pizza.

Cognitive Proceduralisations

The "proceduralised partitive quotient strategy" is an example of a strategy produced by the process of cognitive proceduralisation. This strategy is a truncated, proceduralized version of the "partitive quotient foundational strategy". The "partitive quotient foundational strategy" consists of the following six steps:

Step 1.	recognition of number of people *(n)*,
Step 2.	generation of fraction name from number of people *(nths)*,

Step 3. recognition of relationship between fraction name and number of equal pieces in each whole object *(n equal pieces)*,

Step 4. partitioning of each whole object into equal pieces *(n equal pieces)*,

Step 5. sharing the pieces *(1/n of each object to each person)*, and

Step 6. quantifying each share *(addition of 1/n pieces)*

In the "proceduralised partitive quotient strategy", the foundational strategy is truncated down to the following three steps:

Step 1. recognition of number of people *(n)*

Step 2. recognition of number of analog objects *(x)*

Step 3. quantification of each person's share as x/n

This proceduralisation reduces the load on working memory involved in the solution of a partitive quotient fraction problem. This proceduralisation is also indicative of progress towards the automatisation of the process of abstractly generating a numerical solution to partitive quotient fraction problems. For example, when Caitlin and Sally used this strategy, they no longer felt the need to actually physically partition and share the objects. When Mark and Joshua executed this proceduralised strategy, they also indicated that they were well on the way to automatising the process of abstractly generating a numerical solution to partitioning problems. They executed all three steps of the strategy and then utilised the concrete analogs to confirm the correctness of their solutions. For example, when sharing three pizzas between four people, Joshua and Mark did not cut each pizza into quarters but instead they cut 1/4 from each of three pizzas leaving 3/4 of each of the three pizzas intact. They then placed the three 1/4 pieces together to form the fourth share.

Cognitive proceduralisation truncations such as the "proceduralised partitive quotient strategy" did not cause the two types of problems endemic with the first type of truncated strategies. Instead, the cognitive proceduralisation truncations resulted in the development of more efficient, automatized strategies which could be successfully applied to wide varieties of partitive quotient fraction problems.

CONCLUSION

In this paper, we reported on two subsets of partitioning strategies utilised by a sample of twelve year three students when they were asked to solve sequences of "realistic" partitive quotient fraction problems: *composite strategies* and *truncated strategies*. These two types of partitioning strategies emerged as each of the children progressed through his or her sequence of partitioning problems and began to seek ways of making the process of partitioning more efficient in terms of generating equal shares, reducing the number of steps and reducing the load on working memory.

The composite strategies were very efficient in terms of generating equal shares, particularly in the context of specific partitioning problems. However, they could not be applied generally to all of the partitioning problems. Furthermore, their efficacy in terms of facilitating the abstraction of the partitive quotient fraction construct from the physical activity of partitioning objects was greatly influenced by the abstraction-ability of the component strategies. If the component strategies facilitated the generation of fair shares, the accurate quantification of shares and a conceptual mapping between the physical activity of partitioning and the partitive quotient fraction construct, then the composite strategy was efficacious in terms of facilitating the abstraction of the partitive quotient fraction construct.

Two types of truncated strategies were identified: *simplification of the physical process of partitioning*, and *cognitive proceduralisation*. The former type of truncated strategy, whilst simplifying the physical process of partitioning, more often than not complicated the cognitive processes of accurately quantifying each share and making a conceptual mapping between the concrete activity and the partitive quotient fraction construct. In contrast to this, the latter type of truncated strategies not only generated fair shares, reduced the number of steps, but also facilitated the process of abstracting the partitive quotient fraction construct. Furthermore, the cognitive proceduralisation strategies greatly reduced the load on working memory.

A number of implications for instruction can be drawn from these findings. First, is that the emergence of composite and truncated strategies indicate that children are actively constructing knowledge and are not merely going through the process of completing the partitioning problems. Second, is that children can follow a number of developmental paths as they seek ways of making the process of partitioning more efficient in terms of generating equal shares, reducing the number of steps and reducing the load on working memory. They can construct composite strategies, truncated strategies, and/or combinations of these types of strategies.

However, some of these types of strategies, even if they lead to the successful completion of sharing tasks, do not assist children to abstract the partitive quotient fraction construct from the physical activity of partitioning. Therefore, teachers need to be aware of the strengths and limitations of the composite and simplification of physical process truncated strategies and if possible, actively facilitate the construction of cognitive proceduralised truncated strategies. This can be done by: (i) the careful sequencing of partitive quotient fraction problems in the context of circular region models, rectangular region models, and length models to reinforce the conceptual mapping of the partitive quotient fraction construct from the concrete activities and (ii) the presentation of more complex (interrelated by number of people and number of objects) tasks to generalise the partitive quotient fraction construct from the concrete activities such that x objects shared among n people results in a share of $^x/_n$.

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JUMPING TO CONCLUSIONS: DATA INTERPRETATION BY YOUNG ADULTS

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The increased emphasis on the Chance and Data strand of the mathematics curriculum means students should be approaching adulthood better prepared to interpret data and recognize relationships between variables and between a sample and its corresponding population. This study examines the statistical appropriateness of conclusions drawn by young adults from data in a small sample. While many of the analyses took into account the data's limitations, there were still some erroneous assumptions and omissions.

INTRODUCTION AND BACKGROUND

Statistical study has acquired a more prominent place in the school curriculum in recent years. As a consequence, Australian students approach adulthood with at least some experience in data handling and interpretation, varying in complexity from unsophisticated descriptive statistics and data depiction to hypothesis testing. Thus the depth of statistical understanding exhibited by individuals can vary hugely. Shaughnessy (1992, p.485) categorizes the sophistication of statistical conceptions into four levels, of which the first three are relevant here. The lowest of these is *non-statistical*, in which responses are based on beliefs or causality, and there is no attention to or awareness of chance or random events. In the second, called *naïve-statistical*, the individual has some understanding of chance and random events, can use simplistic judgemental heuristics and can make decisions based on experiences which may or may not be relevant to the data at hand. Individuals at the *emergent-statistical* stage have some training in probability and statistics and can apply normative models in some statistical settings. Australian students at the end of compulsory schooling will exhibit abilities in one (or more) of these levels.

One of the key ideas of statistical understanding, which may receive only intuitive coverage as part of the Chance and Data strand of the mathematics curriculum, is that of the relationship between a sample and a population. Tversky and Kahneman (1971/1982) write that many people view samples as highly representative of the populations from which they are drawn, and that small samples will show no more variability than larger samples. Such views may be held even by those at the emergent-statistical stage.

In studies which have considered students' statistical skills students rarely have been called upon to make their *own* decisions about how to investigate relationships in data. In 1987 Curcio urged that students should be encouraged to verbalize their observations of the relationships and patterns in data, but to date very few researchers have examined the extent to which students can do this. One such study by Lehrer and Romberg (1996) described the work of a group of ten children collecting and analysing data in various ways. Other studies (e.g. Chick & Watson, 1998; Lidster, Chick & Watson, 1997; Watson, Collis, Callingham & Moritz, 1995; Watson & Callingham, 1997) have been conducted using the *data cards protocol*. These studies mainly considered upper primary students and examined their abilities to represent and interpret data. The data cards protocol used there was adapted for the current study and is described in the next section.

Furthermore, while many studies have considered adult students undertaking statistics courses, little work has been done with high school students or those just leaving school on the threshold of adulthood (Batanero, Godino, Vallecillos, Green, & Holmes, 1994). The purpose of this study, then, is to examine the extent to which young adults — with an above average background of high school Chance and Data — can recognize relationships in a set of data, and consider the legitimacy of their conclusions about the data.

METHOD

The 32 students (21 males and 11 females) involved in this study were enrolled in a first year university basic mathematics course intended as a service subject for students needing some mathematics as part of a science or arts degree. It had no statistical content. One of two pre-tertiary mathematics subjects was an acceptable pre-requisite for the university course; each of these subjects had a statistics component. In particular, one of the pre-requisite subjects had, as one of its aims, that students develop skills in collecting, organising, displaying and interpreting numerical data as well as calculating summary statistics from this data. The other subject, used as a pre-requisite for mainstream tertiary calculus courses, incorporated the development of certain specific statistical skills associated mainly with normally distributed data; more descriptive and intuitive data interpretation was a component of mathematics subjects studied prior to this.

The *data cards protocol* used for this study supplied students with details about a set of 16 fictitious Tasmanian juveniles; giving name, age, weight, weekly fast food consumption, favourite activity and eye colour information for each one. The Appendix gives the complete set of data. In previous studies the students were supplied the data on 16 separate cards (hence the name of the protocol); in the current study the information was supplied in a table like that in the Appendix. Students were then asked to investigate the data, in such a way as to allow students to choose their own areas of focus. This made the task exceptionally open-ended, with no discernible end-point indicating when students had achieved "the answer." In this study the protocol was set as an assignment question:

> "Examine the data and produce a report which highlights any aspects of the data which you think are interesting. You are free to choose your approach to this problem and what to include in your report. If you make any claims about the data be sure to back them up with suitable evidence."

In response to this question, students examined various aspects of the data, producing written reports which occasionally included graphs and summary statistics, and which varied in complexity from simple frequency information to investigating associations between variables and discussing causes and effects. For the purpose of this study, the focus is on the conclusions drawn, not on the type of representations used nor the particular relationships examined. Legitimate conclusions which can be drawn about the data include (a) a very strong correlation between age and weight; (b) boys appear to be marginally heavier than girls; (c) boys, and older boys in particular, tend to consume more fast food meals per week; and (d) there is a strong correlation between fast food consumption and the activity or passivity of favourite activity, with high consumption associated with more passive activities, such as TV, reading, and board games. It should be noted that since the data was supplied to students the tasks associated with the protocol involve descriptive statistics rather than hypothesis testing.

It must also be said that the protocol's deliberate avoidance of specific directions means that we have to be cautious about the conclusions that this study might attempt to draw. For example, the students were *not* asked to discuss whether their observations of the sample might apply to a broader population. Consequently, a student's failure to address this does not mean that the concept of the relationship between a sample and the population from which it is drawn is not understood. If, however, we are interested in developing statistical skills in students to the extent that they can make valid judgements about data in various contexts, it is to be hoped that they will realise for themselves that this is an issue that needs to be considered, and will consider it. The extent to which students spontaneously do this, then, is one focus of this study.

RESULTS AND DISCUSSION

In looking at the nature of students' conclusions this study will examine the extent to which students were able to recognize that (i) the small sample size of the supplied data limits the applicability of conclusions to a broader population; (ii) the small sample size of the supplied data may invalidate conclusions drawn about the data itself due to the higher variability of data in small samples; and (iii) relationships between two variables may be influenced by a third or fourth variable, that is, correlation does not imply that one variable is the sole effect on the other. Table 1 indicates what kind of conclusions and clarifications the students made, based on the data in the protocol. Some students' responses were classified as being in more than one category.

Table 1
The nature of students' conclusions on analysing the data cards problem

	Conclusion type	% (N=32)
a.	Over-generalized or made a claim not reasonably supported by the data	34
b.	Specifically recognized that conclusions only apply to the given data	47*
c.	Observed that sample is too small to apply to the real world	31*
d.	Observed that unbiased selection of the sample is required	19*
e.	Some use of external reference or knowledge outside the data itself, either cited or asserted as fact	56
f.	Gave a qualified hypothesis to explain an observed phenomenon (e.g. "this *could* be because of ...")	66
g.	Did *both* f and a (i.e. some hypotheses qualified, other claims were too strong)	25
h.	Mainly listed some summative statistics and observations from the data only	38
i.	Did h but *not* b (i.e. focussed only on data but made no reference to data's limitations)	25
j.	Made an unwarranted assumption about the data (as opposed to an unwarranted conclusion)	9
k.	Made and tested an hypothesis (and may have accepted refutation of the hypothesis)	6
l.	Overlooked an important third component where a two variable correlation exists	41
m.	Recognized a third component influencing a seeming correlation	31
n.	Did *both* l and m	9

* Note that the set of students doing d is a subset of those doing c, which is a subset of those doing b, except for one student in d and b but not c.

The Sample and its Relationship to the Population

As can be seen from Table 1, 47% of the students made a comment to the effect that their conclusions only apply to the given data; of these, 66% went further and stated that the sample was too small to apply to a larger population. For example, in acknowledging that her observations apply to the data only, one student wrote "This shows that *of the students surveyed*, males had the higher number of fast food meals consumed per week [emphasis added]." [In these quotations of student work, spelling and more obvious punctuation errors have been corrected.]

Another student was more specific about the fact that the sample size is too small for conclusions to be drawn. She also expressed the need for a representative sample.

> A sample group of 16 may be a little small for an accurate conclusion to be made on Tasmanian school children; the larger the sample the more accurate a picture will be, and also a variety of children from the North, South, east and western schools including public and private schools would draw more accurate conclusions.

One student did not acknowledge that sample size was an issue, but was also concerned about how representative a sample it was, noting "that the table of results may have invalid data, i.e is the data a true representation of the cross-section of Tasmanian school children?" This point was raised by 19% of the students, with all but this particular student recognizing the issue of sample size as well.

In contrast to these expressions of caution, there were some unjustifiable conclusions and generalizations among the students' responses. In fact, over one-third of the students made some sort of statistically inappropriate interpretation of the data. In some instances students' difficulties were a consequence of trying to reconcile the data supplied with their own knowledge and expectations. One student, for example, wrote that "with 50% of the female population playing netball it is easy to understand how it has become one of Australia's largest followed sports." Interestingly, he was only talking about the younger students at this time. Another noted that males eat more fast foods per week and then concluded "Clearly by this information, it is possible to make the assumption that females take more care in their appearance, and their health." A third student, on observing that swimming was the favourite activity of only one person in the sample, claimed "Swimming is becoming an obsolete favourite activity."

One student drew on some external references to make some sweeping claims, based on her observation that no boys chose reading as a favourite activity. There is an implicit assumption on the part of the student that the data is representative of a larger population, that the small sample really does illustrate the claimed relationship, and that the observations really are a consequence of the claimed causes.

> These findings clearly show the relationship between gender and literacy skills [web site cited]. It can be shown that males are much slower at picking up reading and writing skills than females, especially at primary school level [another citation] ... Thus lower literacy levels in males, and their reliance on television for information, makes them more susceptible to advertising, especially food advertising, and because they are not as good at analysing the information they will be at a higher risk of not understanding the concept of a "well-balanced diet" and the importance of eating healthy food regularly ... Therefore literacy in children affects not just information gathering and analysis, but health and physical shape.

In contrast, one student had a particularly good understanding of the limitations of the data. The final two sentences in the quote below are her report's concluding remarks.

> In such a small sample it is hard to define what is correlated and what is coincidence. For instance, all [those] who enjoy television are males and as it happens, three out of those four children have blue eyes, yet it is likely that with a larger sample size, enjoying television wouldn't be dominated by blue-eyed males (especially the blue-eyed part) ... This type of data possibly demonstrates if you want to find relationships, you probably will, but with little evidence to support it. Without substantial evidence you cannot draw conclusions.

Students' Conclusions About the Sample Itself

Students had no difficulty making observations about various aspects of the data itself. As will be seen later in this report, many attempted to give explanations for the trends that they noticed. Very few students, however, were aware that some of their speculated conclusions could be due to random fluctuations. One student recognized that "The number of students is way too small to make general statements or assumptions from as there could exist ... coincidences within the data or the sample could be unrepresentative of the whole population."

While making general statements about the data, one student did not make it clear that in stating that "the majority of board game players are female," there are only three children in the set of board game players. This, of course, is the minimum-sized set required to have a non-unanimous majority! While his statement is true as a factual observation of the data, there is no indication that he is aware that (a) his statement is very strong in the light of so little evidence, and (b) it only takes a change of mind by one person in the sample to alter his conclusion.

One student made what could be regarded as a valid observation about the data, but backed it up with the example of two individuals rather than attempting a more legitimate statistical analysis. He wrote that "fast food consumers tend to be heavier than lower fast food consumers over the age range for both males and females. i.e. Mary Minski is 13 years old and weighs 55 kg whereas Dorothy Myers is 15 years old and weighs 50 kg." (It is conceded that the student may have confused "i.e." with "e.g." but more substantial evidence should still have been supplied.)

Three students made unwarranted assumptions about the actual data itself. Two asserted that two of the boys in the data were brothers (it *is* true that there is some evidence in favour of this possibility, and an additional student made this distinction), while another student claimed that a variety of children had been selected, with "ethnic background" a factor in that variety, based on the occurrence of some non-Anglo-Saxon surnames.

As a final example of students' difficulties with the sample itself, one student attempted to assign children of particular ages to particular school grades, allowing only one age per grade grouping (thus precluding the possibility that, for example, a grade 6 class could have both 11 and 12 year olds in it). He tabulated his results and then said that the "table seems to suggest there is some data missing. It seems logical that if in some grades one boy and girl were interviewed then there should be a girl and a boy from all grades." His identification here could, in fact, be an implicit expectation that the data be "representative" in some sense.

Influences on Relationships Between Variables

Qualified Hypotheses

Of the 12 students (38%) whose main approach in writing their reports was to list some summative observations from the data, only four were among the 21 students (66%) who proposed *possible* reasons for their observations. The remaining eight of the 12 made no attempt to give reasons for the results at all, just noting their observations without comment. The three students who neither hypothesised tentative reasons nor merely listed factual observations were among those making unjustified sweeping claims. The existence of a cautiously proposed hypothesis indicates that students *are* seeing the data in the broader context of a population, but that they appreciate that there is insufficient evidence to be sure of the claim. As will be discussed later, many students used "outside" knowledge or experience as a source of explanation for identified trends.

After observing that males ate more fast food than females and yet the average weights differed by only 4.4kg, one student wrote "This could mainly be accounted for by the fact females mature faster than males;" his use of the word "could" permits the possibility of an alternative explanation or recognizes the need for appropriate evidence. Another used three qualified statements in attempting to explain the connection between fast food consumption and television watching. It should be noted that very few students, and not this one, recognized that "favourite activity" did not mean "most frequent activity."

> One of the ... assumptions [sic: observations] that can be made from the data is that people who enjoy watching TV, also eat at least 7, and up to 12, fast food meals a week. This could be because of television advertising fast food, such as McDonalds ... The people who watch TV the most, would most likely have no sports commitments and so would not have strict diets to cling by. This could be one reason for being the heaviest in their age group.

After making the connection between the same variables, one student wrote that "it could possibly be stated that television causes children to eat an excess of fast food." Here the conditional "could" again indicates statistical caution, but no further explanation is offered.

Awareness of Factors Which Influence Relationships Between Two Variables

Many students were able to identify some two-variable correlations in the data. In many cases, however, these correlations were influenced by a third variable; this is best exemplified by the possible interconnections between age, weight, fast food consumption and the passivity of the favourite activity. In identifying a particular two-variable correlation, less than one-third of the students were able to recognize the existence of a third variable which would almost certainly influence the relationship, while 41% of students failed to identify such a factor despite noting the two-variable correlation. There were three students who did both: identifying a third variable on some occasions and failing to do so on others, sometimes even in adjacent paragraphs.

One student, who *was* able to recognize a third (or even fourth) correlating variable, observed that higher fast food consumption indicated greater weight, then noted "This could also be attributed to the gender difference (males tend to be heavier than females) and the age difference (the older people get the heavier they get (as they grow up))." Two students considered three variables successfully by considering the age to weight ratio, and then compared that with fast food consumption (in one case) and passivity of favourite activity (in the other).

In contrast, several students failed to take age into account when comparing weight with other factors. One student simply wrote "Those that play sport have a lower weight than those who prefer other activities such as TV or board games." Another actually calculated the average weights of the "active" and "inactive" children, before writing "It can be compared that the average weights of children that are more likely to have an outdoor activity are less than the rest, 34.3 kg [compared with] 52.6 kg," again failing to take age into account (the older, and thus heavier, children prefer the passive activities). Others made similar connections between weight and favourite activity while omitting the age link, yet after making this observation one student's very next paragraph stated that "it becomes apparent that age is also a major factor. It [is] noted that all students of age 13 and over (adolescents) have a weight that is above average, regardless of the amount of fast food meals eaten per week." This student, like one or two others, failed to make any further connection between these observations.

As a final example, one student identifies a possible third factor not included in the data in order to explain a two-variable relationship but overlooks a possible third variable (fast food consumption) which *is* present in the data.

> If we graph the weight to age then as the age rises so does weight as expected, but, in the 12 to 15 age group there seems to be a significant rise above the normal trend. This could be because of puberty and the accelerated growth rate associated with it.

The Use of External Knowledge and Assumptions

As indicated earlier and in some of the quotes already seen, some students interpreted the data in light of knowledge and assumptions based on their experience or other research. In fact, over half the students (56%) made some use of external information, either explicitly citing it as such or incorporating it without comment. The correlation between age and weight, for example, was regarded as unsurprising common knowledge (when it was considered at all); some students also regarded the connection between weight and fast food consumption in a similar light. One student's response illustrates both aspects:

> To think about it logically, the more fast food consumed, the heavier the child is going to weigh. But also the child's age can come into the equation. The older the child the heavier he is more likely to weigh, so it is going to be hard to determine that weight is a major factor involved.

At least three students indicated that eye colour should be irrelevant, although two of them observed that a possible correlation existed. One of these students was also one of two who actually hypothesised relationships in advance before testing them with the data. He wrote:

> Now we can look to see if eye colour has any relation to the amount of fast food consumed per week. My prediction is that it won't have any effect on the result, simply because it is a physical feature of each student. But, in fact, it seems as though it does play a major part in determining the number of meals consumed. [Shows table supporting his claim.] Well, there you have it. Blue eyes seem as though they play a major part, but as pointed out before they are only a physical feature, and it doesn't make sense that it would be possible to make a direct effect on the outcome. In my opinion it is only a coincidence that it appears to show to have any direct effect.

CONCLUSION

The statistical sophistication of the students in this study seems to range from non-statistical to emergent-statistical. There are examples of unsupported conclusions and tendencies to over-generalize, contrasted with examples of deeper awareness of statistically appropriate analysis. Most of the analysis carried out by the students was fairly simplistic, but this is hardly surprising given the size of the sample and the vagueness of the protocol. As mentioned earlier, the deliberate non-specificity of the task has implications for what might be expected from students in their responses. There was no specific request for them to make generalizations from the data nor to discuss implications nor to propose explanations for observed aspects of the data. Consequently it should not be surprising that some students did not address these issues nor that they took a fairly simplistic approach to interpreting the data. Nevertheless, if one of the objectives of the mathematics curriculum is to improve statistical literacy, then the results of this study suggest that students should be given more opportunities to deal with drawing valid conclusions from raw data, to identify relationships between variables, and to address the extent to which claims about a population can be made based on a sample.

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APPENDIX — DATA SUPPLIED TO STUDENTS

Table 2

Table of data supplied to students for interpretation

Name	Gender	Age	Favourite Activity	Eye Colour	Weight (kg)	Fast food meals consumed per week
Adam Henderson	M	12	Football	Blue	45	5
Andrew Williams	M	14	TV	Blue	60	10
Anna Smith	F	11	Boardgames	Brown	32	1
Brian Wong	M	9	Football	Green	26	1
David Jones	M	8	TV	Blue	30	7
Dorothy Myers	F	15	Swimming	Blue	50	2
Janelle MacDonald	F	18	Reading	Blue	66	4
Jennifer Rado	F	9	Boardgames	Green	33	4
John Smith	M	10	Football	Green	29	0
Kathy Roberts	F	12	Netball	Brown	32	0
Mary Minski	F	13	Reading	Green	55	3
Peter Cooper	M	16	Boardgames	Green	54	2
Rosemary Black	F	8	Netball	Brown	24	0
Sally Moore	F	17	Reading	Brown	56	1
Scott Williams	M	17	TV	Blue	66	8
Simon Khan	M	18	TV	Brown	74	12

CONCEPTUAL MODELLING OF FUNCTIONS BY AN EXPERIENCED TEACHER

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An important element in teaching is the quality of content knowledge that teachers use in the design and delivery of their lessons. In this study, we present a framework for investigating how this knowledge is structured. The framework is then used in the analysis of an experienced teacher's knowledge of functions and the teaching of functions. The data show that our teacher has built up knowledge that is dominated by conceptual rather than procedural aspects of functions.

BACKGROUND

Much of the recent research on the learning of mathematics has focussed on students' ability to use previously-acquired knowledge in making progress with the solution of novel problems. An important development in this area has been appreciating that the quality of the knowledge that students acquire may have a significant influence on how well that knowledge is used in the search for solutions to problems. In a classroom setting, teachers play an active role in facilitating not only the acquisition of new knowledge by their students, but also in providing pedagogically valuable experiences that may assist in extending that knowledge into new territories.

An important element in the goals that teachers set for their lessons and the structuring of these lessons is their own understanding of both the subject-matter and their students. Thus, the nature of a teachers' knowledge base underlying a particular mathematical topic and the teaching of that topic can be expected to exert a major influence on the quality of the understanding that students develop about that area of mathematics. While this point about the role of teachers' knowledge base has received considerable support in research findings (Ball and McDiarmid, 1990) and curriculum reform documents (National Council of Teachers of Mathematics, 1989) there is little information about the interaction between teachers' subject-matter knowledge and what students learn. In the present study, we begin a process of addressing this issue by examining an experienced teacher's knowledge about algebra and how that knowledge is used in modelling functions.

Schemas as Structured Mathematical Knowledge

Network theorists have advanced several theoretical frameworks in which to investigate concepts and their development. According to one view, conceptual growth and mathematical understanding can be interpreted in terms of conceptual nodes and relations between nodes (Anderson, 1995). As students' experiences with a concept or a set of concepts increase, they come to form organised meaningful wholes. Various attempts have been made to elucidate such cognitive structures. Among these, the notion of *schemas* has gained considerable support amongst researchers.

In the context of learning, schemas have been given a number of interpretations in the psychological literature. Skemp (1985) describes how we construct 'what we already know' by engaging in mental construction of reality by building and testing a schematic knowledge structure, where a schema is "a conceptual structure existing in its own right, independently of action" (Skemp, 1979:219). In the context of problem solving, Paas (1992:429) describes how a schema "can be conceptualised as a cognitive structure that enables problem solvers to recognize problems as belonging to a particular category of problems that require particular operations to reach a solution", while Sweller (1992:47) defines a schema as "a cognitive construct that permits problem solvers to categorise problems according to the moves required to solve them." Because our existing schemas serve either to promote or restrict the association of new concepts, the quality of what an individual already knows is a key determinant of our ability to understand, or as Skemp (1979:113) concludes "our conceptual structures are a major factor of our progress".

What then do schemas comprise? Dubinsky and others (Dubinsky, 1991, Cottrill *et al.*, 1996) use the acronym APOS to describe the four components of Action, Process, Object, Schema in the building of mathematical knowledge. The chain of events, they suggest, develops as follows. Actions, when applied to objects become processes, which in turn become *encapsulated* as mental objects. In turn examples of these three link together to form cognitive structures or schemas. Thus, conceptual entities in mathematics often present themselves with two distinct but complementary faces; they may be viewed as dynamic processes or as static objects. To make a mathematical idea readily manipulable and applicable in other contexts, it must be available internally in a concise form and the encapsulation of the process as an object is one way of accomplishing this. The relations that are constructed between the conceptual objects forming a schema could represent, for example: similarities and dissimilarities between concepts; instances of a concept; procedures for using concepts for solving problems; or affective factors related to those concepts.

According to Anderson (1995), two variables determine the quality of a schema: the spread of the network and the strength of the links between the various components of information located within the network. A complex schema can be characterised as having a large network of ideas that are built around one or more core concepts. Further, the links between the various components in the network are robust, a feature which contributes to the accessing and use of the schema in problem-solving and other situations. A well structured schema can also benefit students by helping them assimilate incoming new mathematical ideas because such a schema can be expected to have many conceptual points to link with. As a theoretical construct schemas provide a useful way to interpret the growth of mathematical knowledge.

Teacher Knowledge and Schema Induction

Applying the above ideas about the nature of mathematical schemas and their formation to teaching and learning in the classroom it becomes apparent that the quality of the mental schemas of the teacher may be a key component in a) what is learnt and b) how it is learnt by their students. When we examine mental schemas of teachers in any given content area we become aware of what the important links which we want students to build into their knowledge structures are, and we could structure the learning environment towards these.

Leinhardt (1989) have suggested the existence of strong links between teachers' subject-matter knowledge, their explanations and the type of representations generated by students. A schema-based analysis, therefore, suggests that teacher actions could promote the construction of powerful schemas that would benefit student learning in two important ways. Firstly, students would better access prior knowledge and integrate that with incoming information. Secondly, students could be expected to deploy acquired knowledge flexibly during the process of problem analysis. Hence the question is, what is the nature of a teacher's knowledge that would promote the construction of type of schema that are useful?

The aim of this research study was to examine the above issue by characterising the schemas of experienced teachers in the content area of function. This is part of a larger project in which we will compare and contrast schemas of the experienced teachers with those of teachers who are new to the teaching of algebra and functions.

One major difference we have hypothesised has to do with encapsulation of processes as objects. Many individuals appear not to progress to the point where they can think in a dual *proceptual* (Gray & Tall, 1994) or *versatile* way (Tall & Thomas, 1991; Hong & Thomas, 1998), about mathematical symbols, seeing them either as a process invoked by the symbol or as the concept represented by it. Instead, they are *process-oriented* (Thomas, 1994) in their thinking, constrained primarily to mathematical processes. For a teacher the absence of the object view in their schemas may structure their thinking, causing them to over-stress procedural methods. In contrast the *versatile* teacher, with a global view of a concept, is able to see its components, or constituent processes, and relate these to the whole, rather than seeing only the part in the context of limited, procedural understanding.

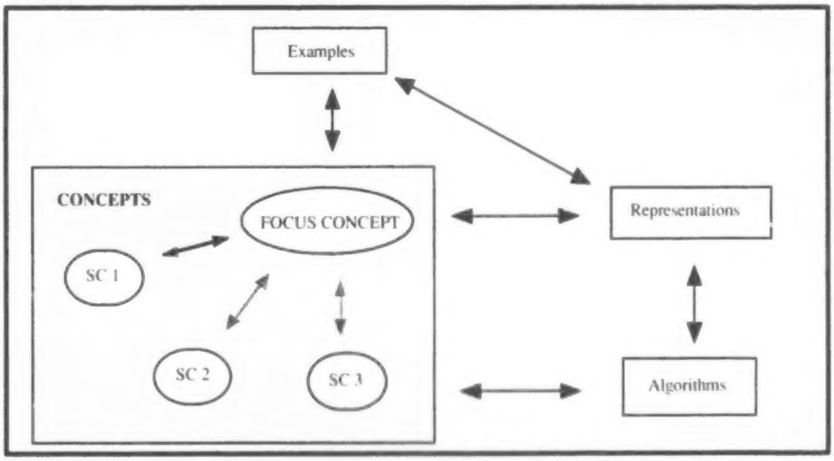
Considering this in the context of function, how would the schemas and teaching approach of a teacher with a conceptual view of function diverge from those who have a primarily procedural base? An example could be the construction of a composite function fg from the functions f and g where

$$f(x)=x^2 \text{ and } g(x)=x+1, \text{ so that } fg(x)=(x+1)^2.$$

One might be able to carry out such a procedure without actually having a concept of what a composite function actually is. Our view is that it would be difficult for a procedurally-oriented teacher to engage students from a conceptual perspective. What does this kind of teaching involve?

We present in Figure 1 a theoretical view of a modelling approach to teaching. According to this model, the emphasis is on a progression from the *formal aspects* to the *algorithmic components* associated with the focus concept (Fischbein, 1994). It should be noted that this process is usually cyclical. This model may be contrasted with a procedural perspective which starts with a symbolic representation for a function, seen as representing a procedure and either operates on it (as above) or uses it as an algorithm in the form of value in and value out.

Figure 1
A macro model of the structure of teachers' mathematical knowledge



Note: SC 1 stands for subsidiary concept one etc.

METHOD AND RESULTS

This research employs a case study methodology, examining the conceptual structures of individual teachers and their influence on their teaching. The research is very much still in progress and this paper describes the results in relation to one participant experienced teacher.

Margot[1], an Experienced Teacher

A number of criteria were set up for defining whether a teacher could be categorised as experienced, particularly with respect to the use of technology in their teaching, which is to be a future focus of the study. Margot, a secondary school, teacher in Auckland, New Zealand, fulfilled each of these, having 31 years teaching experience, including 15 years using technology in mathematics teaching. She has been active in promoting the use of calculators and computers to other teachers in her school, has attended professional development courses including study for a Diploma in Mathematics Education, where she took a paper on Technology in Mathematics Education. In addition, she has run advisory courses on using graphic calculators and was recently seconded to a one-year appointment as a mathematics adviser.

Procedure

Margot took part in a non–structured, free recall interview in January 1999, where she was asked to talk freely about functions and polynomials. The interview was recorded on audiotape and afterwards transcribed for analysis. Later we were able to go into Margot's school and observe and videotape two lessons on function. This data is still being analysed and the discussion presented here is based solely on the free recall interview.

Margot's Schematic Structure

Margot had a conceptual view of function underpinning her teaching. She saw function very much as a relationship between two variables and commented that, when teaching about functions "you tend to concentrate on is the relationship between two variables the fact that there is one variable affecting the outcome of another variable. . . So you're encouraging relationships between variables." Further this relationship was about change. She commented that "You could just say that there is a connection between these two variables, which one is causing the change, and what is the result of the change?" and "So what we tend to do is do this practical type work first where they're getting the idea of this variable changing and this one resulting." This relationship for her was strongly based on the idea of a one-to-one mapping between the values of the two variables. Her comments were often expressed in terms of practical examples, which she clearly saw as very important for her own understanding and her teaching, and as modelling the function concept. For example she talks about beakers of water, kangaroo jumps, pendulums and costs etc.

Margot emphasises throughout the one–to–one nature of a function, "one in, one out" clearly excluding a one–to–many relationship (this emphasis should not necessarily be taken to mean that she excludes a many–to–one function, since this was not specifically mentioned). Furthermore, the distinction between discrete and continuous variables is something she is very conscious of.

> One of the things that I find that causes confusion is the distinction between discrete and continuous. You take them away and you do, you know a quadratic patterning, your kangaroos jumping or whatever and that's a discrete pattern. And then all of a sudden you produce a parabola which is continuous, and I don't think myself at the moment that I'm yet very good at making the distinction there for them between the two, and I think a lot of them lose that. . . Yes, yes, because the kangaroos they can see, where the kangaroo, the jumping one, they can see that this is a discrete, you know, you take ten jumps, you take five jumps you take four jumps, whereas with this they will be able to see that with that fact that what you're graphing is a continuous, a continuous movement.

What subsidiary concepts did Margot have embedded in her overall conceptual view of function? Clearly from her comments variable is the primary subsidiary concept, but in any modelling episode there are others which emerge, as the example below demonstrates.

Modelling - An Illustrative Example

We can illustrate Margot's approach to teaching via modelling, and the subsidiary concepts supporting it, by detailed reference to one example from her interview. She describes at some length one way to approach the teaching of rates of change with sixth form students (age 17 years), in this way:

> What we then try to do again is to make the work as practical as possible, and last year what we did was we took the coil of rope, we took a coil of rope and we, I went out to [store name] and brought all this bits of string and they mark off. So this is bits of rope being round onto a coil and they mark off with pen, and they get a table for the number of the coil and the length of the string.
>
> So what they're actually doing, is they're modelling rope being wound by machine or onto a spool or whatever. Worked beautifully, it was perfect, and then we gave them questions that, we asked them to graph it, so they were graphing, and then we asked them to estimate the rate at which the rope was going on between two integer values, so that they could work from the table.
>
> We then asked them to work out the rate at which it was going on two interpolated values so they actually had to work from that curve, and then we asked them to work out the instantaneous rate at which it was going on, so that they had to have the idea of a tangent. But again, this is all functions because again we're looking at one variable resulting in a change and another variable and the resulting graph and how you interpret it, and when we modelled that on the graphics calculator, it was just beautiful.

The emphasis on modelling is clear here. Not just in Margot's use of the word which indicates that she believes she is encouraging modelling, but in terms of the whole approach. Her aim is to take a real world 'practical' situation and represent it mathematically. In terms of our theoretical model, the focus concept of function is supported by, and related to, subsidiary concepts, each of which has a number of different representations. The underlying subsidiary concepts which she specifically mentions or alludes to for function are: average and instantaneous rate of change; interpolation; chord, tangent; gradient; and variable. The major representations employed are symbolic, tabular and graphical. Margot often sees the value of technology in enabling manipulation of the mathematical concepts both within and between these different representations (Thompson, 1992), but whether the technology is used or not, the transition between representations, preserving the conceptual structure of the mathematics is a crucial one in her schemas. In this example, the variables are first symbolised (one representation) with the function being a relationship between the independent and dependent variables, and then there is a move between representations as the symbols enable access to a tabular representation. Working within this tabular representation the average rate of change is calculated from the two sets of values present. As a next step, Margot uses two values which have not been directly measured to stimulate the use of a graphical representation, with a co-ordinate or ordered pair representation of the data as the link between table and graph. Once in the graphical mode an algorithm to find the gradient of a chord from the use of interpolated values is employed (we note that although this can be done by linear interpolation from the table this was not mentioned here). Finally, also working within the graphical representation, the concept of instantaneous rate of change requiring the graphing of a tangent and an algorithm to find its gradient, was introduced. This example does not involve algebraic symbolisation other than of the variables, since the situation has been adequately represented mathematically without recourse to this (although, of course, one could have gone on to model a polynomial function for the data). The final step in the modelling process involves working within a representation to carry out algorithmic processes, in this case to calculate gradients or rates of change, namely between two points and at a point.

This example, which as stated, is lacking a symbolic, algebraic representation of function is still completely about function for Margot, since as she says "this is all functions because again we're looking at one variable resulting in a change and another variable". This fundamental conceptual mental construct, the linking of an independent and a dependent variable, runs through all her ideas on function.

When we isolate some of the key concepts which Margot is building into the modelling she is doing in the classroom what kind of rich relationships do we see? Figure 2 presents an attempt to represent a macro view of this modelling example and its relationship to our theoretical model. We are currently investigating Margot's schematic thinking by looking at the micro level too, taking some of the focus concepts, such as those in this example, and drawing simplified schemas for them based on Margot's comments in the free recall task. The quality of some of the links in Margot's schemas may be observed in the comments she makes. For example, Margot is definitely not limited to thinking in one representation.

Figure 2

A Macro View of Margot's Modelling Example and its Relationship with the Theoretical Model

Concepts	Physical Examples	Representations	Algorithms
Function	Wind string on spool	Symbolise variables	Calculate gradient of a chord
Variable	Mark length	Construct table of values	Calculate gradient of a tangent
Rate of change (average and instant)	Count number of coils	Convert table to ordered pairs	
Chord, Tangent		Plot Graph	
Gradient			

When she talks she moves seamlessly between them. One significant episode showing the linking between concepts and representations is the way that Margot connects the symbolic form of variable and function, with parameters and the inverse function.

> Well, where you have two variables where x and y are related to one another through a third variable often denoted by the letter t. So for example if you were to graph $x=t$, $y=t^2$ you would actually be graphing $y=x^2$. To get your inverse, all you then have to do is make $x=t^2$, $y=t$, and you get that mirror image. So that works really nicely.

Notice here also her use of the terms "graphing" and "mirror image" referring to a graphical representation, in the context of a manipulation within the symbolic representation whereby she combines $x=t$ and $y=t^2$ to get $y=x^2$. This demonstrates that her schemas are flexible enough to allow her to think and work between representations and that she has schematic links between these and variable, function, parametric form, and inverse function.

DISCUSSION

In this paper we have introduced a model for studying teachers' conceptual knowledge in a given domain in relation to mathematical modelling. In this model we have considered the role of schematic knowledge and how it is important to have high quality, strong connections between the concept under focus and both subsidiary concepts and their representations. It is our contention that it is the richness and robustness of the structure of the teachers' schemas which are a primary influence on whether their teaching of the focus concept is procedural or conceptual. The data presented here suggest that experienced teachers may have more links to subsidiary concepts, and place greater emphasis on these and we hypothesise that these links enable them to move between representations more easily, keeping the focus concept intact, while gaining the advantages that each has to offer. In contrast, an emphasis on procedural aspects of the focus concept would make it more difficult to move across representations and hence tend to anchor one in one representation, for example, the symbolic one. This difference may be illustrated with regard to function by considering two approaches to teaching a graphical solution to the problem: "Find graphically where $y=x^2-3$ is zero"

A teacher, such as Margot, who has a rich schema for function with many subsidiary concepts, will keep alive the concept of a one–to–one (or one–to–many) relationship between variables as the representations are traversed. The table of values, the set of co-ordinates and the graph are each simply viewed as another representation of the symbolically presented functional relationship which assigns a value x^2-3 to a value x, in a one–to–one manner. The question (which could be answered in any representation) then becomes 'Which value of x produces the value 0?' In contrast, a teacher who has less rich schemas may concentrate on a procedural approach which lacks this underlying linkage. A sequence of conceptually unconnected procedures, in or between representations, is the result. Thus, students may calculate values of y given certain values of x. They may transfer these to a graph by a matching procedure which aligns the first number to the x–axis and the second to the y–axis, giving a sequence of points, and then join these up. The final procedure, in the graphical mode, sees them read the value where the curve crosses the x–axis. While the solutions may be the same, the conceptual knowledge built is quite different. The results of our study indicate a potential relationship between teachers' schema and the quality of schemas constructed by students. Chinnappan (1998) showed that, even when students had built up a reasonable number of schemas in the domain of geometry, the quality of their problem search was not flexible enough to construct alternative representations of the given problem. It would seem that these students' schemas had more procedural than conceptual information. We suggest that teachers need to draw more on conceptually-dominated schemas of the type revealed by our experienced teacher in order to promote a more flexible approach to mathematical learning and problem solving by students.

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Endnote

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THE USE OF CONCEPT MAPPING PROCEDURE TO CHARACTERISE TEACHERS' MATHEMATICAL CONTENT KNOWLEDGE

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This paper describes the use of a method for analysing concept map data to generate information on quantitative and qualitative features of a mathematics teacher's knowledge base in the area of geometry. The system of analysis used in this study provides a useful way to represent the state and connectedness of teachers' knowledge in the area of geometry and trigonometry. Use of this system indicates that this experienced teacher has a rich store of knowledge which is complete and accurate. His knowledge base shows a high degree of branching with depth of knowledge within each branch.

INTRODUCTION

Many recent discussions of the role of the mathematics teacher emphasise the importance of teachers helping students to develop effective knowledge structures, representations of mathematical content that will allow the student to productively explore a suitable range of mathematical problems. The sense of this perspective is clearly articulated in the recommendations such as that of the National Council of Teachers of Mathematics (1989, p. 128) that there is a need for teachers to 'shift from dispensing information to facilitating learning'. Knapp (1997) views the adoption by teachers of this shift in conceptualisation of teaching as one of the central planks of the broad movement to generate systemic reform in mathematics and science teaching. This change in emphasis for teachers' actions recognises the crucial role that teaching plays in influencing what knowledge is constructed by students. Even though the ultimate control of the knowledge construction process rests with the student, it is clear that the actions of the teacher in a lesson have substantial impact on the outcome of that process.

In the wider project in which this study is embedded we are interested in drawing links between the knowledge and actions of mathematics teachers and the mathematical knowledge and actions of their students. In undertaking this research one of our major assumptions is that the use of connecting, or linking, actions by the teacher is affected to a substantial extent by the state of connectedness of the teacher's own knowledge. Thus one of our first concerns has been to attempt to develop a procedure for representing that state of connectedness. The purpose of this paper is to describe the procedures we have developed to represent the quality of connectedness of teachers' mathematical knowledge. We will illustrate the procedures by focussing on one teacher's knowledge of a) the schema of square and b) relations between a square and a number of other geometrical schemas.

Knowledge Connectedness

An important consideration in the study of mathematical knowledge is the degree of connectedness that exists among the various pieces of that knowledge. Labels for the structured representations vary between research programs but it is common for the "packets of integrated information" (Hunt, 1993, p. 530) to be referred to as schemas, or schemata. When viewed at a larger grain size, a network of schemas can be thought of as *mental spaces* , structured "conceptual packets constructed as we speak, for purposes of local understanding and action"(Fauconnier & Turner, 1998, p. 137) The quality of connections among knowledge components is assumed to influence the ease with which the presence of one element aids in the retrieval and use of another in a problem environment.

Mayer (1975) made use of the notion of connectedness in his description of the accumulation of new information in long term memory as adding new 'nodes' to memory and connecting the new nodes with components of the existing network. 'Internal connectedness' was referred to the degree to which new nodes of information were connected with one another to form a single well-defined structure. The degree to which new nodes of information were connected with information already existing in the learner's cognitive structure, Mayer referred to as 'external connectedness.'

In a further analysis of mathematical knowledge structure, Greeno (1978) noted that within a schema the learner could also be expected to establish connections of how the schema relates to specific problems and situations. This issue was examined further by Chinnappan (1998a, 1998b) in which he found evidence to support the view that the linking of the different pieces of knowledge of geometry and trigonometry into schemas could play an influential role in the construction of appropriate representations for problems. Use of the term connection by both Mayer and Greeno also involved an attempt to establish a link between knowledge of a schema and knowledge of how to use that schema, thus being concerned with the proceduralization of declarative knowledge.

We have used the notions of internal (within-schema) and external (between-schema) connectedness as the core of a conceptual framework for representing the qualitative characteristics of the teachers' knowledge. We see this as useful for two main reasons. First, it allows us to represent the complexity of a knowledge base in a way that focuses on the state of organisation of that knowledge. The connectedness framework also provides a basis for establishing relationships between knowledge states and teaching actions. In relation to this second purpose we suggest that one of the key motivations of the teachers, and one of the most important sets of their actions, can be seen as attempting to facilitate the establishment of both internal and external connections in the knowledge bases of the students. In this way we argue that it makes sense to seek out such relationships in the teachers' knowledge base, for it seems reasonable to expect that it is from these connections that teachers will generate actions that are designed to improve the state of connectedness of the students' knowledge base.

Representation of Structural Knowledge by Concept Maps

As the study of connectedness involves the examination and specification of schemas and relations that teachers have constructed over a period of time, we need tools that would help us represent the organisational features of teachers' knowledge base. A number of different mapping procedures have been used for this purpose including concept maps. Concept maps, in general, are graphs consisting of nodes and labelled lines. The nodes are used to indicate concepts while the lines correspond to a relation between pairs of concepts. The label on the line tells how the two concepts are related by the individual. Shavelson, Lang and Lewin (1993) referred to this relation as a proposition and argued that concept maps represent some important aspect of learner's propositional knowledge in a domain. In this sense, a concept map could be used to probe teachers' understanding of not only relationships among concepts but also information about how they perceive and use those concepts to explain other related concepts.

Despite their popularity in teaching and research, attempts to use concept maps to represent the quality of knowledge organisation in quantitative terms have, so far, not been very successful (Lawson, 1994). It seems, therefore, that the potential of a concept mapping procedure for representing knowledge organisation can be further exploited and in this paper we have attempted to realise more of this potential by developing a range of indices of knowledge organisation from a teacher's concept maps. In using this form of map we are not intending to suggest that we have captured an enduring representation of an individual's conceptual network. We assume that all conceptual networks are dynamic mental spaces, so that we are representing the conceptual space that has been constructed across the times of our interaction with the participant.

Williams (1998) examined the value of concept maps as instruments for the assessment of conceptual knowledge of a group of professional mathematicians and students. In that study participants were asked to generate a list of terms about functions and organise them into maps. Each of the maps contained function-related concepts in ovals, with words indicating relationships among concepts appearing on the line connecting concepts. The maps showed evidence of participants' relating concepts in the form of chains. The maps, however were not designed to allow for the representation of cross linking between items of information between chains. Further, Williams did not attempt to represent the quality of conceptual understanding. Her analysis was also designed to be more global, as it allowed her to make judgments about the algorithmic nature of students' understanding as opposed to deeper level or categorical understanding shown by the experts. Therefore, although William's(1998) study showed the value of use of concept maps in the study of conceptual knowledge it suffered from the lack of detail about the various levels of connections and their influence on conceptual understanding.

In the present study we were interested in describing the quantity and quality of the organisation of the teacher's knowledge base about the topic of geometry in more detail. In doing so we were interested in representing the configuration of the connections that teachers have built among these concepts. Unlike that of Williams's(1998) study, our design used more than one context within which participants could externalise their knowledge. It was, therefore, reasonable to expect the concept maps to be more varied and complex. Thus, the structure of the concept map for the present study was designed to allow for the visualisation of the hierarchical organisation of geometric concepts and subconcepts as well as lateral links among these concepts and information about the teaching of the concepts. We will now describe the design and development of a concept mapping procedure that can be used to characterise the organisational state of teachers' knowledge of key concepts in geometry and trigonometry.

PROCEDURE

The teacher whose knowledge is the subject of this report was one of five experienced teachers recruited, along with five novice teachers, for the larger project. The experienced teachers were selected on the basis of two criteria: First they had at least 15 years of mathematics teaching experience at the high school level; and second, they were all recommended by their peers or professional subject associations as exemplary teachers. The concept map developed for this report was based on the knowledge exhibited by one expert teacher, Gary. In addition to having 20 years of teaching experience, Gary was the head of the mathematics department at his high school and was also involved in the writing of mathematics textbooks for high school students.

Gary was interviewed individually during school hours and his responses were audio- and video-recorded for later transcription. Gary was told that the purpose of the study was to find out what teachers might know about topics in geometry and the teaching of these topics.

Three interviews were conducted, each lasting about one hour. During the first interview Gary was asked to talk about a list of focus schemas in the areas of geometry, trigonometry, and coordinate geometry. The list included 13 focus schemas which were required to solve the four problems given to the teachers during the second interview, and six related schemas.

The 13 focus schemas were squares, rectangles, lines, similar triangles, congruent triangles, parallel lines, area, coordinates, triangles, right-angled triangles, regular hexagons, regular octagons, and circles. The six related schemas were plane figures, trapezium, polygons, rhombus, quadrilateral, parallelogram. Gary was invited to use diagrams to explain his thoughts if he wished to do so.

During the second interview, he was asked to solve four problems, each of which required use of schemas in the above three areas. Gary was asked to talk aloud as much as possible during the problem-solving attempts . During the final interview Gary was required to expand on items that he mentioned during the first interview when he freely recalled information.

In addition to the experienced teachers recruited for the project information was also sought from an academic mathematician. This mathematician was a member of a university mathematics department whose area of professional expertise was in the area of geometry. The focus of our interview with this mathematician was generation of a list of features and relationships for each of the focus schemas. For this study the list of features and schemas related to the focus schema square developed from this interview was used in evaluating the completeness and accuracy of the teacher's knowledge base.

Structure of the Concept Map

A primary consideration in the design of the concept maps was to capture the detail of schemas that teachers have acquired and relations that they have built within and between these schemas. The boxes and circles, or nodes, in the concept maps indicate schemas and features and the lines joining the boxes/circles show that a relationship was expressed between schema and features or between schemas. We identified four types of information about a focus schema and the relations that had been built around that schema.

1. *The defining features of the focus schema*

2. *The related features of the focus schema*

3. *Relationships between the focus schema and other schemata*

4. *Other representations of the focus schema*

Scoring of Concept Maps

In scoring the concept maps we adopted a strategy that would enable us generate a comprehensive description of the qualities of connectedness of teachers' knowledge. The connectedness of knowledge can be described both in terms of the number of knowledge components present as well as in terms of the qualitative relations that exist among the knowledge components. In our concept maps the qualitative character of the knowledge base refers to the pattern and extent of connections that were constructed by the teachers.

The basic units of analysis used to construct the maps were nodes and links. Nodes were established for all schemas associated with geometric terms or features, or attributes, associated with these terms that were mentioned by the participant in any of the tasks. The task on which the node was first identified was also recorded on the map. Links were interpreted as propositions. All propositions expressed in words or actions by the participant in any of the tasks were recorded, along with any labels expressed for the linking relationship. Nodes and links were established by consensus using two coders. The measures used to score maps and their definitions are available from the authors.

The following measures were derived for each concept map:

1. Quantity

We interpreted quantity as having two sub-categories:(a) number of nodes , and (b) number of links.

2. Quality

The quality of the concept maps were analysed for the two broad dimensions of integrity and connectedness. Integrity was analysed in terms of (a) completeness and (b) accuracy. The idea of integrity of knowledge was not used here in an absolutist sense. In scoring teachers' responses for completeness and accuracy we used the list of ideas generated the expert mathematician as a reference point. For completeness our scoring was not bounded by this list. If teachers produced a node or link that was not found in the mathematician's list but one that was mathematically acceptable this was also identified and scored.

Accuracy refers to the degree of correctness of the information provided by teachers. Information that is not correct may be manifest itself in various forms. An incorrect piece of knowledge could be a misconception (McKeown and Beck, 1990) or it could be 'underdifferentiated' or 'garbled' (Perkins and Simmons, 1988). For example, some students may not be able to differentiate the attribute of the schema of area and the measurement of area. This would be a case of a misconception. An example of garbled knowledge could be: 'Diagonals of a square are lines of symmetry for squares. Therefore, diagonals of rectangles are lines of symmetry'.

Connectedness was represented by four sub-categories: (c) depth, (d) branching, (e) crosslinking and (f) quality of relationships. Depth here refers to the extension of connections in a concept map in vertical directions along a single path. Within schema depth is a measure of the degree of vertical connection in a schema. We refer to this spread as occurring over a different vertical layers of the nodes in the concept map. Between schema depth is a measure of vertical connections among schema.

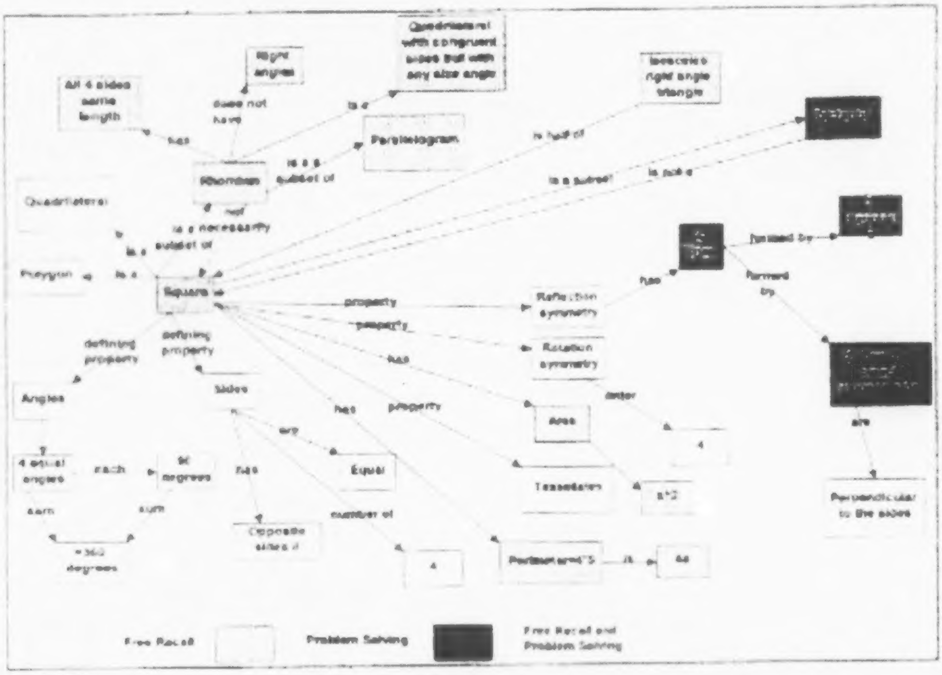
Branching is a measure of the number of paths associated with a feature. Crosslinking is a measure of horizontal linking between paths or between schemas. The quality of relationship measure attempts to rate the complexity of the label given to a particular link. For the purposes of scoring the map, we found it convenient to divide the organisational features of the information into two types of schematic links: within schema and between schema (Marshall, 1995).

RESULTS AND DISCUSSION

Gary's Concept Map

Figure 1 shows the concept map that was constructed for the focus schema of square for our expert teacher, Gary. The most striking impression of this map was that Gary has built up a large number of connections involving squares, as shown by the number of arrows emanating to and from the Focus schema. The bottom left hand corner of Figure 1 showed that his knowledge about the properties of squares is extensive, for both defining and related features. The top-half of the map indicated that Gary has been able to link squares with other polygons such as rhombus and isosceles triangles. Gary's understanding of defining properties of a square was extended to more advanced features such as symmetry and area. This component of his knowledge is depicted by the right-half of Figure 1. The upper left section of the concept map provides information relevant to the contexts in which the above knowledge components were activated and the ways that these would be expressed by this teacher.

Figure 1
Gary's Concept Map for a Square


All 4 sides same length
Right angles
does not have
Quadrilateral with congruent sides but with any size angle
is a
Isosceles right angle triangle
has
Parallelogram
is half of
Rhombus
is a subset of
is a subset
is not a
Quadrilateral
not
is a necessarily subset of
is a
Polygon
is a
Square
formed by
has
formed by
property
Reflection symmetry
property
Rotation symmetry
defining property
defining property
has
order
Angles
Sides
has
property
Area
4
are
4 equal angles
each
90 degrees
has
Equal
Tessellates
s^2
Perpendicular to the sides
sum
sum
number of
Opposite sides //
= 360 degrees
4
Perimeter = 4 S
4s
Free Recall
Problem Solving
Free Recall and Problem Solving

The concept map of this experienced teacher revealed that, in a quantitative sense, he has a rich store of knowledge as manifested by the multiple links he had constructed with many facets of a square. These links suggests that the teacher not only has a well-developed understanding about the properties of square but more importantly, he has an extensive network of connections between squares and other geometrical figures. For example, he was able to represent a square as a polygon and as a quadrilateral. He saw square as part of a rhombus but noted correctly that a rhombus was not a square. His ability to recognise a right-angled triangle as forming part of the square also indicated that the teacher was able to use basic properties such as right-angles to create other figures. Gary's knowledge base indicates that he was able to highlight differences and similarities that exist between square and related figures.

Gary commented that a square has symmetrical properties and that this property can be created by more than one way. This understanding of symmetry is extended to his observations that squares tessellate. Here one can detect not only an understanding of the geometric properties but also practical applications of squares.

Within-schema Ratings

Results of analysis for the Focus schema of square in terms of quantity and quality of the links are provided in Tables 1 and 2. Table 1 shows connections made by Gary, that are internal to square, links that indicate within schema organisation. Table 1 shows that Gary's concept map contained a total of nine defining, and 12 nodes associated with the related features of the focus schema. Table 1 also shows that the number of links was almost the same as the number of defining and related features nodes.

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Table 1
Analysis of Concept Map for a Square: Within Schema

FEATURES	QUANTITY		QUALITY							
			Integrity		Connectedness					
	Number of nodes	Number of links	Completeness	Accuracy	Depth	Branching	Cross-linking	Quality of relationships		
								Simple	Moderate	Complex
Defining Features										
Angles	5	5			Level 3	Moderate	None	2	1	
Sides	4	4			Level 2	Moderate	None	3		
Subtotal	9	9	High	High				5	1	0
Related Features										
Area/ Perimeter	4	4			Level 2	Moderate	None		2	
Symmetry	7	7			Level 3	High	None		2	
Tesselates	1	1			Level 1	None	None	1		
Subtotal	12	12	High	High				1	4	0
Applications										
Subtotal	0	0	Low	x		x	x	0	0	0
TOTAL	**21**	**21**						**6**	**5**	**0**

In the qualitative analysis there is, first, a high level of integrity in the defining and related features components of Gary's map when judged against the knowledge base of the expert mathematician, as indicated by high scores for both accuracy and completeness. However, there were variations in the levels of depth of links of his knowledge base across the defining and related features of the focus schema . There was a greater depth of connectedness for angles and symmetry than for tessellation, area or perimeter. Secondly, there is a low level of completeness in the applications of the square. Because Gary did not present an example of how the square can be applied in the real world, we were unable to make any judgements about the accuracy of his knowledge about the applications of squares.

Gary's map showed high levels of branching especially for several features, with tessellation again being the least developed feature. Figure 1 shows that he can differentiate between rotational symmetry and reflective symmetry, sides and angles, and area and perimeter. Within each of the two symmetry branches, the knowledge is well structured and organised into a highly integrated chunk. However, the lack of crosslinking between these two symmetry branches indicates that there seems to be little integration between knowledge about line symmetry and rotational symmetry.

The pattern of scores in the last three columns of Table 1 indicate that the quality of the descriptions used by this teacher to label the relationships among nodes defining and related to the focus schema were either simple or moderate. For example, Gary mentioned that an isosceles triangle is half of the focus schema, square. This is a correct statement and there was an attempt to elaborate this relationship, as indicated by the comments he made about the relative size of the two figures. However, he did not go further and establish a bidirectional connection about how a square might be related to the isosceles triangle, or discuss the implications of such relationships in using or deriving, say, the Pythagoras' theorem.

Overall, Gary's within schema knowledge about square appears to be extensive and is well differentiated across several layers and along several branches. However, the level of integration between the different branches of knowledge about the square could only be described as minimal, as evidenced by the scores for crosslinking.

Between Schema Ratings

Table 2 shows the ratings given to the various dimensions of Gary's knowledge about square and other focus schemas. The quantitative analysis showed that Gary had made links between the focus schema (square) and four other schemas. Of these the highest number of links and nodes involved the rhombus schema, followed by rectangle. The qualitative analysis indicates that the links made were complete and accurate, suggesting a high level of integrity about this aspect of his knowledge base. With regard to depth, there were fewer instances of spread of links at level 2 or higher than in his within-schema knowledge structure. The links at the highest depth was between square and rhombus, rhombus and parallelogram, and rhombus and quadrilateral, thus indicating the existence of a hierarchical understanding of these schemas.

Gary provided two instances of simple relationships and two complex relationships in his discussion of these schemas. For instance, he mentioned that a square is a subset of a rhombus but not all rhombuses are squares suggesting an understanding of the complexity of relations between the two geometrical figures. He further commented that all squares are rectangles but not vice versa, again indicating a complex relationship about differences and similarities between the two figures.

Table 2
Analysis of Concept Map of a Square: Between Schema

FEATURES	QUANTITY		QUALITY							
			Integrity		Connectedness					
	Number of nodes	Number of links	Completeness	Accuracy	Depth	Branching	Crosslinking	Quality of relationships		
								Simple	Moderate	Complex
Quadrilateral	1	1			Level 2			1		
Rhombus	5	6			Level 2					1
Rectangle	1	2			Level 1					1
Isosceles right-angled triangle	1	1			Level 1			1		
Polygon	1	1			Level 1			1		
TOTAL	9	11	High	High		Yes	None	3	0	2

In summary an analysis of the quantitative and qualitative dimensions of the knowledge shows that Gary has a rich set of connections with evidence of complex differentiation in some instances. Overall, his knowledge has high integrity and shows evidence of substantial branching in certain areas. As an experienced teacher it might have been expected that Gary would have shown evidence of more crosslinks between branches. Half of the relations that were discussed by Gary could be categorised as complex in nature. In general, however, Gary's knowledge network is not well integrated between branches. This lack of integration between the different branches of knowledge from an experienced teacher may influence his teaching in ways that students may not be able to draw out important differences between key features about squares and other geometry figures.

CONCLUSION

The purpose of this paper was to describe the use of a method for analysing concept map data for generating information on quantitative and qualitative features of a mathematics teacher's knowledge base in the area of geometry. By way of illustration we described the integrity and the levels of connectedness of the knowledge one teacher has built up around the concept of square. The representation of the teacher's knowledge was built up from his involvement in three different performance contexts in each of which the teacher could have displayed his understanding of square and its applications.

We contend that the system of analysis used in this study takes us beyond that used in other studies that have employed concept mapping or similar graphical systems as a means of representing knowledge states. The current system takes us beyond the point reached by Williams (1998), enabling us to present more detailed and differentiated description of this teacher's knowledge base in this area of geometry.

Use of this system indicates that this experienced teacher has a rich store of knowledge which is complete and accurate. His knowledge base shows a high degree of branching with depth of knowledge within branching. The relationships between features and schemas shows that he does have a range of complexity. Analysis also suggest that the degree of crosslinking could be higher.

Overall, we have delineated a set of dimensions to describe a subject-matter knowledge in a quantitative and qualitative sense. We have identified areas where the quality of this teacher's knowledge could be improved. The relatively low levels of crosslinking could have implications for the quality of explanations provided, and it could also influence the accessing of knowledge.

Experience with this procedure suggests that there is value in use of the idea of connectedness to make judgements about the qualitative features of a knowledge network. The procedures we have developed here extend the utility of Mayer's (1975) distinction between internal and external connectedness. The levels of connectedness of this framework allowed us to make a distinction between the richness of the internal and external connections associated with square that were made by this teacher. In a practical sense this analysis seems to have provided a way for use to establish a qualitative estimate relevant to the teacher's knowledge base that can be used in other parts of the project where relationships between mathematical content knowledge, pedagogical knowledge and teaching actions are of concern.

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EQUALS, EXPRESSIONS, EQUATIONS, AND THE MEANING OF VARIABLE: A TEACHING EXPERIMENT

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A teaching experiment was designed to facilitate Year 8 students' understanding of algebraic expressions and equations through the use of unknowns, patterns, relationships, and concrete materials. This paper discusses the study's theoretical framework, the teaching episodes relating to expressions, equations, and equals, the students' reactions to this instruction, and their initial and final understandings of algebraic expressions, equations, and equals. Exploration of the successful and unsuccessful teaching episodes emphasises the relationship between instruction, prior knowledge and learning.

Algebra is an abstract system in which components interact to reflect the structure of arithmetic. Understanding algebraic expressions requires abstract schema (Ohlsson, 1993) of the arithmetic *operational laws* and *equals*, combined with the algebraic notion of *variable*. For example, the distributive principle holds for whole numbers [e.g., $2\times(3+4)=(2\times3)+(2\times4)$], decimal numbers (e.g., $4\times4.7=4\times4+4\times0.7$) and mixed numbers (e.g., $5\times3\,{}^{2}/_{11}=5\times3+5\times{}^{2}/_{11}$), and it also holds for algebra [e.g., $4(x+y)=4x+4y$]. Thus, the distributive principle is an isomorphic structure between arithmetic and algebra. It is an abstract schema because its meaning lies in terms of relationships [$a(b+c)=ab+ac$], not the particular content (e.g., fractions).

Difficulties in learning algebra have long been documented (e.g., Thorndike et al., 1923) and more recent research (e.g., Boulton-Lewis et al., 1998; Linchevski & Herscovics, 1996) continues to show that achievement rates in algebra are poor. Research indicates that instruction does not seem to be bridging the gap between arithmetic and algebra, particularly in: (a) developing meaning for variables (Booth, 1988; Cooper at al., 1997; Linchevski & Herscovics, 1996) and for the equals sign (Herscovics & Linchevski, 1994); (b) connecting the knowledge required to solve arithmetical equations by inverting or undoing (backtracking), and the knowledge required to solve algebraic equations by operating on or with the unknown (Booth, 1988; Herscovics & Linchevski, 1994); (c) overcoming the syntactic similarity between the algebraic notation for $3x$ and the arithmetic notation for 2-digit place value (Stacey & MacGregor, 1997); and (d) abstracting the properties and conventions of operations (Herscovics & Linchevski, 1994). In response to the poor achievement rates, instructional practices which focus on patterns and physical materials (e.g., cups to represent variables, counters to represent numbers, balance beam for equals) to introduce algebra have been developed (e.g., Quinlan et al., 1993). However, patterns may not be effective as they do not easily lead to the generalisations required for algebraic understanding (Boulton-Lewis et al., 1997; MacGregor & Stacey, 1995). The use of physical materials may impose additional cognitive demands (Halford & Boulton-Lewis, 1992), contain intrinsic restrictions (Behr, Lesh, Post, & Silver, 1983), and have limited connections to symbols (Boulton-Lewis et al, 1998; Hart, 1989).

To overcome the above problems, Boulton-Lewis et al. (1997) proposed a *two-path* instructional model (see Figure 1). The model was based on the belief that understanding of complex algebra is the end product of a learning sequence of mathematical concepts that includes: *binary arithmetic*; *complex arithmetic* (a series of operations on numbers); and *binary algebra*. It

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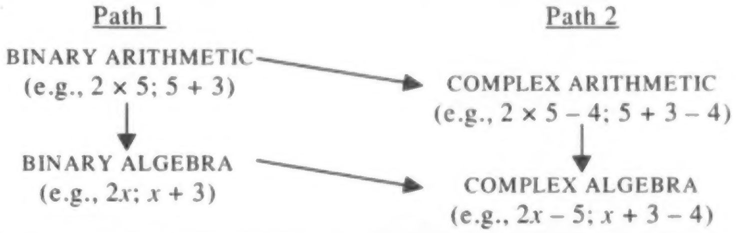
means that 2x5 and 5+3 (binary operations) are a prerequisite for $2x$ and $x+3$ (binary algebra) while, in turn, 2x5-4 and 5+3-4 (complex arithmetic) forms an important prerequisite to understanding $2x-4$ and $x+3-4$ (complex algebra). It also means that understanding operational laws should be applied to series of operations as well as individual operations, and that learning complex algebra is facilitated by understanding similar (isomorphic) structures in complex arithmetic.

Figure 1

Two-path Model (Boulton-Lewis et al., 1997) for Algebra Instruction



Sometimes it is difficult to classify activities into the four areas of Figure 1 (e.g., equations with one unknown which can be solved by *backtracking*, a method from complex arithmetic). Thus, instruction to develop algebraic knowledge should be seen as encompassing three stages, namely, arithmetic through pre-algebra (where arithmetic techniques are used with letters, e.g., $3(x+2)=18$) to algebra (where operations act on variables, e.g., $x+7=3x-1$) (Boulton-Lewis, Cooper, Atweh, Pillay, & Wilss, submitted).

THE STUDY

The study was a teaching experiment (Romberg, 1992) and an intervention design (Hiebert & Wearne, 1991) undertaken with 51 Year 8 students (two classes) at a middle-class suburban state secondary school. Twenty 40-minute episodes (separated into two two-week units, beginning two months apart) to introduce early algebra were taught to one class and repeated with the second. One of the researchers did the teaching whilst the class teacher and another researcher observed. Each teaching episode was videotaped, all students' work was collected, all students were surveyed at regular intervals (including the beginning of the teaching episodes), and a representative sample of 14 students (7 from each class) was interviewed after each unit. The monitoring of student responses and reactions facilitated the modification of succeeding teaching episodes, and permitted the study of the relationship between teacher actions and student learning.

The major purpose of the teaching episodes was to have students reflect on their experience of arithmetic in order to draw out generalities (e.g., those that differentiate between addition, subtraction, multiplication and division and those that underlie the procedures used in simplifications and equation solving in algebra). For the purposes of this paper, the planned sequence of teaching episodes in the study was as defined below. The teaching episodes were informed by the findings of Kaplan, Yammamoto and Ginsburg (1989) and Linchevski & Herscovics (1996) in that they aimed to underpin formal mathematical knowledge with informal mathematical notions, particularly with respect to the repeated addition notion of multiplication (e.g., $3x=x+x+x$).

1. *Review of the four operations and equals.* This aimed at highlighting the structure of the operations, as discussed at the beginning of this paper.
2. *Introduction to the notions of expressions (e.g., 34+58) and equations (e.g., 4+5=9, 34+28=31x2).* This also aimed at emphasising the legitimate changes that can be made to expressions (doing and undoing so they stay the same value) and equations (do the same to both sides so the sides are equivalent). For example, expressions such as 24+58 remain the

same value if multiplied by 10 and then divided by 10, or increased by 15 and then decreased by 12 and by 3. Equations such as 25+11=36 are equivalent if both sides are multiplied by 17, i.e., (25+11)x17=36x17.

3. *Development of the notion of variable.* This was done through Usiskin's (1988) three approaches of unknowns (e.g., $3x=6$), patterns (e.g., 3, 7, 11, 15, ...) and relationships (e.g., 2→5, 8→23, 5→14), and through materials (e.g., cups and counters). Usiskin's (1988) approaches were introduced within complex arithmetic (see Figure 1) and then revisited in turn with concrete materials to introduce variable. Unknowns were taught first (because of their pre-algebraic focus), this teaching based on the transformational approach to arithmetic (e.g., 2x3 is viewed as a transformation from 2 to 6 through multiplying by 3) (Cooper & Baturo, 1992). This method has been shown to be effective in teaching complex arithmetic and introducing *backtracking* (see Figure 2), the method most students use to solve algebraic equations such as $3x+4=19$ (Boulton-Lewis et al., 1998; Herscovics & Linchevski, 1994).

Figure 2
Transformations and Unknowns

$4 \rightarrow \times 3 \rightarrow 12 \rightarrow +2 \rightarrow 14$	Transformation
$? \rightarrow \times 3 \rightarrow ? \rightarrow +2 \rightarrow 20$	Unknown
$6 \leftarrow /3 \leftarrow 18 \leftarrow -2 \leftarrow 20$	Backtracking
$x \rightarrow \times 3 \rightarrow 3x \rightarrow +2 \rightarrow 3x+2$	Variable as unknown

For patterns, a sequence of numbers was to be generated and related to the numbers of the term (directly, or by using a geometrical construction, e.g., the number of matchsticks needed to make 5 squares in a row). The nature of the *n*th term would be discussed. For relationships, numbers would be related to other numbers (either directly, or through a "guess my rule" game) and generalisation sought for the relationships. The ways in which the three approaches are related to variable are highlighted in Figure 3 below. Binary and complex examples are illustrated. Modelling of the expressions was to be undertaken in both directions; the teacher directing the students to show an expression with cups and counters, and the teacher modelling expressions with cups and counters with students giving the expressions.

Figure 3
Three Approaches for Introducing Variable

UNKNOWNS	PATTERNS	RELATIONSHIPS
		$\boxed{\times 3}$
$3 \rightarrow \times 3 \rightarrow 9$	Term: 1, 2, 3, 4, . . . , n	$8 \rightarrow 24$
$11 \rightarrow \times 3 \rightarrow 33$	Value: 3, 6, 9, 12, . . ., $3n$	$4 \rightarrow 12$
$? \rightarrow \times 3 \rightarrow 21$		$13 \rightarrow 39$
$x \rightarrow \times 3 \rightarrow 3x = 21$		$y \rightarrow 3y$
		$\boxed{\times 3 + 2}$
$3 \rightarrow \times 3 \rightarrow 9 \rightarrow +2 \rightarrow 11$	Term: 1, 2, 3, 4, . . . , n	$8 \rightarrow 26$
$11 \rightarrow \times 3 \rightarrow 33 \rightarrow +2 \rightarrow 35$	Value: 5, 8, 11, 14, . . ., $3n+2$	$4 \rightarrow 14$
$? \rightarrow \times 3 \rightarrow 21 \rightarrow +2 \rightarrow 23$		$13 \rightarrow 41$
$x \rightarrow \times 3 \rightarrow 3x \rightarrow +2 \rightarrow 3x+2 = 23$		$y \rightarrow 3y+2$

4. *Extension of variable to more complex expressions [e.g., $3x+2$; $3(x+2)$] and equations (e.g., $3x+2=11$).* As each of the approaches moved towards *variable*, cups and counters were to be used to model expressions of the form $3x$ and $x+3$ (binary algebra) and $3x+2$ and $3(x+2)$ (complex algebra).

Fourteen students were surveyed at the beginning of the intervention regarding their understanding of equals, expression $2x+3$, and equation $3x-4=11$. The same students were interviewed at the end of the intervention regarding equals, expressions $3x$, $3x+1$ and $3(x+2)$, and equation $3x-4=11$. This interview also focused on students' ability to represent the expressions and equations with materials and to relate them to patterns.

RESULTS AND DISCUSSION

The students' responses for the pre-survey and post-interview are presented first, followed by the results of the teaching episodes.

Survey and Interview Responses

Table 1 summarises the fourteen students' responses to the initial survey tasks and the final interview tasks. The discussion focuses on the intervention's effectiveness for developing understanding of equals, expressions, and equations.

Table 1
Students' Appropriate Initial and Final Responses to Items Associated with Equals, Expressions, and Equations (n=14)

Appropriate Initial Responses	No.	Appropriate Final Responses	No.
Equals		*Equals*	
Meaning of the sign "="	3	Meaning of the sign "="	6
Expressions		*Expressions*	
Meaning of $2x+3$	1	Meaning of $3x$	13
		Meaning of $3x+1$	12
		Meaning of $3(x+2)$	12
Equations		*Equations*	
Meaning of $3x-4=11$	1	Meaning of $3x-4=11$	11

At the beginning of the teaching episodes, three students could interpret the equals sign as "the same as" or "equivalence"; others were concerned with getting an answer. (These results concur with previous findings by Behr, Erlwanger, and Nichols, 1992, and Cooper et al., 1997.) Only one student was able to explain the meaning of an algebraic expression or equation. Inappropriate responses were associated with getting an answer (expressions only), doing something (e.g., replacing x), or incorrectly interpreting the coefficient of x (e.g., twenty something for $2x$). Over half the students offered no responses, suggesting a low understanding of, or lack of familiarity with, algebraic expressions and equations at the start of the study.

As Table 1 also shows, there was an improvement in students' explanations of the "=" sign after the intervention. However, over 50% of students gave no response or an inappropriate response. Contrary to this, there was a clear improvement in students' ability to explain the meaning of algebraic expressions such as $3x$, $3x+2$, and $3(x+2)$. In particular, students were comfortable with the meaning of variable as "any number," and there was no confusion with place value (thirty something for $3x$). However, explanations involving multiplication and grouping were more common than those involving repeated addition. Most students providing appropriate explanations could also represent the first two expressions with cups and counters. Less than one third of the students could correctly use them to represent the third expression, $3(x+2)$. Pattern construction was also found to be poor.

At the end of the teaching episodes, the majority of the 14 students were able to find an expression and an equation, and use real world examples. Fewer students could provide an adequate rationale for their choice of expression or equation. The greatest difficulties seemed to be associated with the allowable changes that can be made, particularly with respect to expressions.

Teaching Episodes

To address the problems highlighted in the previous section, the teaching episodes emphasised the notion of equals as equivalence (e.g., $36 \div 9$ is the same value as 2×2), and focused on the differences between expressions (no equals, containing operations) and equations (equals, operations on one or both sides of equals). The episodes also developed the notion, somewhat informally, that equals "balances" (mathematically) both sides of the equation. The equals lessons appeared to be successful, the students particularly enjoyed doing the worksheets (which were in a game environment) associated with the lessons, and they completed them correctly. However, the expression and equation lessons were less successful. Discussions with groups of students during the worksheet activity indicated that there was little understanding of what was expected, nor of the differences between expressions and equations. To overcome this, the introduction to transformations (including *backtracking*) was brought forward to help students with the idea of doing and undoing arithmetic operations. The second attempt at teaching the difference between expressions and equations appeared to be more successful, and students seemed to experience no difficulties with the transformation worksheets. However, observations of the worksheet activity on expressions and equations continued to show that a significant proportion of the classes exhibited little understanding of the difference. An additional worksheet was designed to relate them to real world instances, as shown in Figure 4 below. These worksheets were completed eagerly and correctly.

Figure 4
Expressions and Equations

Expression	5x4	There were 5 packets with 4 chocolates in each packet
Sum	5x4=	There were 5 packets with 4 chocolates in each packet – how many chocolates?
Equation	5x4=20	There were 5 packets with 4 chocolates in each packet – this made 20 chocolates.
Equation	5x4=18+2	There were 5 packets with 4 chocolates in each packet – this was the same number of chocolates as John ,who originally had 18 and was given 2 more.

Observations of teaching episodes and analysis of worksheets relating to introducing variable indicated that the students' responses to the episodes were mixed. The transformational episodes were again well received in that students appeared to like and understand the teaching of transformations with unknowns. In fact, the interview students commented that the transformational activities were their best lessons, and their favourite worksheets were those concerned with equals, transformations and relating expressions and equations to real life situations. The visual representation of change as an arrow seemed to assist in understanding the effect of operations on variables, particularly the difference between $3x+2$ and $3(x+2)$. The episodes dealing with patterns and relationships were not as well received, and students appeared to have difficulty determining the generalisations for the patterns and relationships, particularly when there were two operations. Hence, the teaching episodes were modified to include hints (e.g., the hint "multiply by 2" was placed beside relationships, $3 \rightarrow 7$, $12 \rightarrow 25$, $5 \rightarrow 11$, $a \rightarrow ?$).

The episodes in which cups and counters were used to introduce variable and to represent algebraic expressions were also not as successful as expected. The students tended to model $3x$ and $x+3$ with the same material (3 counters for the 3 and a cup for the x) and, when this was overcome, to repeat the error for $3x+2$ and $3(x+2)$ (3 cups and 2 counters for both expressions). They appeared not to like using the materials. This was reinforced by the interview, where 9 of the 14 students felt that the cups and counters were a negative part of the teaching. Only 3 of the 14 students liked any of the worksheets concerned with introducing variables; this was equally the case for unknowns as with patterns and relationships.

The students' responses with the cups and counters showed that they did not have a facility with multiplication in terms of repeated addition. (This phenomenon may have been a consequence of the focus on the *array* model of multiplication, rather than on the *repeated addition* model, in Queensland schools). Therefore, extra time was scheduled in the next episode to teach understanding of multiplication as repeated addition, and representing $3x$ as $x+x+x$ (similar to Linchevski & Herscovics, 1996). After this, the teaching episodes on variable, including representation with cups and counters, appeared to be more successful. According to the observers, the students' responses during these teaching episodes indicated that most students were grasping the meanings being portrayed.

Overall, a difficulty with sequencing emerged. The episodes were designed so that all the complex arithmetic work in unknowns, patterns and relationships was completed before variables were introduced using these approaches. When the episodes returned to each approach for the development of variable, the students appeared to have forgotten the approach and the arithmetic activities had to be repeated. As a consequence of this, later episodes were designed so that arithmetic work led straight on to variable work for each particular activity.

CONCLUSIONS

Several main conclusions could be made following the results of the study.

1. *Arithmetic as a basis for algebra.* The interaction between the understanding of arithmetic and the understanding of algebra was more complex than expected. The students' reactions to the teaching episodes revealed that reflecting on arithmetic to build algebra generally worked, but only if the arithmetic lead straight to the algebra generalisations for each activity. The two path model (see Figure 1) appeared to provide a framework for effective teaching. However, rather than consisting of four separate steps to be performed across time, it represents a framework that should be followed for each separate notion and principle.

2. *Equals*: While there was improvement in students' understanding of the "=" sign during the study, inadequate responses persisted. The correct meaning of equals appears to need ongoing reinforcement.

3. *Expressions and equations.* The episodes were successful in helping students to gain understanding of the meaning of expressions and equations. However, one problem often encountered in secondary school mathematics is the use of incorrect procedures in making changes to expressions and equations. Attempts to address this problem here were not overly successful. It may be that more complex understanding is required to understand operations an expressions and equations. The findings supported Linchevski and Herscovics (1996) in stressing the importance of understanding $3x$ as repeated addition. Although extra work was undertaken, repeated addition was still a weakness of the students in examples which included more than one operation (e.g., interpreting $3(x+2)$ as $(x+2)$ plus $(x+2)$ plus $(x+2)$). However, there appeared to be a relationship between understanding multiplication as repeated addition and correctly using cups and counters for multiplication situations such as $3(x+2)$.

4. *Usiskin's approaches*: Usiskin's approaches (e.g., unknowns, patterns and relationships) seemed a useful way to introduce algebra because of the way they reflect the different notions of generality. However, students had problems with patterns and relationships for more than one operation. Unknown as transformation was well received by the students and appeared in most cases to be understood. Therefore, as found by Herscovics and Linchevski (1994), variable as unknown and arithmetic transformation appear to offer good opportunities for the transition from arithmetic to algebra and seem to fit within the "pre-algebraic" level between arithmetic and algebra, as proposed by Boulton- Lewis et al, 1997.

5. *Cups and counters*: The use of materials (cups, counters) acted as a conduit between arithmetic and algebraic notions. They appeared to work successfully for $3x$ (binary algebra) and $3x+1$ (simple complex algebra). Although it appeared useful for students to differentiate between multiplication then addition (e.g., $3x+1$) and addition then multiplication [e.g., $3(x+2)$] with cups and counters, understanding was not evident in the final interview. This could reflect the criticisms of materials given by Halford and Boulton-Lewis (1992), Boulton-Lewis et al. (1998), and Hart (1989) that children see little connection between materials and symbols due to the additional cognitive demands that materials bring.

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ASSESSING DIAGRAM QUALITY: MAKING A DIFFERENCE TO REPRESENTATION

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The effectiveness of a diagram in problem solving is dependent on its utility as a cognitive tool. In order to develop students' ability to use diagrams as cognitive tools, teachers need to assess the quality of students' diagrams and provide them with the necessary support. However assessing the quality of diagrams is problematic. This paper discusses how theoretical prototypes and exemplars of level of performance provide a practical and effective avenue for assessing the quality of students' diagrams.

Mathematics educators strongly advocate the use of the strategy *draw a diagram* for mathematical problem solving (e.g., National Council of Teachers of Mathematics [NCTM], 1989). This perspective is strongly grounded in the belief that generating a diagram facilitates the conceptualisation of the problem structure (van Essen & Hamaker, 1990). Although the use of a diagram as a tool of mathematics can empower primary students to deal with novelty (NCTM, 1989), effective problem solving depends on the quality of students' diagrams (Yancey, Thompson, & Yancey, 1989). Thus, the generation of high quality diagrams should be a goal in any instructional programme on diagram use. The salient question that emerges, which is the focus of this paper, is how to assess the quality of students' diagrams.

THE USE OF THE DIAGRAM IN PROBLEM SOLVING

The advantages of generating a diagram are related its utility as a cognitive tool (e.g.; Larkin & Simon, 1987; van Essen & Hamaker, 1990). For example, diagrams act as an external sketch pad where interconnected pieces of information can be chunked together in a holistic manner (van Essen & Hamaker, 1990). Thus, implicit information within a problem may become explicit to the solver on a diagram (Larkin & Simon, 1987). However not all diagrams have the potential to be cognitive tools. For example, some students generate diagrams that focus on the surface (literal) features of the problem at the expense of representing the problem structure (e.g., Dufoir-Janvier, Bednarz & Belanger, 1987).

Problem representation can be enhanced by knowledge of general purpose diagrams, namely, *matrices*, *networks*, and *hierarchies*, and a range of diagrams that exhibit *part-whole* characteristics (Diezmann, 1999; Novick, Hurley & Francis, in press). These diagrams assume an important role in mathematics because they provide representational frameworks that are appropriate for a range of problem structures. For example, because of its particular spatial characteristics, a matrix can be used to represent the problem structure in combinatorial tasks (English, in press) or in deductive reasoning tasks (Novick, in press). Networks and hierarchies are also spatially-oriented diagrams that have unique visual characteristics. A network is a path-like representation (e.g., a train line map), whereas a hierarchy is a tree-like representation (e.g., family tree) (Novick, in press). Knowing the conditions of applicability for each of these representations is advantageous in selecting an appropriate diagram type, which is the first step in generating a successful diagram (Diezmann, 1999). Part-whole diagrams, unlike matrices, networks and hierarchies, have no unique visual characteristics. For example, both a diagram showing a "pie" with a quarter of the pie missing and a Venn diagram could be categorised as part-whole diagrams. There can be considerable variation between correct representations of a particular "part-whole" problem however. For example, in representing the number of tennis players in a class, there are various ways to draw the tennis players and the whole class, and to represent the relationship between them. Due to the range of diagram types and the variation possible within some diagram types, assessing the quality of diagrams can be problematic. However to support the development of students' use of the diagram as a cognitive tool, teachers need to be able to assess the quality of students' diagrams.

ASSESSING THE QUALITY OF A DIAGRAM

The quality of a diagram can be determined by the correctness of the diagram as a representation for the problem (Lindvall, Tamburino, & Robinson, 1982; van Essen & Hamaker, 1990). Correctness involves selecting an appropriate diagram type and generating a diagram that accurately represents the problem structure (Novick, in press). The appropriateness of the diagram can by determined from the conditions of applicability of particular diagrams (Novick, in press). For example, a hierarchy is an appropriate diagram for representing a knock-out tennis competition. However determining the accuracy of the diagram is more complex, due to the surface variation that may occur between diagrams, as discussed previously.

Lindvall, Tamburino, and Robinson (1982) proposed that the focus in assessing the accuracy of students' diagrams should be on the representation of the quantitative and qualitative components of the problem structure. Thus, for each problem situation, there are essential components of the problem that should be accurately represented on a diagram. These components constitute a theoretical prototype for the problem representation. Prototypes are particularly useful for ascertaining expertise when the degree of similarity between the exemplars (i.e., diagrams) may be low (Sternberg & Horvarth, 1995). The congruence between the diagram and the prototype indicates the students' level of performance in diagram generation and additionally, identifies aspects of diagram generation that need further development.

The purpose of this paper is to explore how the quality of diagrams can be assessed using theoretical prototypes, and specifically, how prototypes can be used to identify different levels of performance. Pragmatically, it is useful for teachers to have a set of performance benchmarks for assessment purposes (Maher & Martino, 1992). This "benchmarking" process will be illustrated using one problem in which a spatially-oriented diagram (e.g., matrix) is appropriate and another in which a conceptually-oriented diagram (i.e., part-whole diagrams) is appropriate.

METHOD

The establishment of performance levels for students' generation of general purpose diagrams was a component of a case study (Yin, 1994), which evaluated the effectiveness of instruction on various aspects of diagram use in novel problem solving (Diezmann, 1999). In the case study, it was hypothesised that there would be an improvement in students' generation of diagrams after instruction. Hence, it was necessary to develop a means of ascertaining the quality of students' diagrams on the pre- and post-instruction isomorphic tasks in order to compare students' performance and test the hypothesis.

The participants in the case study were 12 Year 5 students with a mean age of 10 years 3 months from a moderately sized school in a lower socio-economic suburb in Brisbane, Australia. They represented a cross section of students, who were high and low performers in problem solving, and had high and low preferences for a visual method of solution. The instruction consisted of twelve half-hour whole class lessons and addressed the generation and use of the four general purpose diagrams on novel problem solving tasks. Tasks comprising isomorphic sets of five novel problems were presented to each participant during 30 minute interviews conducted before and after instruction. The interviewer was known to the subjects through prior classroom involvement. The interviews were videotaped and subsequently transcribed. As the participants were not specifically instructed to use a diagram, those participants who did not spontaneously use a diagram were given further opportunities to generate a diagram. However the data used in the establishment of levels of performance for the accuracy of diagrams, is restricted to the diagrams that were spontaneously generated by the students.

The students' diagrams from the interview tasks were classified according to the degree of congruence between their diagram and the theoretical prototype on the following basis. When no diagram was produced Level 0 was assigned. Students' diagrams that could plausibly be considered to be of the appropriate type but had no structurally accurate components were assigned Level 1. The diagram was assigned Level 2 when at least one but not all of the structural components was represented accurately. Diagrams in which all structural components were accurately represented were assigned Level 3.

RESULTS AND DISCUSSION

Two examples of the diagram classification process are presented for discussion. The first example involved the use of a spatially-oriented diagram (a matrix) and the second example involved a conceptually-oriented diagram (a part-whole diagram). Both the problems that are discussed were post-instruction tasks because on these tasks the students generated diagrams for the full range of levels of accuracy.

The Sports task is a deductive problem with a factorial structure comprising two sets (see Figure 1). Tasks with this structure can be represented using a matrix (Novick, in press). The theoretical prototype for this task consisted of *two distinct sets*, which are drawn as a two-dimensional representation consisting of *rows and columns*. One of the sets should be represented on a row and the other on a column. Although the location of each of the sets, either on a row or a column, is irrelevant, each member of the set needs to be represented in the same location as the other members of that set. For example, on the *Sports* task the four people should be represented either on a row or a column (see Figure 2).

Figure 1
The Sports task.

***Sports:** Four friends like different sports. One likes tennis, one likes swimming, one likes running and one likes gym. Each person only likes one sport. Use the clues to help you find out which sport each friend likes. Sally and Rick met when one of them won a swimming race. Tara and Greg met when one of them was exercising at the gym. Sally is not a swimmer or a runner. Greg is a friend of the gymnast's brother.*

Students' performance on this task confirmed that four levels of accuracy in the generation of a matrix could be identified. On Figure 2, these levels are described and students' diagrams are presented and annotated.

The assignment of levels for the matrix task was straightforward because a gradual development in the accuracy of the matrix was evident from Levels 1 to 3, which is the optimal level. The only variation in the representation of information on a matrix is in the positioning of each set of information. For example, in each of the diagrams on Figure 2, the students positioned the sports in the columns, however, this set could also have been positioned on the rows. A change in position of the sets would not affect the assignment of levels because position is not a critical aspect in the representation of sets on a matrix. Hence, despite differences in accuracy, the students' "matrices" were visually similar.

In contrast to the *Sports* task, *The Park* problem can be represented with a part-whole diagram (see Figure 3). A theoretical prototype for *The Park* comprised the *representation of parts of the total* (i.e., legs), and the *representation of the total* (i.e., total number of legs). As with Task 1, the students' diagrams varied substantially in their congruence with the prototype (see Figure 4). However, the assignment of levels on the part-whole tasks was more complex than on the matrix task (Task 1). Although there were the same number of levels of accuracy on both tasks, there was considerably more surface variation in the part-whole diagrams compared to the matrices. For example, on Figure 4, students represented "legs" as lines, as part of an animal, and as dots. Additionally, there was variation in the grouping of these legs to make a person or a dog. Groupings were represented by circling, by attachment as part of a person or dog, and by the positioning of the legs (dots) near lines. Furthermore, there was ambiguity in the use of graphic components. Whereas, Ian and Gemma used a detached line to represent a "leg", Damien used a detached line to represent either a person or a dog. Whether Damien's line was a person or a dog depended on the number of dots near the line. Two dots near a line represented a person and four dots near a line represented a dog.

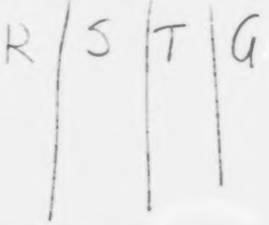
Figure 2
Four levels of performance for student-generated matrices.

Level 0

No diagram was drawn

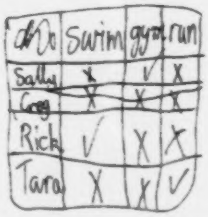
Level 1

Two distinct groups are not represented, however there is some similarity between the diagram and a matrix. *Lisa's explanation of this diagram revealed that she had represented the different sports using their initial letters but had not represented the people.*



Level 2

Two distinct groups are represented but there is an error in the representation. *Gemma has correctly represented the people on the matrix but has omitted tennis from the set of sports*



Level 3

Two distinct groups are represented correctly. *Helen has correctly represented the sets of people and sports, and the relationship between each set.*

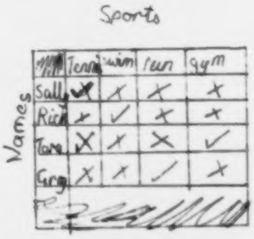


Figure 3
The Park task.

***The Park:** Jane saw some people walking their dogs in the park. She counted all the legs and found that there were 48 legs altogether. How many people and how many dogs?*

Visual differences in the surface details of students' part-whole diagrams complicate the assessment of the quality of the diagram. For example, although Ian's and Gemma's diagrams were visually similar, Gemma's diagram was of higher quality than Ian's because she accurately represented the total number of legs whereas he did not. Furthermore, although Candice's and Gemma's diagrams were visually distinct they were both assigned the same level albeit for different reasons. Thus, in determining the "accuracy" of part-whole diagrams attention to the theoretical prototype is required.

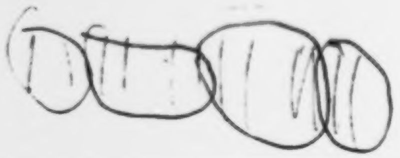
Figure 4
Four levels of performance for student-generated part-whole diagrams.

Level 0

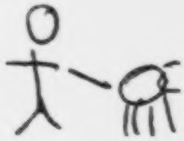
No diagram was drawn.

Level 1

A diagram was generated but neither the sets (the parts) nor the total (the whole) were correctly represented. *Ian's diagram was intended to represent the groupings of legs. However neither the number of legs nor the groupings was correct.*

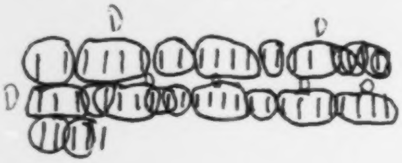


Both of the sets were correctly represented but the total was either omitted or incorrect. *Candice represented one of each set of legs correctly, however the total was not represented*



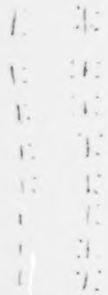
Level 2B

The total was correctly represented but at least one of the sets was incorrectly represented. *Gemma represented the correct number of legs however the grouping of legs was incorrect because there was a leg that was not included in the groups.*



Level 3

Both the sets and the total were correctly represented. *Damien correctly represented the sets of legs and the total number of legs. A line with four dots represented a dog and a line with two dots represented a person.*



CONCLUSION

The development of theoretical prototypes and levels of performance has the potential to make a difference to teachers' ability to assess the quality of diagrams and students' use of the diagram as a cognitive tool in the classroom. Knowing a student's level in diagram generation for particular types of problems and across the range of general purpose diagrams enables the teacher to pinpoint a student's difficulties, to provide effective intervention, and to monitor the development of diagrams as a cognitive tool in mathematics.

The theoretical prototypes for these tasks provided a useful guide for establishing the quality of students' diagrams. However the need for the prototype varied. On the matrix task, the theoretical prototype provided the guidance needed to establish a series of levels of performance. However after establishing these levels, the theoretical prototype is essentially redundant because subsequent diagrams can be classified visually without reference to the prototype.

On the part-whole task, the prototype assumes an ongoing role. Due to surface variation that occurs with part-whole diagrams, the prototype provides a needed reference point during the assessment process. For example, the prototype provides the justification for assigning Candice's and Gemma's diagrams the same level despite their substantive differences (see Figure 4). The prototype also indicates the particular support they need to develop their ability to generate effective diagrams. Whereas Candice needs to represent the total in her diagram, Gemma needs to represent the parts of the total accurately. Knowing this information is advantageous for teachers because specific components can be addressed during instruction and students' progress in representing these components accurately can be monitored.

Differences in the ease of assessing matrices and part-whole diagrams can be explained by the orientation of the diagram. Because matrices have unique visual characteristics, they are easier to assess visually, than part-whole diagrams that lack unique visual characteristics. The relative ease of assessing hierarchies and networks, which are also spatially-oriented diagrams (Diezmann, 1999) suggests that the various spatially-oriented diagrams are relatively easier to assess visually than conceptually-oriented diagrams.

Formative assessment serves a crucial role in the learning process. If students are to use diagrams as a cognitive tool, teachers need ways to determine what support is required by learners to achieve this goal. By assessing diagram quality and providing the necessary support, teachers can make a difference to problem representation (Diezmann, 1999), which is a crucial aspect of successful problem solving (Nickerson, 1994). Developing prototypes of essential problem components and identifying levels of performance in diagram generation provides teachers with a practical avenue for understanding and addressing some of the issues inherent in assessing the quality of students' diagrams and improving the effectiveness of diagrams as cognitive tools.

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ENHANCED MATHEMATICS LEARNING: DOES TECHNOLOGY MAKE A DIFFERENCE?

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This paper investigates the effectiveness of technology-based instruction in secondary mathematics, by comparing students' achievements resulting from technology-rich assignments with those achievements resulting from equivalent assignments presented in traditional format. In addition, the development of the technology-rich assignments, from traditional paper-based instruction and within existing curricula, provides an example of the relative ease of integrating technology into the curriculum. Within the context of mathematics, issues of attitude towards computers, motivation and gender differences are examined.

Background

Since educational computing today is still in the formative stages (Kaput, 1992; Joiner, 1996), its final impact on the structure of education cannot be completely extrapolated from current experimentation. Clearly there is a need for further research, so that bounds can be established as the field matures. Although there are a number of studies comparing the effectiveness of traditional instruction with computer-based instruction (for example McCoy, 1991; Morrell, 1992; Stick, 1997), none have been sighted that are specific to the area of Australian secondary mathematics. The honours thesis reported here (Dix, 1998) is therefore potentially important and of particular relevance in that it responds to the void of qualitative and quantitative data in this domain.

Research Questions

The broad issue of whether the use of technology enhances mathematics learning can be broken down into a number of specific inquiries, providing a foundation for the current study:

1. Is there a difference between achievement of students using computers to generate work and the achievement of students using the traditional methods of pen and paper?
2. Does the use of computers motivate students and influence their subsequent level of achievement?
3. Do attitudes towards computers differ between male and female students, and how are attitudes affected following a number of technology-rich lessons?

The Study Sample

The study sample involved students from two Year 8 Mathematics classes at a public Secondary school in the Adelaide metropolitan area. This school was selected on the basis that it typified current secondary education practice, and the computing facilities were both extensive and available for this investigation. Class A comprised 28 students, 11 female and 17 male, and Class B comprised 19 students, 8 female and 11 male. The differences in average age of students, on class and gender comparisons, were considered to be of insufficient size to bias the study in any way.

Of greater concern was the determination of any differences in mathematical ability between the two intact classes. The Standard Progressive Matrices Test (Raven, 1960) was selected as an appropriate test measure. Commonly referred to as Raven's test, it utilises pattern recognition to test non-verbal reasoning ability. Although no significant difference (at the pre-designated level of 0.05) in age was determined, a similar finding for ability levels between classes was ideally required. The results of the Student's two-tailed t-test for independent groups ($t = 1.17$, $p = 0.25$) suggest no significant difference. Mean class scores of 47.4 (SD = 5.6) and 45.7 (SD = 4.8) confirm the similarity in ability with a discrepancy of only 3% in favour of Class A.

The Comparative Design

In essence, two different instructional methodologies (technology-rich and traditional) were used to teach two classes of similar age and mix of ability. Within a curriculum topic, the majority of the unit was taught in exactly the same way to both classes. The assessable variation involved two assignments designed to satisfy the two formats, technology-rich and traditional, each requiring several lessons to complete. Results from the assignments and a common end-of-unit test provided a comparable measure of the effectiveness of the methodology (Morrell, 1992).

The Assignments

The use of the geometry-related topic, *Constructing and Drawing*, was predetermined by the school's curriculum. However, the topic was highly suitable for flexible delivery and for the purposes of this study. Within the topic, two assignments were designed which utilised and fulfilled the requirements of existing curriculum content and allowed for easy presentation in the two formats, technology-rich and traditional. The assignments, *Tessellations* and *Angle Sum in a Polygon*, had been taught successfully in previous years using traditional methodology. The conversion of the two assignments into a technology-rich format required little modification. To draw comparisons between achievement levels in each assignment, classes received both assignments but in alternating format. Thus, neither class was disadvantaged since each was given the opportunity to use computers in mathematics, and curriculum coverage was equally achieved. Special attention was given to ensure that methods used in the two formats fulfilled the same objectives, preventing study bias. Furthermore, the possibility of bias arising from differing teaching styles between classes was minimised by having the same teacher teach both classes for the respective units of work. A visual representation of the research design is presented in Figure 1.

Figure 1.
A Visual Interpretation of the Research Design

	Two assignments in Geometry	
Two formats	**Tessellations**	**Angle Sum in a Polygon**
Technology - rich method	Class A	Class B
Traditional method	Class B	Class A

Generally when assignments are given, students have the opportunity to work on them at home. In the present study, such a procedure would unfairly disadvantage those students doing the technology-rich assignment since they, more than likely, would not have access to the relevant software, or perhaps even to a computer. To prevent this possible source of bias, students were only given class time to complete the work.

The length of each assignment was designed to be achievable in the three lessons allocated. It was anticipated that not all students would finish at the same time, and therefore extension work was provided for those who completed the assignment quickly. By the end of the third lesson, however, all students were required to submit their assignments, incomplete if necessary. Approximate length of time on task was monitored, and all efforts were made

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to provide an equal amount of support to every student. The assignments were structured so that marks were specifically allocated for method as well as accuracy, reducing the possibility of personal judgement.

Geometry Software

The software package, *Geometer's Sketchpad* (Klotz, 1991), used for both technology-rich assignments was chosen, principally, because it was judged to be pedagogically appropriate (Pegg and Daveys, 1991; Kaput, 1992; Arnold, 1996). A second factor in selecting the software package was its ease of installation on the school network.

Although subjects in the two classes considered themselves to have at least some level of computer experience, none had used this particular software package. Three orientation lessons were therefore conducted for each class. These highly structured lessons, while still covering the curriculum, exposed all students to the basic geometric manipulation tools needed for the subsequent assignment.

Anecdotal Assignment Questions

To gain a greater understanding of how each student felt specifically about mathematics and the assignments, the following questions were asked at the completion of each assignment.

1. If this assignment had to be done *with/without* the computer, do you think it would be easier or harder? Explain why.
2. Write down what you currently think of maths and how it makes you feel.
3. What did you like/dislike about this assignment?

Computer Attitude Survey

To assess general attitudes about computers, an attitude scale by Jones and Clarke (1994) was adopted in a slightly modified form. The adapted scale was distributed to both classes before the start of the study, and again at its completion. The instrument comprised 38 questions to be answered on an ordinal scale of 5 choices, ranging from strongly agree to strongly disagree. Although the primary purpose of the scale was to establish the effects of motivation on gender and achievement, the readministration of the scale provided a brief insight into longitudinal changes, if any, of these attitudes.

End-of-Unit Test

The final measure used in this study made use of the traditional pen and paper, end-of-unit test. To be completed in a single 45-minute lesson, the test was restricted to 10 questions that were posed in a variety of formats to accommodate a range of learning styles. Although the 10 questions were written to assess all major objectives covered in the topic, two questions were designed as specifically pertinent to the assignments. By assessing the level of retention and understanding, the test results provided another means to gauge the effectiveness of computer technology.

Statistical Analysis

Descriptive statistics (means and standard deviations) were used to describe the central tendency and dispersion on all measures. To test for differences between groups, Student's t-test was used. This was the appropriate statistic as just two groups were compared in each case. The 0.05 level of significance was set for the rejection of all null hypotheses.

The assignments and end-of-unit test were graded and the raw scores provided the database upon which statistical analysis was performed. To assist in analysing the student attitude questionnaire, the five categories of student responses on the survey were assigned numerical values: strongly agree = 5, agree = 4, undecided = 3, disagree = 2, and strongly disagree = 1. To generate all statistical calculations, Microsoft Excel spreadsheet was used. In particular, the Student's t-test and its associated probability were utilised to test for gender and classes differences between groups.

Similarly, correlational analysis was applied to check validation of the assessment tools. Pearson's product moment correlations between the Raven's test, assignments, and end-of-unit test are presented in Table 1. For the purposes of correlation, subjects with missing data were removed thereby reducing the degrees of freedom (with 72 df, a correlation of 0.23 is required at the 0.05 level).

Table 1

Correlation Matrix for Five Measures of Achievement – the Raven's test (standardised measure), the Tessellations and Angle sum in a Polygon Assignments, the end-of-unit test, and Q.1 (angle sum) and Q.2 (tessellations)

	Tessellations	Angle Sum	End of unit test		
			Total	Q.1	Q.2
Raven's Test	-0.01	0.11	0.45*	0.37*	0.23*
Tessellations		0.31*	0.38*	0.07	0.21†
Angle Sum			0.25*	0.29*	-0.14

* denotes significance at the 0.05 level † denotes significance at the 0.10 level

Using Raven's test as a standardised test measure, comparison with the end-of-unit test questions, reveals that the significant correlation of 0.45 indicates that the developed test measure is an adequate index of mathematical achievement. Internal consistency between the assignments, irrespective of methodology, is indicated by a significant correlation of 0.31 at the designated level. Although the assignments do not significantly correlate with the Raven's test, they do so with the end-of-unit test, suggesting they are adequate assessment tools for the purposes of this study.

Results

Statistical analysis of the scores resulting from the various test measures, and qualitative analysis of the anecdotal questions and informal observation, provide answers to the research questions.

Differences in Achievement

Analysis of the assignments suggest no significant statistical bias towards traditional or computer based methodology. However, assignments completed via traditional methods achieved marginally higher averages within each class.

Clarification of the assignment findings is supported by the results from two questions in the end-of-unit test, specifically pertaining to the assignments. Irrespective of class, the answers to those questions learned through traditional methods marginally outperform those learned through technology-assisted methods. The differences between these values, however, are not significant.

Thus, in response to the first research question, there are no significant differences between the achievement of students using computers to generate work and the achievement of students using the traditional methods of pen and paper. These finding, however, should be viewed in the light of the circumstances under which the computer-enriched assignments were completed. Prior to this study, the subjects had no previous experience with the computer software, *Geometer's Sketchpad.* Comments from a number of students suggest that they were actually learning about two things, both the assignment and how to use the program. The achievement of outcomes, gained through unfamiliar software, similar to those gained through traditional methods, reflects the relative ease of using the software package. Accordingly, on the basis of this study, *Geometer's Sketchpad* commends as an appropriate and effective mathematics tool for teaching and learning geometric principles.

The Influence of Computers on Motivation

Motivation towards learning due to the use of technology has been qualitatively analysed though the end-of-assignment questions and through informal observation. In general, both classes:

- agree that computers made the task easier;
- value higher levels of accuracy gained through using a computer;
- are less willing to leave their computer-based work to attended break or other classes;
- find *Sketchpad* easy to manipulate and readily achieve success independently;
- produce more computer-based work (not evident in final marks); and
- feel negative about maths but less so when computers are used.

In the light of the findings, response to the second research question is mixed. The use of computers in mathematics does appear to positively influence student motivation. However, a corresponding shift in achievement is not apparent.

Gender Differences and Changes in Computer Attitude

Administration of the Computer Attitude survey, prior to and after the study period, supports quantitative analysis of gender differences and changes in attitude over the course of teaching. Statistical analysis of results, presented in Table 2, indicates that differences in attitude between male and female students are significant, for both the pre- and post-computer surveys. In addition, longitudinal changes in attitude reveal a significant positive change in male attitude towards computers, whereas a similar shift for female students is also evident, though not at the 0.05 level of significance.

Table 2
Mean and Standard Deviations on the Computer Attitude Scale together with t-test Results for Gender Differences and Differences Between Pre- and Post-Computer Surveys

	Pre-computer survey Mean (SD)	Post-computer survey Mean (SD)	t-test (paired)	probability (2-tailed)
Male	149.3 (16.0)	153.9 (16.7)	2.80	0.007*
Female	137.6 (15.3)	140.6 (15.6)	1.50	0.14
t-test (unpaired)	2.51	2.77		
probability (2-tailed)	0.02*	0.01*		

* denotes significance at 0.05 level

In response to the final research question, differences in attitudes towards computers do exist between male and female students, and are positively affected, significantly so for male students, following a number of technology-rich lessons.

Discussion

In addition to the findings relating specifically to the research questions, a number of equally important observations were made.

Although results indicate that students achieved similar scores for the technology-rich and traditional assignments, the quantity of work produced for each assignment varied. It was evident that much more exploration and investigation by students occurred in the technology-rich environment, resulting in the completion of several versions of each assignment, and extension of the task beyond the original scope (eg. animating the tessellation pattern). Due to the limited method of assessment, there was no component in the marking scheme to accommodate such achievement, effectively biasing the study against technology-rich methodology, a problem alluded to by Lesh (1990). However, it does provide a direct example of the development of new curricula that would otherwise be impossible without the use of technology, and the need for authentic assessment methods that respond to these changes.

This study observed clear differences in attitude and approach between male and female students to the use of computers in mathematics. Male students were more willing to experiment with the software, deviating from the assigned task, while female students were much more task focused and particular about the appearance of their work. Despite the differences, a significant positive change in attitude of the participants resulted.

Through the use of anecdotal questions, this study revealed that most students begin their high school mathematics career with a general dislike for mathematics. Encouragingly, the inclusion of computers in the mathematics curriculum appeared to alleviate, to some degree, negative attitudes felt by students towards mathematics.

Findings from this study conclude that the computer-enriched lessons provided a positive learning experience for the participants. The preference for computer-assisted learning is not surprising, considering the generally positive attitudes towards computers, evidenced by the results of the computer attitude survey. Although not an objective of this study, it should be noted that from a teaching perspective the computer-enriched lessons provided an equally positive experience for the teacher.

Implications for Education

As a result of this study, a number of implications for education, and for junior secondary mathematics in particular, arise. These findings, however, should be considered in light of the limited nature of the study.

Above all else, this study highlights the need for authentic assessment, especially when a dynamic software package, such as *Geometer's Sketchpad*, is involved in the learning and assessment process.

The wide acceptance of *Geometer's Sketchpad* by participants provides confirmation and recommendation of the program as a pedagogically appropriate mathematics construction and exploration tool.

By allowing students to pose, investigate, and extend problem situations, an environment can be established in which students (and teachers) can recognise the power and usefulness of mathematics.

Gender differences in attitude and approach to computers in mathematics identifies the need for teachers to provide an environment that supports and encourages a diverse range of learners and their needs.

Integration of technology into mathematics, by modifying existing curricula, is a viable and effective method for current curriculum development. In the longer term, however, complete curriculum reform may be required on a whole school level.

Recommendations for Future Research

Due to the limited nature of this study, and the apparent absence of similar studies, multiple directions for future research exist.

Clearly, further research examining different age groups, and a larger population, is needed. The issue of equity, arising from computer illiteracy, poverty, disability, or non-English speaking background, was not applicable to students in the current study. However, the same may not be so for students in other year levels, or in other schools. Accordingly, research into the effect of these possible causes of inequality should be considered.

Although the use of technology in education has traditionally predominated in mathematics, its occurrence in other curriculum areas, such as art, science, and English, is increasing. Further research is necessary into the effectiveness of educationally appropriate software packages and their application within their respective curriculum areas.

For many students the use of computers in any curriculum area, aside from technology classes, may be a novel experience and a source of motivation. Reflecting a concern of this study, changes in motivation due to the novelty of the experience could not be completely discounted. Whether levels of motivation towards mathematics (or any curriculum area) are maintained or increase as the novelty *wears off*, provides a further avenue for research.

The research and development of effective technology-rich curricula requires a corresponding adjustment in the assessment process. Due to the nature of dynamic software, current forms of assessment may no longer be appropriate indicators of achievement. Although authentic assessment provides direction, further research is needed to validate it as an appropriate method of assessing computer-assisted achievement.

In many evaluative studies, the computer is perceived as an independent variable, the direct effects of which can be observed and quantified. Due to unavoidable limitations, the current study also adopted this approach. However, the introduction of technology into the classroom clearly changes the whole learning environment. Thus, there is a need for evaluative research into the impact of technology on learning that focuses on the total system (Rowe, 1996; Neal, 1998). Such research may most effectively be achieved though in depth qualitative rather than quantitative research.

Finally, the concern of context validity resulting from the rapid pace of curriculum development highlights the need for ongoing research and affords the unique opportunity for school teachers, students and university researchers alike, to participate in the process of curriculum design and reform.

The expectation that a clear result would emerge from a study of such short duration was considered optimistic. However, the findings do provide greater insight as to the effectiveness of computers as a tool to enhance mathematics learning. In short, this study suggests that computers do have the potential to make a positive difference in both the learning and the teaching of mathematics.

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OF COURSE I CAN['T] DO MATHEMATICS: ETHNICITY AND THE STEREOTYPING OF MATHEMATICS

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A new instrument aimed at measuring the extent to which mathematics is stereotyped as a gendered domain was recently trialed. Included among the participating schools were two with distinct student populations from strong, but different, ethnic and religious backgrounds. In this paper we report the findings from comparisons made between the responses of students from the two schools (N = 75 and N = 67) and those of students from other schools participating in the trial (N = 394). The results indicate that ethnic/cultural backgrounds influence stereotyping of some dimensions of mathematics education.

INTRODUCTION AND PREVIOUS RESEARCH

In the past, the study of mathematics and participation in related jobs were viewed as more suitable for boys than for girls. This was a strongly held belief in society at large. More boys than girls were found to study mathematics and, on average, boys outperformed girls. In the mid 1970s, girls were identified to be educationally disadvantaged with respect to mathematics and much effort was expended to re-dress inequities (Leder, Forgasz, & Solar, 1996).

Times have changed. Traditional societal expectations of men's and women's roles at home and in the workforce are now challenged. Although mathematics participation rates, particularly in the most demanding mathematics options offered at the school level continue to favour boys, the gap in performance levels has closed. The recent *Third International Mathematics and Science Study* [TIMSS] results, for example, revealed relatively few differences in the performance, on average, of Australian girls and boys in the middle school years. Far greater differences in performance were noted when comparisons were made between students who spoke English at home, and those who did not, an indicator often used as a crude measure of cultural background. At both the lower and upper year levels, students born in a non-English speaking country but using English as their main language at home, on average, did better than students born in an English speaking country and using English as their home language. The latter group, in turn, did better than students born in an English speaking country but using a language other than English in the home (Lokan, Ford, & Greenwood, 1996). However, in other countries involved in TIMSS - including Belgium, Ireland, Hong Kong, Switzerland and England (Beaton et al, 1996) - more traditional gender differences favouring boys continued to be reported. Collectively, these findings suggest that factors other than gender alone contribute to the observed performance levels of boys and girls.

The stereotyping of mathematics as a male domain is often included in models postulating explanations for the under-representation of women in the mathematics and science fields (see Leder, 1992). More recent research into equity issues within mathematics education (Secada, Fennema, & Adajian, 1995) recognises the interacting roles of gender and cultural backgrounds on individuals' beliefs and attitudes. Interactions between gender and cultural background were also found by Bishop and Leder (1999) in a recently completed study involving students from eight co-educational secondary schools in the metropolitan area of Melbourne. For example, there appeared to be greater differences in self appraisal on a host of variables between boys from Anglo cultural and non-Anglo cultural backgrounds than between girls from these groups.

The Fennema-Sherman *Mathematics Attitude Scales* [MAS] (Fennema & Sherman, 1976) have often been used in research on gender differences in mathematics learning outcomes. The need for a new instrument to replace the 'Mathematics as a Male Domain' subscale was recently established (Forgasz, Leder & Gardner, in press). These authors argued that the assumptions underpinning the development of some of the items on the MD scale were anachronistic and that the interpretation of responses to some items were no longer valid.

A new instrument, loosely based on the 'Mathematics as a Male Domain' subscale, and aimed at measuring the extent to which mathematics is now stereotyped as a gendered domain has recently been trialed (Forgasz, Leder, & Barkatsas, 1998). More specifically, the themes explored overlapped with those tapped by the 'Mathematics as a Male Domain' scale but the response format differed substantially. The results of the trial of the new instrument challenged some previous research findings while others were confirmed (Forgasz, Leder & Barkatsas, 1998). Taken together the data strongly implied that views on the stereotyping of mathematics are changing. Given Australia's multicultural profile, it was of interest to know if varying views would be found among students of different cultural backgrounds. Included among the schools participating in the trial was one school affiliated with the Jewish community (school J) and one associated with the Greek community (school G). Students attending these schools share strong, yet vastly different, ethnic and religious backgrounds. In this paper, we present findings comparing the views of students at each of these two schools with the views of students from the other schools (combined as school O) in the trial.

THE STUDY

The Instrument

In developing the items, we drew on previous research findings about gender issues in mathematics learning. Completed items were read and modified, if appropriate, by 10 volunteer mathematics educators. Remaining items were attempted by some two dozen volunteer grade 7 to 10 students. Various items were omitted or further modified on the basis of reactions obtained from this group. Following these trials, 24 items remained for inclusion in the instrument (see Table 1). On the survey, each item took the form of a statement to which students were asked to select one of the following response categories:

BD: Boys definitely more likely than girls
BP: Boys probably more likely than girls
ND: No difference between girls and boys
GP: Girls probably more likely than boys
GD: Girls definitely more likely than boys

Table 1
Items included in the survey instrument

Qn	Item	Qn	Item
1	Think mathematics will be important in their adult life	*13*	Think they did not work hard enough if they do not do well in mathematics tests
2	Think the mathematics test must have been easy if they do well	*14*	Give up when they find a mathematics problem is too difficult
3	Get on with their work in mathematics classes	*15*	Tease girls if they are good at mathematics
4	Are not good at mathematics	*16*	Care about doing well in mathematics
5	Need mathematics to maximise future employment opportunities	*17*	Think it is important to understand the work in mathematics classes
6	Are thought of as 'nerds' if they do well in mathematics	*18*	Expect to do well in mathematics
7	Worry if they do not do well in mathematics	*19*	Think it is OK to be excellent at mathematics
8	Are asked more questions by the mathematics teacher	*20*	Mathematics teachers spend more time with them
9	Get the wrong answers in mathematics	*21*	Distract other students from their mathematics work
10	Mathematics teachers think they will do well	*22*	Tease boys if they are good at mathematics
11	Need more help with mathematics	*23*	Likely to work with computers in future jobs
12	Find mathematics easy	*24*	Consider mathematics to be boring

The Sample

In all, 536 (281M, 255F) Grade 7-10 students from eight Australian schools completed the survey. Of these, 75 (34M, 41F) were from school J and 67 (35M, 32F) from school G. Results from these two schools were compared with each other and with data from the other participating schools (combined as school O; N=394: 212M, 182F).

Test Administration

After obtaining the necessary ethical consent the instruments were distributed to the participating schools. Administration procedures were discussed with the teachers who administered the questionnaire within class time.

Data Analysis

For each survey item, a chi-square test was conducted to determine if there was a statistically significant difference in the distribution patterns across the five response categories for the three groups of schools. Significance was set at the .05 level.

RESULTS

The results can be summarised as follows:

- For each item on the survey, the most frequent response category was generally that there was 'no difference' between girls and boys
- On 16 items (Qns 2, 6, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 22, 24), more students from school J than from schools G and O responded that there was no difference between girls and boys. On two items (Qns 3, 13) more students from school G than from the other schools responded that there was no difference between

boys and girls. No difference was scored higher by students from school O than by students from schools J and G on the remaining six items (Qns 1, 4, 5, 7, 21, 23).

- For 15 items there were no statistically significant differences in the distribution patterns of the responses of students in the three groups. The nine items (Qns 1, 3, 6, 7, 11, 12, 15, 20, 23) for which statistically significant results were obtained and descriptions of how the groups differed on these items are shown on Table 2.

Table 2
Items with statistically significant differences

Qn	Result	Sig. level
1	Overall, students felt that boys and girls would not differ in thinking that mathematics was important to their adult lives (75%). However, a higher proportion of students from school J (22%) than from schools G (14%) and O (16%) considered boys more likely to hold this view, and proportionally more school G (15%) students that it would be girls (school J: 5%, school O: 9%).	$\chi^2=16.01$ $p<0.05$
3	The majority of students from all schools indicated that girls (61%) were more likely than boys (7%) to get on with their work in mathematics. Students from schools G (66%) and J (69%) believed this more strongly than students from school O (58%). No student from school G believed that boys get on with their work in mathematics. Only about 1/3 of all students perceived that there was no difference between boys and girls.	$\chi^2=20.46$ $p<0.01$
6	The data indicated that many more students overall considered boys (47%) than girls (12%) to be thought of as 'nerds' if they did well in mathematics. Students from school O (54%) held that view more strongly than students from schools G (31%) and J (33%). Students from school J (61%) believed more strongly than at schools G (47%) and O (35%) that there is no difference between boys and girls being considered 'nerds'.	$\chi^2=35.36$ $p<0.01$
7	Overall, girls (49%) were considered more likely than boys (12%) to worry if they did not do well in mathematics. Students from school G (25%) believed that this was definitely the case more strongly than students from schools J (10%) and O (10%).	$\chi^2=17.93$ $p<0.05$
11	At all schools, boys (37%) were considered more likely than girls (8%) to need help in mathematics. This belief was held more strongly at schools O (35%) and G (36%) than at school J (22%). Compared to schools G (57%) and O (58%), students at school J (75%) strongly believed that there is no difference between boys and girls.	$\chi^2=15.62$ $p<0.05$
12	Much higher proportions of students from schools O (29%) and G (32%) than from school J (9%) believed that girls were more likely than boys to find mathematics easy. Students from school J (75%) were more likely than students from schools O (60%) and G (61%) to indicate that there is no difference between boys and girls.	$\chi^2=24.91$ $p<0.01$
15	The data indicated that many more students from all schools considered boys (44%) more likely than girls (12%) to tease girls who are good at mathematics. Students from school G (60%) held that view more strongly than students from schools O (41%) and J (41%). Students from school G (31%) also perceived less strongly than students from schools O (46%) and J (51%) that there is no difference between girls and boys.	$\chi^2=20.15$ $p<0.01$
20	A higher proportion of students from school O (26%) than from schools J (7%) and G (17%) responded that teachers spend more time with boys than with girls. Students from school J (74%) believed more strongly than students from schools O (56%) and G (60%) that there was no difference between girls and boys in the time teachers spent with them.	$\chi^2=24.83$ $p<0.01$
23	Many more students overall considered that boys (29%) were more likely than girls (12%) to work with computers in their future jobs. The pattern of responses was similar at each school. But, the largest difference favouring boys (54%) over girls (3%) was found at school J. Students at school O (54%) believed more strongly than students from schools J (43%) and G (35%) that there was no difference in the likelihood of girls and boys working with computers in the future.	$\chi^2=29.00$ $p<0.01$

Key: school J: affiliated to Jewish community; school G associated with the Greek community; school O = all other schools

DISCUSSION

There were 16 items for which there were no statistically significant differences in the response patterns by school. Although the general distributions of responses across the five categories were similar for the remaining nine items, statistically significant differences were found. The nine items related to: future careers (1, 23), classroom factors (3, 15, 20), mathematical ability (6, 11, 12), and confidence (7). Cultural factors may be implicated in understanding the differences that emerged.

How Students from Ethnic Background Schools Differed from 'Others'

Responses to items 3, 6, 20 and 23, indicated that students at the two ethnic schools, J and G, held different beliefs from the others. They believed more strongly that girls got on with their work in class (O: 58%: G: 66%, J: 69%), and less strongly that teachers spent more time with boys (O: 25%, G: 17%: J: 7%), that boys were more likely than girls to be considered 'nerds' if they were good at mathematics (O: 54%, G: 31%, J: 33%), and that there was no difference in the likelihood of girls and boys working with computers in their future careers (O: 54%, G: 35%, J: 43%).

Although the students at the ethnic schools viewed some aspects of mathematics education differently from students at other schools, there was a clearer pattern in the differences that emerged when the data were examined to see how students at each ethnic schools differed from the others.

How Students at School J Differed from the Others

For 16 of the 24 items, including five out of the nine items on which significant differences were found - the three 'ability items' (6 - 'nerds', 11- need more help, 12 - mathematics easy) and two items related to classroom factors (15 - tease successful girls, 20 - teacher time) - school J students believed more strongly than students at schools G and O that there was no difference between girls and boys (see Figure 1, for example). Students from Jewish schools are known in the broader community to achieve excellent results in grade 12. In explaining their outstanding performance on the 1996 Victorian Certificate of Education, Lee Dow (1997) claimed that "the achievement of Jewish schools… points less to social and financial advantage than to a community… recognised for its respect and value of educational excellence" (p.19). Does this community's high expectations contribute to the apparently more equitable views about the gender-appropriateness of many aspects of mathematics education held by the students at school J? Simultaneously, however, does the community hold strongly to more traditional views of future career paths? It was noteworthy that with respect to items related to future careers (3, 23), school J students believed more strongly than students from the other schools that boys were more likely than girls to consider mathematics important in their adult lives and to use computers in their future work (see Figure 2).

Fig. 1.
Frequency Distributions for Item 11 by School and Gender

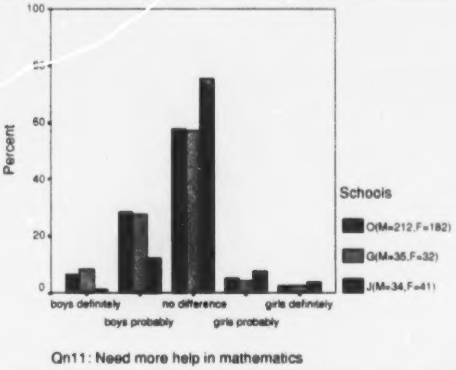
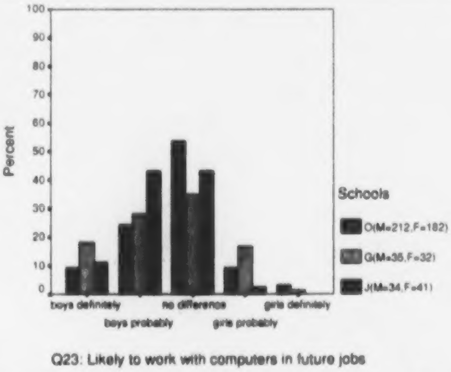


Fig. 2.
Frequency Distributions for Item 23 by School and Gender



How Students at School G Differed from the Others

Students at school G differed from the others in their beliefs on three items - two classroom-related factors (3, 15) and the confidence item (7). They believed most strongly that girls would definitely be more likely than boys to worry about not doing well in mathematics, and that boys were more likely than girls to tease girls who did well in mathematics. Overall, only 7% of students believed boys were more likely than girls to get on with their work in mathematics classes. However, there was not a single student at school G who indicated that this was the case.

In a study which focussed on the experiences of Greek students in a government high school located in a low socio-economic area, Polesel (1997[illegible]ed that:

> It would seem that certain characteristics of Greek culture, which value and support the education of boys at the expense of girls, combine with school-based gender stereotypes to undermine the confidence and expectations of these Greek girls, particularly in the gender sensitive domain of mathematics. (pp.41-42)

The implications of school G students' response patterns on the classroom-related items are that boys 'muck around' in mathematics classes, do not get on with their work, and tease capable girls. This scenario has 'macho' overtones. The stereotyped directions of these responses appear to support Polesel's (1997) claims and are also consistent with previous findings that reflect classrooms which can be considered 'male domains' (e.g., Forgasz & Leder, 1996).

In summary, the aspects of mathematics education that have traditionally been viewed to reflect mathematics as a 'male domain' were explored in the present study using a new instrument. In general, the patterns found for the two ethnic schools were similar to those in all the other schools. Some specific differences were noted, however. In the school associated with the Jewish community, students were less stereotyped in their views than the other students on the majority of items. However, they reflected more traditionally stereotyped beliefs that mathematics was more likely to feature in the future careers of males than of females. At the school affiliated with the Greek community the image of mathematics classrooms portrayed was more strongly consistent with a 'male enclave' than could be inferred from the responses of students at the other schools. Taken together, these findings suggest that cultural background has a role in shaping students' belief systems.

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FINAL WORDS

Findings from the present study support the hypothesis that gender and culture interact to influence students' beliefs about the gender-appropriateness of mathematics. However, more work is needed to confirm the findings reported here and to explore patterns among students from a range of different cultural backgrounds. In the present study, data were gathered from only two schools with strong ethnic associations. The results were also based on findings from a trial version of a new instrument which requires further refinement and a second administration to establish its reliability. However, a number of questions are raised by the findings. Do students in schools with strong ethnic associations hold the same views as their counterparts who are enrolled in other schools with a mixed ethnic enrolment profile? Can equity be attained in schools with strong ethnic associations if their purpose is to perpetuate, and not to challenge, the culture they represent? Do male and female students from the same ethnic background differ in their beliefs about the stereotyping of mathematics?

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STUDENT PARTICIPATION: PHENOMENON AND ANALYSIS

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Students' participation in classroom activities is considered to constitute cognition, and is evidenced in student-teacher interaction. In this paper, elements of the complex phenomenon 'student participation' are described with reference to an episode where a student was discussing her solution of a three-dimensional vector problem with her teacher. The multiple-paradigmatic research method used is considered in relationship to how the episode has been portrayed, and is proposed as being suited for both the teaching and research purposes of a teacher-researcher.

INTRODUCTION

This paper reports the beginning of an ongoing study where students' classroom participation is being explored. The study has started with mapping the nature and development of students' participation, and observing students' associated cognitive outcomes, over 18 lessons, when students were learning a topic on three-dimensional vectors and I was the teacher. The problem that initiated the research was that, in the past, my students have had difficulty understanding some of the abstract vector concepts that are part of the Year 11 Geometry and Trigonometry course in Western Australia, and the literature (Dreyfus, 1994) suggests other students have similar difficulties.

The teaching/research agenda of encouraging students' classroom participation was in order to explore the suggestions that the learning processes of *participation*—knowing socio and socio-mathematical practices (Cobb, 1998), and *acquisition*—developing personal understanding (Sfard, 1998), are inter-related, and that both metaphors should be adopted for teaching and research: "Each has something the other cannot provide. . . . When two metaphors compete for attention and incessantly screen each other for possible weaknesses there is a much better chance of producing a critical theory of learning" (Sfard, 1998, pp. 10-11). To motivate students' participation, I adopted a problem-centred learning approach (Wheatley, 1993) where students worked together on 'real-life' problems to develop their understanding of three-dimensional vectors.

One challenge was to establish a research methodology that did not interfere unduly with my teaching, but instead informed it. The personal experience methodologies of hermeneutic phenomenology (Pinar & Reynolds, 1992; van Manen, 1990) and narrative inquiry (Connelly & Clandinin, 1988), and the attitude of 'reflective practitioner' (Schön, 1987) guided, and continue to guide the research. This paper describes the research method that emerged early in the study. The method is discussed in relationship to an episode where I was working with a student on a one-to-one basis, during the first lesson on the vector topic. Discussion of the episode illustrates how I used the literature to explicate the nature of participation as experienced in the classroom, in order to refine my research/teaching focus. The purpose of the paper is to illustrate how personal experience research methodologies can be used to develop understanding of professional practice from both teaching and research perspectives.

THE RESEARCH METHOD

Simply, hermeneutic phenomenology (van Manen, 1990) is experiencing a phenomenon and interpreting it through writing, while narrative inquiry (Connelly & Clandinin, 1988) is writing about experience over time. The methods of data generation, the quality criteria and the theoretical framework for my interpretive inquiry are provided below.

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Data Generation

The site of the teaching/research was a private, all girls' college in Western Australia. The sources of data were seventeen students in a Year 11 Geometry and Trigonometry class, myself as their temporary teacher replacing their regular teacher who was on leave, and a colleague who was teaching a parallel class. All eighteen lessons on three-dimensional vectors were video-taped from a back corner of the room, with the video-recorder set to record continuously. The dialogue on the video-tapes was transcribed. A journal was kept, recording my impressions of lessons and conversations with the teacher of the parallel class, with whom I discussed the teaching and learning outcomes. Other data, which have been used minimally for what is reported here and are not described in detail, were student responses to three questionnaires, interviews with students. As well, students' worksheets were collected and photocopied.

Quality Criteria

For a personal-experience research approach, van Manen (1990) suggests that to "render a human science text a certain power and convincing validity . . . . our texts need to be orientated, strong, rich and deep" (p. 151). The episode that is presented here is orientated around the research question "What *is* participation?" and, while writing it, I paid attention to the student it concerns, in what I see as an honest way so as not to misrepresent her. The criteria I used were that the portrayal of classroom life is strong, believable and coherent and, as a check that this episode and others 'rang true', I adopted McNiff's (1995) suggestion to have a critical friend, a past teaching colleague read them.

A Framework for Interpreting Classroom Interaction

As the reader, you may choose to refer to the next section as you read this one, because the classroom episode that is described there contains (a) text-boxes indicating theoretical perspectives, which describe components of the writing. In the episode, my experiences are represented with (b) conversation sequences from the video-transcript and with (c) reflective comments. The episode itself is broken into segments, each accompanied by (d) a discussion in terms of the literature. In addition, (e) my interpretations of elements of 'participation' are inserted in italics into the discussion, an approach I adopted from reading Lakoff & Johnson (1980). The italicized inserts are intended to capture the elements of participation in the paragraph they are attached to, and as thought provoking for the reader. They contributed to the description of 'participation' in the conclusion.

The lines of dialogue that form the basis of the portrayal of the episode are from the video-transcript and are linked together with comments, e.g., 'Jane continued...', rather than being reported in the bald form usual for research reports. This linking is to make for smooth reading as in the genre 'short story'. Some of the comments are a record of my pre-reflective perceptions (Pinar & Reynolds, 1992) and reflective responses while living the classroom experiences, which I re-viewed or re-lived by watching the video recording. The episode or story, therefore, expresses personal 'knowing-in-action' and 'reflection-in-action' (Schön, 1987), where action includes conversation. The text also includes elements of 'reflection-on-action' (Schön, 1987), reflection which involved looking back at the incident, but which didn't impact on it.

I purport to portray only one of the many possible realities of the classroom: the voice of a teacher at a particular time, the time of writing the account. This research approach is in accordance with a constructivist epistemology (Noddings, 1990; Taylor, 1998) and assumes understanding is personal: that there are multiple realities for any event, none privileged over others.

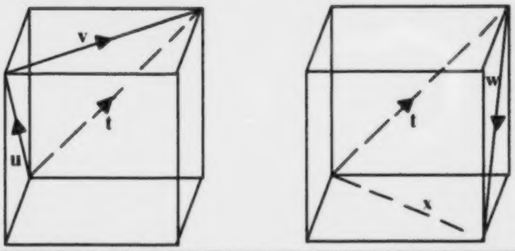
However, interpretation did not rely solely on personal experience. My classroom experience was shared with students. There was shared meaning in our participating in conversation and activity together. This shared understanding can be cast in terms of the constructionist metaphor of mind "actors in conversation" (Ernest, 1995, p. 482) where understanding rests in the "dialogic space" (Gergen, 1995, p. 46) between the actors. Resting in space implies understanding that wasn't personal. It was 'knowing in participation' (Sfard, 1998), in co-ordinating our interactions with each other. Elements of this knowing are evident in the comments that are used to link the lines of transcribed dialogue in the episode.

THE EPISODE: STUDENT-TEACHER PARTICIPATION

The episode illustrates a student's difficulties when the class was involved in small-group problem solving. The student was working by herself, as her working partner was absent. First, the problem (Schools Council, 1978) that students were working on is provided so that events can be followed by the reader.

A VTOL (vertical take-off and landing) aeroplane, in a display of precision flying, is to take off at an angle of 45° in a vertical plane, which has a bearing of 180° (vector **u**). After flying 100m it is to turn, and fly horizontally for 100m on a bearing of 045° (vector **v**). The final movement is to land at 45° to the horizontal, again flying at 180° (vector **w**). What is the resultant **x**, of the vector addition **u** + **v** + **w**. Give the magnitude and direction.

Figure 1
Diagram Provided with the Problem



I was helping Cheryl interpret the diagram for the problem. Jane, who was sitting in front of Cheryl, turned around to join in.
I was saying:
"You have to work out where it [the plane] ends after the three legs [of the flight path]", when Jane, pointing to the angle that x made with the lower back edge (see Figure 1), asked:
"Will it be 45°?
I answered:
"Yes", then paused, unsure, and continued:
"Yes, it will be 45° because they are equal. But will it? Not necessarily?"
Jane asked:
"Isn't it a cube?"

Reflection-on-action.

I had been thinking of individual vectors each framed by a separate face of the figure, rather than viewing the figure as a whole, so now had to switch to Jane's approach:
"But here you have got a diagonal, not a side of 100. No it's not a cube. Why isn't it a cube? Or is it?"
Jane was certain:
"It's a cube", but I was still doubtful, replying:

Knowing-in-participation.

"Okay."
Jane argued:
"The diagonals are equal", convincing me she was right:

Reflection-in-action

"Okay. Yes it is, because all the diagonals are equal."
Participation is negotiation.

Gergen's (1995) social constructionist view is "what we count as knowledge are temporary locations in dialogical space" (p. 30) and he suggests that:

> authority [for knowing] is socially accorded, and within most academic spheres it is typically given to those occupying a given discursive position. Thus, anyone may be an authority who occupies the position, if permitted to do so by a relevant social group. The ideal in this case is not knowing that something is the case, but knowing how to produce language that will be accorded status. (p. 31)

In being able to articulate her knowing that the figure was a "cube", Jane shifted the authority from me to herself. Her seeing the diagram as a cube, instead of as separate sectors of the flight path, potentially reduced the cognitive demand of the problem because of the geometric properties that might be assumed: Eylon and Lynn (1988) describe how the "working memory demands of problems" (p. 266) are different for students according to the way their knowledge is organised and the "task representation that they choose" (p. 266). However, it wasn't sufficient that Jane recognised the more efficient representation, she had to convince me of it, in order to continue. *Participation is constituted by and constitutes cognition.*

For some, including me initially, a cube brings up the image of a block with edges of equal length and the given information in the question didn't directly satisfy this requirement. Jane established a higher level of thinking by justifying her classification of "cube" with the identification of equal diagonals. My confirmation was a signal for Jane to go on, and implied that we were talking about the diagonals drawn on the faces—that the internal diagonal through the centre of a cube is longer than the other diagonals wasn't an issue.*Participating is interpreting language.*

> Jane continued:
> "I think it [the resultant vector] goes from there to there."
>
> Element of participation.
>
> I moved forward to see her work, and she explained:
> "So that's 100 and it is a bearing of . . . . So that's a bearing of 315°."
>
> Knowing-in-action to let Jane try to find her mistake.
>
> She had made a mistake, which I thought was to draw the bearing at the end instead of the beginning of the resultant, so I answered:
> "Yes. That's a bearing of 315°. But where do you measure your bearing from?" and she replied:
> "From North. From there", pointing to the diagram. Jane talked through her solution again but it was still based on a wrong assumption.
> I repeated:
> "Yes, that's definitely 315°. I agree with that, but where do you measure bearings from? From." She was sure that she was right, replying:
> "North."
> I told her:
> "From North. Okay. But this is the endpoint (pointing to the diagram). And bearings are measured at the start."
> *Participating is persevering.*

Gergen (1995) again provides a basis for analysing the exchange:

> The success of the dialogue depends on one's ability to (a) single out from the others' words (and actions) those elements that can be coordinated to what has preceded, (b) generate words (and actions) that grant to the interlocutor's offerings the status of authority and coherence, (c) extend the preceding patterns of words (and actions) in the dialogue in ways that are useful to the exchange, (d) leave space in the exchange for the other's participation, (e) furnish backchannel information that helps retain the trajectory of the exchange, (f)avoid moves that terminate the discussion (e.g., personal threats, insults), and so on. (p. 34)

Discussion earlier in the lesson had established that bearings were measured from North, which explains Jane's association of 'from' with direction, rather than with position. Despite the misunderstanding, the conversation proceeded smoothly with Jane and myself taking turns at putting forward our ideas and having them acknowledged as authoritative and coherent. Then, the authority moved back to me with the clarification of what I meant by 'from' and with the suggestion of a source of error: an extension of dialogue, past reiteration, to open up the possibilities for resolving the misunderstanding, but without the terminationof an instruction that would directly yield the answer. *Participating is assuming / relinquishing authority.*

> Jane argued:
> "But that vector is starting from there and goes to there."
> Now a different source of error was apparent to me.
> "Okay. That's the vector going back to the beginning. But vectors always start at the beginning and go to the end."
> She paused, and answered:
> "Oh, yes. So we have to do the bearing here?"
> "Yes, so you have to go back and adjust your angle."
> *Participation constitutes cognition.*

It turned out that Jane's mistake was not to do with her understanding of bearings, but with the direction of the resultant vector. Later Jane, when asked, explained the solution to the class. Then, issues from the one-to-one conversation with Jane became teaching points: a constructivist attitude resulted in "teacher as learner" (Tobin & Tippins, 1993, p. 9) with an understanding of student difficulties that mightn't have otherwise been considered by the teacher and which might have been widespread. Whether Jane and the class would be able to transfer (Lave, 1988) any new understandings to a different problem was a separate issue. *Participating is being open to others' ideas.*

Jane's involvement could be seen as *competent participation*. "[W]e talk about competent participation. Knowing physics therefore means to participate in *talking* about relevant objects and events in the way physicists do, using acknowledged words, sentences, gestures, inscriptions, and so forth" (emphasis added, Roth & Tobin, in press). Having initiated her own involvement in the discussion, Jane's competence was constituted by mutual exchange of ideas with me: for both of us, " 'I know' when I speak in ways that enable you to treat me as if I know and vice versa" (Gergen, 1995, p. 34). However, competence in Roth and Tobin's use of the word, is more than appropriate use of classroom mathematical language. It is measured by the extent to which *authentic participation* in the mathematising of mathematicians is engaged in. Perhaps the applied problem solving lent authenticity to the classroom endeavors. *Participating is being competent.*

Lave and Wenger (1991) also discuss authenticity or *legitimacy*. Jane's participation was legitimate, in Lave and Wenger's definition, to the extent that it matched the practices of the outside mathematical social world, or alternately was typical of the Year 11 class. Evidence suggests that Jane explaining her solution to the class was not typical of the way lessons had been facilitated with the students' regular teacher: "Like, Mr Brown does the method on the board" (Student Interview). So the criterion of legitimacy, what was normal for the students, might not have been met in the lesson. *Participating is being adaptable.*

In the collaborative achievement of solving the problem, learning was mutual, not of the synthetic type where I had worked the problem previously and 'held back' to allow students to explore it, but instead Jane and I genuinely sorted out the detail together with control shifting between us. My not bringing a prepared solution perhaps encouraged teacher-student *co-participation* (Lave & Wenger, 1991), where co-participation is learning in partnership through conversation or activity, which is enhanced by the absence of teacher domination (Taylor, 1998). Gergen (1995) describes how, in preparing for lectures, the "hours of preparation—the rereading of texts, scanning of notes . . . are essentially removed from student view. Such removal is essential, of course, in sustaining the myth of authority as an individual possession: 'My lecture demonstrates the superiority of my mind' " (p. 31). A teacher in a secondary-school classroom having content worked out to the last detail might have the same effect that Gergen describes for the lecturer. *Co-participation is a symmetrical relationship.*

However, Schoenfeld's (1996) experience suggests that what matters isn't a teacher's lesson preparation but the way the lesson is played out. He compares his group-sessions with post-graduate students where he is "frequently at the limits of my competence . . . . nor am I usually on familiar turf" (p. 11) to his problem-solving, undergraduate course where he is on "familiar ground most of the time . . . and even if the students don't know where they are going, I usually do" (pp. 11-12). His aims for both groups include authentic participation: "I want my students to emerge as skilled practitioners of the discipline . . . . [and he suggests that] in a particular kind of environment, the acknowledged artifice [of the instructional approach] can become irrelevant, as can the question of whether the instructor knows 'the answer' " (p. 12). Although his two teaching environments are different, Schoenfeld feels there are many similarities in the outcomes for both, and attributes this in his problem-solving course to: "The atmosphere is open; the students are generating ideas that are fresh to them; and I am working on the spot with what they generate" (p. 14). *Participating is being listened to.*

Jane's persistence and motivation to solve the problem and to engage in discussion of it did not seem to be hampered by the abstractness of the worksheet diagrams, which were void of angles, lengths and compass directions. For her, the diagram acted as a *coherence object* (Roth & Tobin, in press) around which she negotiated her understanding. In contrast, analysis of other episodes suggests the limitation of the diagram acted as a *boundary* or *border* (Roth & Tobin, in press) to some students participating as successfully as Jane: they couldn't interpret the diagram in order to discuss adequately the problem. *Participating is interpreting inscriptions.*

Lave and Wenger (1991) discuss learning in terms of apprenticeship so that in a classroom the teacher is seen as the *expert* or master and the students as *peripheral participants.* The expert, taken to mean the person with wider knowledge or more experience, enculturates students into mathematising practices. In this first lesson it seemed as though some students, like Jane, were already nearly experts in the practices (explaining, justifying and listening) of a mathematics classroom. Yet, I wrote initially about others rather than Jane, not recognising the efficacy of her involvement. *Participation is present in every classroom but varies in quality.*

CONCLUSION

In summary, What *is* participation? Briefly, participation involves taking part with others: working together, one speaking, another listening, taking turns and waiting. It is evidenced in language and in action or gesture. It can centre on inscriptions such as diagrams and algebraic formulations. Not understanding the language and accepted actions, or the inscriptions, can be barriers to an individual participating. Participation moves through time: it is initiated, engaged in and terminated. It goes hand-in-hand with mental activity: cognition, interpretation and making sense. It concerns emotion and attitude: being motivated and confident. Participation can be legitimate and/or authentic and can vary in quality measured by relevance and coherence. It concerns and disperses authority. Input might be mainly from one party or it might be co-participatory, with contributions spread across a group. Last, participation can result from both taking and making opportunities for involvement.

After looking at the episode related above, and at others from the first lesson, the question for me was: What sort of participation is desired by the teaching goal of working towards students' active participation in classroom activities? What captures the essence (van Manen,

1990) of the phenomenon 'optimal participation' in the way that Noddings (1990) classifies students' cognition as strong? Perhaps it is that participation is voluntary and constructive. Voluntary to the point that if a student has ideas or understanding that might benefit the group, then she shares them, and constructive in that it adds to the understanding of the group, or is part of a process of a student developing confidence. Exploring the nature of students' participation in the first lesson with the Year 11 class, allowed me to define that voluntary and constructive student participation was one of my teaching objectives and was a forerunner to encouraging its development and inquiring into its relationship to students' cognition of vectors.

Schön's (1987) epistemology of practice, which encourages reflection while engaged *in* teaching and reflection in retrospect *on* teaching, and the research methodologies of hermeneutic phenomenology (van Manen, 1990) and narrative inquiry (Connelly & Clandinin, 1988) continue to enrich my understanding of teaching and learning processes. Other teacher-researchers may find the methodologies similarly accessible.

Note: I sincerely thank my doctoral studies supervisor, Dr. Peter Taylor, for his supervision and ideas for my study.

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A TALE OF TWO CITIES: WHEN MATHEMATICS, COMPUTERS, AND STUDENTS MEET

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"It is a far, far better thing that we do now than we have ever done" Well, probably not quite – but we report on the extension of a project that addresses the potential of attitudinal factors to impact on situations involving the use of computers to learn mathematics. The construction of six attitude scales is discussed, together with response data from students in London and Brisbane. The data point to separate attitudinal dimensions related respectively to mathematics and to computers. This has implications for learning situations in which they interact.

BACKGROUND

Undergraduate mathematics courses in Australia have for some time been integrating symbolic algebra software into their teaching programs at an increasing rate (Pemberton, 1996). This is a local reflection of an international trend. Students are being asked to relate to computers and to mathematics in new ways, and their response to these demands forms an important focus for research. In this paper we discuss an approach to investigating attitudinal factors that emerge as consequences of interactions endemic to such an enterprise.

While the study of *attitudes* in mathematics learning has a substantial history, the relationship between *attitude* and *performance* is not clear cut although positive correlations have often been noted between these characteristics. Early claims that affective variables can predict achievement (e.g. Fennema and Sherman, 1978) have been balanced by later comments (e.g. Schoenfeld, 1989) indicating that research does not give a clear picture of the direction of causal relationships. The Tartre and Fennema (1995) comment that described *confidence* as the affective variable most consistently related to mathematics achievement is probably a safe summary of the position.

More recent studies have continued to pose the direction of the relationship between *attitude* and *performance* as an open question. Thus while Tall and Razali (1993) argued that the best way to foster positive attitudes is to provide success, Hensel and Stephens (1997) concluded that "it is still not totally clear whether achievement influences attitude, or attitude influences achievement", while Shaw and Shaw (1997) noted that among engineering undergraduates the top performing students (at entry) had a much more positive attitude to mathematics, and lower performing students a commensurately negative one – again leaving the direction of causality open.

The study of attitudes towards information technology (most frequently computers) has a shorter but more intensive history, probably because information technology, while newer, is all pervasive and permeates all curriculum areas. In fact the emphasis has been overwhelmingly towards the interaction with computers as such, rather than on the use of technology in particular learning contexts, although as Selwyn (1997) noted, awareness of students' attitudes towards computers constitutes a "central criterion in the evaluation of computer concerns and in the development of computer based curricula". Among the plethora of studies that have been conducted, Francis (1993) observed that the integration of findings is difficult because of the lack of common agreement regarding what it is that computer-related attitude scales set out to measure. In considering attitudes to information

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technology among graduates, studies involving mathematics students appear relatively hard to come by, although several studies have included affective variables when evaluating outcomes (see below). It is this very breadth of discipline background which has served to keep the investigation of attitudes to information technology at a general level, appropriate to the majority who will not be called upon to use computers in the same technical sense as mathematics students working intensively with specialised software.

The importance of studying attitudes to information technology in conjunction with those relating to mathematics is emphasised and re-inforced by the increasing use of technological devices in mathematics instruction, and while there have been many enthusiastic claims for the positive impact of technology on the teaching and learning of mathematics, systematic evaluations of impact are difficult to assess. Several studies refer incidentally to attitudinal impacts as well as proficiency measures and Mackie (1992) in an evaluation of computer-assisted learning in a tertiary mathematics course indicated six positive learning outcomes, three of which were related to attitudinal factors. Park (1993) in comparing a Calculus course (utilising Mathematica) with a conventionally taught program, found some improvement in disposition towards mathematics and the computer in the experimental group. However Melin-Conjeros (1992), in comparing the performance of a group of Calculus students (equipped with limited access to Derive) with a control group, noted that the attitude of both groups decreased slightly.

In particular it is not generally clear in the mathematically focused studies just which 'attitudes' have been affected by technology, as the reporting tends to be non-specific. By inference it appears that it is 'attitude' to mathematics that is referred to, and we are led to consider the implications of technology in impacting upon component attributes. The consistent and strong relationship between mathematics confidence and performance noted previously (whatever the direction of causality), means that the implications of a nexus between technology and mathematics needs specific research attention. The broad reporting of studies on the use of technology in mathematics instruction make it difficult to disentangle whether reported affective outcomes are associated with changed attitudes to mathematics, or are linked directly to the technology. Theoretically we are moved to ask about the interpretation of outcomes if students possess high mathematics confidence and motivation, but low computer confidence and motivation, and vice versa. These interests have motivated the construction of attitude measures to address such questions.

The purpose is to design attitude scales for use in programs in which computer technology is specifically directed towards assisting mathematics learning.

The context within which the scales have been designed is that involving undergraduate students newly enrolled in introductory mathematics subjects at two universities viz. City University (London), and the University of Queensland (Brisbane). The subjects involved major use of symbolic algebra packages.

THEORETICAL BASIS OF SCALE DESIGN

Given our purpose of developing scales for use in settings involving interaction between technology and mathematics learning, we now discuss the rationale behind our choice of items and methods. We have found the positions articulated by Hart (1989), Mandler (1989), and McLeod (1989, 1994) to be helpful in fashioning the approach to our definition of terms and hence instrumentation. We shall refer to this as the HMM classification in which the ordering *beliefs, attitudes, emotions* represents increasing affective involvement, decreasing cognitive involvement, and decreasing stability. Beliefs are viewed as mainly cognitive in nature being built up slowly over time, while attitude may be viewed as the end result of emotional reactions that have been internalised and automatised (McLeod, 1989) to generate feelings of moderate intensity and reasonable stability. With the decay of the emotional content with time, the response becomes more stable, and hence amenable to access through questionnaires and interviews. Emotions, as hot reactions, cannot reasonably be measured by such dispassionate means as questionnaires, and their role has been relatively underplayed in terms of research. We have adopted the HMM classification as the basis from which to develop our instrumentation. The distinction between an *attitude* and a *belief* is tenuous to a degree – we have endeavoured to seek an *attitude* focus by wording items so that the respondent is personally involved:

e.g.	*I feel more confident of my answers with a computer to help me;*
rather than	*Computers help people to be more confident in obtaining answers*

THE ATTITUDE SCALES

The students for whom our measures are designed are tertiary undergraduates in mathematics courses; having made this a conscious career choice-whereby mathematics has been selected as both useful in pursuing career aspirations, and as a subject compatible with themselves as individuals. Hence while we retain an overall monitoring interest, gender and usefulness, which have figured prominently in other attitude studies (e.g. Fennema and Sherman, 1976) do not play a dominant role in our design. Two of the nine attributes (confidence and motivation) represented in the Fennema-Sherman formulation have been reflected in scale development, with appropriate items constructed for use by undergraduates. The choice of these attributes was influenced strongly by the total purpose of designing instruments for use when computer technology is used in the teaching/learning context. We have chosen *confidence* and *motivation* because of their extensive appearance in the literature for both mathematics and technology, and because of their potential for discriminating between attitudes when technology and mathematics interact. These four scales are designed to measure attitudes on both dimensions so that such differences can be identified and their implications noted. In particular the choice of *confidence* and *motivation* enables two circumstances of particular interest to be identified viz situations where students hold strong positive feelings towards mathematics and negative feelings towards technology, and vice-versa. And of course *confidence* and *motivation* are two constructs that have been strongly and consistently linked with mathematics achievement over many years as discussed previously.

Two further scales address factors, important for the learning context, that are not accessed by the separate computer and mathematics scales. These are designed respectively to measure the degree of interaction between mathematics and computers that students perceive they apply in learning situations, and the degree of engagement that the students bring to their mathematical learning. The theory underlying the construction of these scales is summarised below.

The interactive significance of the learning and instructional context has been emphasised in general (e.g. McLeod 1989). Given the significance of *engagement* for effective learning (e.g. Anderson, 1995), whereby its importance is recognised by scholars as disparate as constructivists and time on task proponents, the extent to which students interact with learning materials is of central interest. In a computer environment students may simply respond to the screen or be active in note making, summarising, and experimenting. Indeed they may choose not to utilise technology when it is available and relevant (Boers and Jones, 1994). The physical separation of the learning components; pen and paper, computer screen, and human brain adds a further dimension to the co-ordinating processes required for effective learning strategies. The computer-mathematics *interaction* scale assesses the extent to which students bring their mathematical thinking into active inter-play with the computer medium.

Within each scale the (six or eight) items were arranged randomly with half requiring the reversal of polarity at the coding stage. Students were asked for a measure of their agreement (or rejection) with respect to item wording, which resulted in a 13 point Likert scale. The item groups were presented such that the constructs underlying each respective group were not known to the students. The scale items were theoretically determined from the respective underlying constructs and from cognate literature.

ADMINISTRATION & OUTCOMES

The instrument was given initially in October 1994 to 156 first year students on entry to courses in engineering, mathematics and actuarial science at City University, London, and to 56 students in a general first year mathematics subject at the University of Queensland in 1997. This represents an extension of the study described in Galbraith & Haines (1998).

The responses displayed similar patterns for both cities. To illustrate sample items we include below the items whose responses contributed most strongly to the scale score; polarities have been adjusted so that a higher score means more of the property described by the scale label. We include for each scale, the positively worded item(s) attracting the strongest support, and the negatively worded item(s) invoking the strongest rejection. (B=Brisbane, L=London). B1&L2 means that the item was the strongest choice of Brisbane students, and second strongest choice of London students.

mathematics confidence:	I can get good results in mathematics (B1&L1) *No matter how much I study, math is always difficult for me (B1&L1)
mathematics motivation:	If something about mathematics puzzles me, I find myself thinking about it afterwards (B1&L2) *If something about mathematics puzzles me I would rather someone gives me the answer than to have to work it out for myself (B1&L2)
mathematics engagement:	I find it helpful to test understanding by attempting exercises and problems (B1&L1) *I find working through examples less effective than memorising given material (B1&L1)
computer confidence:	I am confident I can master any computer procedure that is needed for my course (B1&L1) *As a male/female (*cross out that which does not apply*) I feel disadvantaged in having to use computers (B1&L1)
computer motivation:	I will work at a computer for long periods of time to successfully complete a task (B1&L2) * The way that computers force you to follow a procedure annoys me (B1&L2)
comp/math interaction:	I find it helpful to make notes in addition to copying material from the computer screen, or obtaining a print out (B1&L2) *I rarely review material soon after a computer session is finished (B1&L2)

* negatively worded item involving scale reversal

Scale Means

These are provided in Table 1 (Brisbane data in brackets).

Table 1
Scale Means

mathematics confidence	7.9 (5.6)	computer confidence	7.3 (6.7)
mathematics motivation	8.5 (6.2)	computer motivation	7.3 (6.6)
mathematics engagement	8.1 (8.4)	comp/math interaction	7.3 (6.6)

The study did not set out to compare the *level* of student response between groups. Rather interest is in the structural relationships between the mathematics and computer responses at each location. We note that the London students feel more positively about mathematics than the Brisbane students. This is to be expected, given the strong engineering representation in London versus the service course orientation of the Brisbane group. It is observed that the relative ordering of means has a similar pattern within each group, except that mathematics engagement is relatively higher among the Brisbane students.

Scale Reliabilities

These were obtained for each scale as follows-Brisbane data in brackets (see Table 2).The scales are coherent with reliabilities from strong to moderate, and with all items contributing. The bringing together of theoretically based disparate properties to address engagement and interaction issues, has unsurprisingly resulted in lower as than for more closely defined concepts like confidence and motivation.

Table 2
Scale Reliablities (Cronbach)

mathematics confidence	0.77 (0.87)	computer confidence	0.82 (0.90)
mathematics motivation	0.80 (0.86)	computer motivation	0.85 (0.91)
mathematics engagement	0.57 (0.60)	comp/math interaction	0.70 (0.69)

Scale Validity

This rests primarily upon the theoretical base behind the construction of the scales. Additional structural evidence may be inferred from the sample items given above. For example the two items attracting the strongest responses for *mathematics confidence* (expecting good results, and rejecting that mathematics is difficult irrespective of effort), are both centrally to do with confidence. The coherence of the scale as indicated in the a value then supports the argument for validity without examining each additional item. Similar arguments apply to the other scales.

DIFFERENCES IN ATTITUDES TO MATHEMATICS AND COMPUTING

A main purpose in this research is to investigate the extent to which attitudes to computer use and to mathematics represent different inputs into technology based teaching contexts involving mathematics learning. In this section we analyse the student responses to address this issue further. Consider first, correlations between the six scales (Table 3).

Table 3
Correlation Between the Attitude Scales

	mconf	mmotiv	mengag	cconf	cmotiv	cmint
mconf		.47(.81)	.08(.31)	.29(.16)	.14(.09)	.13(.14)
mmotiv			.46(.37)	.25(.41)	.29(.37)	.35(.40)
mengag				.06(.18)	.09(.04)	.26(.16)
cconf					.71(.90)	.61(.77)
cmotiv						.68(.85)
cmint						

The entries in Table 3 indicate that the confidence and motivation scales are strongly associated within mathematics (0.47,0.81) and within computing (0.71,0.90) but they are less strongly associated across the areas. This is shown by the weak correlation, for example, between mathematics confidence and computer confidence (0.29,0.16). Mathematics engagement is solidly associated (0.46,0.37) with motivation. The computer-mathematics interaction scale is more strongly associated with the computer confidence (0.61,0.77) and computer motivation (0.68,0.85) scales than with the mathematical scales. This suggests that computer attitudes are more influential than mathematical attitudes in facilitating the active engagement of computer related activities in mathematical learning. These results suggest that Factor Analysis using the six scales as input variables with a two-factor solution as goal is appropriate. Using oblimin rotation (SPSS) following a principal components analysis the loadings shown in Table 4 were obtained. The two-factor solution confirms that the computer and mathematics related scales define different dimensions with computer properties dominant in the interaction scale. Brisbane data again in brackets.

Table 4
Factor Pattern Matrix

	Factor 1	Factor 2
mconf	0.11(-0.12)	**.55(.96)**
mmotiv	0.14(.02)	**.85(.90)**
mengag	-0.17(.08)	**.79(.45)**
cconf	**.89(.92)**	-03(.05)
cmotiv	**.92(.97)**	-.05(-.09)
cmint	**.80(.89)**	.13(.07)
Percentage of variance 67.2(75.2)		

REFLECTIONS

Characteristics associated with high scores on the respective scales are given below. These are based on the scale items, some of which were included in the earlier discussion. Space does not allow a more comprehensive listing. Low scoring characteristics may be inferred by reversing the positive properties noted.

Students with high mathematics confidence feel that they obtain value for effort, do not worry about new topics, expect good results, and feel generally good about mathematics as a subject.

Students with high mathematics motivation enjoy challenging mathematics, stick at problems until solved, puzzle over problems away from formal classes, become absorbed in their mathematics, and would rather work a problem out than be told the answer.

Students with high computer confidence feel comfortable about operating computers, believe they can master any computer procedure required of them, are more sure of their answers when supported by a computer, and are confident of resolving mistakes that occur during computer activity themselves.

Students with high computer motivation find using computers makes learning more enjoyable, like the freedom to experiment that a computer provides, will spend long hours at a computer, enjoy thinking up new ideas to test on a computer, and do not feel limited by the computer environment

Students with high computer-mathematics interaction feel that computers enhance mathematics learning by providing many examples, enable the user to focus on major ideas by reducing mechanical toil, use note-making to augment screen based information, review material following each computer session, and find computers helpful in linking algebraic and geometric ideas.

Students rating high on mathematics engagement prefer working examples to memorising material, like to test understanding through exercises and problems, try to link new knowledge to existing knowledge, elaborate given material with their own notes, and review work regularly.

Pedagogies to support the new focus in undergraduate teaching are still in the process of development or refinement, and within this enterprise the interaction between mathematics and technology is of significant importance. Regarding the current work, we note firstly the properties independently confirmed among students with different backgrounds in different locations. We believe this increases confidence in the robustness both of the instrument, and of the outcomes. Secondly the strong correlations between *confidence* and *motivation* within mathematics and computing respectively, suggest that one such scale might suffice if a concise instrument is envisaged. Both *mathematics engagement* and *computer/mathematics interaction* should be retained as these represent important indicators of student involvement. An issue for future tracking arises from the current work. This is whether structural differences between mathematics and computer based affective responses on attributes such as *confidence* and *motivation* will diminish with time, or whether they represent distinctive sets of characteristics with a permanent presence in computer assisted mathematics learning. That this issue is a real one was confirmed by the 34 Brisbane students who responded to an invitation to answer the open ended question "How do you feel about using computers to learn mathematics?" There were 15 positive responses, 14 negative responses, and five that contained both positive and negative comments!

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TECHNOLOGY, MATHEMATICS, AND PEOPLE: INTERACTIONS IN A COMMUNITY OF PRACTICE

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This paper describes our sociocultural perspective on learning in secondary mathematics classrooms where technology is integrated as a central resource. We propose four roles for technology in relation to student learning: Master, Servant, Partner and Extension of Self. One classroom episode is analysed to reveal the different 'voices' that emerge through the interaction of mathematics, people (students and teacher) and technology. We are using this approach to develop a framework for describing and analysing the characteristics of a classroom community of practice.

The sociocultural orientation that underpins our research emphasises the socially and culturally situated nature of mathematical activity, with learning viewed as a collective process of enculturation into practices of mathematical communities. The classroom as a community of mathematical practice supports a culture of sense making, where students learn by immersion in authentic practices of the discipline. This is distinguished from the constructivist position (Cobb and Bauersfeld, 1995), in which priority is given to the individual construction of understanding, and where social interaction acts principally as a source of cognitive conflict to force the reorganisation of personal mental structures.

Our approach is predicated on three basic assumptions of sociocultural theory: (1) Human action is mediated by cultural tools, and is fundamentally transformed in the process. (2) The tools include technical and physical artefacts, but also concepts, reasoning, symbol systems, modes of argumentation and representation. (3) Learning is achieved by appropriating and using effectively, cultural tools currently recognised and validated by the relevant community of practice. To understand how this appropriation occurs, we make use of Vygotsky's notion of the zone of proximal development in order to examine the social and communicative conditions of learning in classroom settings. In particular, we draw on Bakhtin's theory of voice (Renshaw & Brown, 1998) to investigate interactions between people, technology and mathematics in a senior secondary classroom.

THE CONCEPT OF 'VOICE'

In studying classroom interactions we have found it productive to draw on Mercer's (1995) work on the nature of student talk. *Exploratory talk* is characterised by constructive and critical engagement with ideas presented by 'others' in the group or community. Ideas are challenged and defended by proposing justifications, explanations and alternatives. Claims are publicly accountable, and reasoning, being visible, is therefore open to evaluation and critique. In neo-Vygotskian terms this is associated with the internalisation of social forms and processes. Exploratory talk may be contrasted under the Mercer (1995) construction with *disputational talk* characterised by disagreement, assertion and counter assertion among participants, leading to entrenchment of individual perspectives; and *cumulative talk*, characterised by the compiling and accumulation of ideas leading to uncritical concurrence or consensus. In theorising diversity as it applies within the concept of exploratory talk, it has proved useful to develop Bakhtin's theory of voice (Renshaw and Brown, 1998). This theory emphasises the active, situated and functional nature of speech as employed by various groups. The term *personal voice* is given to words used in conversation that are employed by a wider community of users. So in any individual utterance, a number of voices can be heard—the voice of the speaker, but also voices of other community members who have used similar words or constructions to convey meaning. Then the extent to which a speaker uses such expressions can be used to theorise different levels of performance (e.g. explanation, justification, problem solution) that are accepted as valid by the community. However, not all sources and modes of communication may be equally privileged, and not all 'voices' convey values consistent with community ideals.

We consider a classroom in which the pursuit of mathematical competence and understanding occurs within the framework of our three assumptions underlying the sociocultural perspective. This involves mutual interactions between the teacher and individual students, the total group, and subgroups of students; interaction between individual students and peers involving both cognitive and interpersonal exchanges; individual action and reflection; and interaction between all human participants and artefacts such as text material, and in the context of our special interest, technology as in calculators and computers.

The teacher's 'voice' may be heard in several roles. Firstly, the teacher is a facilitator of mathematical progress in which the 'trace' of her/his membership of a mathematical community will influence strongly the manner in which this role is enacted, through the quality of the mathematics that is demonstrated and facilitated. Secondly, the teacher acts as a custodian of the learning climate in which a productive working atmosphere is maintained in the community. A third avenue for the teacher's 'voice' is through the exercise of pedagogical knowledge that is not specifically content related. This is knowledge applied through such skills as modes of questioning and explanation, achieving student participation, and organising the classroom for learning. This 'voice' will echo the values of professional educators and will interact with those referenced above, for the teacher's 'voice' and utterances are indicative of the extent to which community of practice values are at work, and may of course convey images that are inconsistent with such values. Collective argumentation and dialogue may be curtailed or replaced, for example, by transmissive practices emphasising the unquestioned acceptance of rules and formulae, without opportunity for question or debate. Such 'voices' are heard in many classrooms.

Moving now to peer interaction, we note that interchanges between students provide for the flexible exercise of a variety of mutually reciprocal roles such as tutor, critic, audience, checker and so on. While some research on student-student interaction has examined the effectiveness of assigning such roles to students working in groups, our approach differs in the degree of structure imposed on the peer interactions. Our community of practice view looks at the 'wholeness' of the enterprise. Collaborating mathematicians take different roles at different times with respect to their joint projects, and such decisions are part of the collaborative decision making. Hence students must also be provided with the opportunity to adopt different roles in peer transactions, which will be related to (but distinct from) the content being addressed. Good mathematical practice is modelled by teachers not only in applying mathematical expertise, but also through procedural questioning to make public the choices and actions called for in following a certain solution path. And reciprocally the teacher acts as a coach to highlight strategic and procedural decisions when individual students or groups lead a solution process. In this way, the variety of activities through which clarification is sought are constantly raised and reinforced.

So the 'voices' that can be heard during peer interaction are many. For example, the trace of mathematics as previously learned, understood and passed on, will be audible through the voices of teachers and students in prior learning episodes. Other traces will involve the 'voice' of text material as students articulate its perceived authority. Finally, 'voices' that emerge in relation to technological artefacts are of significant interest for their potential impact on the learning culture.

TECHNOLOGICAL 'VOICES'

We theorise four roles for technological devices such as calculators and computers, in which different 'voices' may be heard.

Technology as Master ("What is it allowing me to do?")

Here the student is subservient to the technology—a relationship that can be induced by either technological or mathematical dependence. If the complexity of usage is formidable, student activity will be confined to those limited operations over which they have technical competence. Alternatively if necessary mathematical understanding is absent, the student is reduced to blind consumption of whatever output is generated, irrespective of its accuracy or worth.

Technology as Servant ("Tell the thing what to do!")

In this role technology is basically used as a reliable timesaving replacement for mental, or pen and paper computations. The tasks of the mathematics classroom remain the same—but now they are facilitated by a fast mechanical aid. Unlike the previous category the user is in control, and 'instructs' the technology as an obedient but 'dumb' assistant. Trust in the reliability of the servant means that the output is regarded as authoritative, although the discerning user will continue to monitor reasonableness of outcome against the possibility of keying errors.

Technology as Partner ("How can we do this together?")

Here a 'rapport' has developed between the user and the technological device—which may be addressed in human terms. A graphics calculator, for example, becomes a friend to go exploring with, rather than merely a producer of results. The user is still in control, but there is appreciation that authority of outcome needs to be judged against criteria other than that a required response has been obtained. Explorations, for example in graphical work, lead to situations where the output needs to be checked against the known mathematical properties of related graphical forms. It is possible for the calculator to be 'wrong', and a feature of use in this mode is the way in which the respective 'authorities' of mathematics and technology are balanced.

Technology as an Extension of Self ("Come fly with me!")

This is the highest level of functioning, and involves users incorporating technological expertise as an integral part of their mathematical repertoire, so that the partnership between student and technology merges to a single identity. Here powerful use of calculators and computers forms an extension of the user's mathematical prowess. Rather than existing as a third party a calculator may be used to share and support mathematical argumentation on behalf of the individual, as when students share and compare computer output as part of their *own* contribution to a solution process. The technology becomes as much a part of the user's catalogue of resources as tabled information and mathematical know-how inside the head.

In a community of practice classroom technology may be involved in any of the latter three roles. At times it will be appropriate to treat technology as a reliable servant for obtaining results necessary to progress a line of development. This may occur frequently during teacher (or student) led group teaching episodes when common output is needed as a focus for discussion. Technology as a partner is important during individual or group exploratory work, and in widening the options in problem solving situations. Technology as an extension of self enters at the higher levels of mathematical activity, where the emphasis is on forms of argumentation, and other characteristics of the discipline of mathematics. Note that there is no necessary connection between these successively higher forms of technological functioning, and the level of mathematics involved, or the seniority of grade level. The 'voices' heard in relation to technology will reflect the mode of use—from despair of a dependent with a flat battery, to instructions given to a servant, through personal conversation with a partner in an investigation, to confident authority in which the contribution of the calculator is merged in the wider argumentation of the user.

RETHINKING THE ZONE OF PROXIMAL DEVELOPMENT IN A COMMUNITY OF PRACTICE SETTING

The different voices that emerge as students appropriate more formal and powerful modes of thinking can be analysed in terms of the Vygotskian notion of the zone of proximal development (ZPD). The most widely known definition of the ZPD is that associated with scaffolding the learning of an individual with help from the teacher or more capable other. However several other contexts have been postulated for the ZPD (Goos, Galbraith, & Renshaw, in press), and in this work we aim to explore the potential of one that addresses learning in the social contexts of classrooms.

The ZPD is normally applied to individuals, but recently it has been applied to whole groups in a way that seems to be both consistent with sociocultural theory, and of practical significance in removing the implication that effective teaching in the ZPD requires sensitive diagnosis of the diverse levels of development of students, followed by one to one instruction (Mercer & Fisher, 1992). Students as participants in a learning community are viewed as having partially overlapping ZPDs that provide a changing mix of levels of expertise that enables many different productive partnerships and activities to be orchestrated (e.g. Brown & Campione, 1995). Through the establishment of a small number of repeated participation frameworks such as teacher-led lessons, peer tutoring, and individual and collaborative problem solving, students become enculturated into taken-for-granted aspects of classroom life that promote a shared knowledge base, a shared system of beliefs, and accepted conventions for communicating and verifying knowledge claims. The lived culture of the classroom becomes in itself a challenge for students to move beyond their established competencies, and adopt the language patterns, modes of inquiry, and values of the discipline.

THE STUDY

We seek to understand and map the changing relationships that characterise classrooms designed on community of practice lines. Complex interactions between students, teacher, the discipline of mathematics, and technology form the context for our study. We aim to identify characteristic behaviours, modes of thinking, and action through the interpretation of classroom events. Included in our range of indicators are the 'voices' identified in the respective dialogues that are a central component of the mathematical exchanges, and the ways that these are elaborated, harnessed, or modified to enhance the quality of learning.

The first year of the study has focussed on a single Year 11 Mathematics C classroom in a co-educational independent school. (Mathematics C is an advanced subject taken by students intending to pursue further study of mathematics at tertiary level.) A series of lessons on matrices was observed and videotaped in order to analyse students' use of technology in learning this new subject matter. Students were also interviewed to seek their interpretations of particular lesson events. Three features of the teaching approach warrant special mention. First, matrix algebra was not taught as a series of algorithms, but instead was developed by presenting students with life related problems from which matrix representation and manipulation arose naturally. Second, students had constant access to graphing calculators to assist them in performing complex matrix calculations. Finally, the teacher regularly asked students to present their solutions to matrix problems by plugging their calculators into an overhead projector panel, so that the whole class could follow the working as it was reconstructed and displayed on the calculator screen.

The following excerpts from a classroom segment (together with commentary) illustrate preliminary attempts to develop a framework consistent with a sociocultural approach to engaging the quality of classroom learning.

Lesson Segment

In this excerpt (in which selected dialogue is labelled in sequence), a student (Paul) leads the development, with other students and the teacher entering the discussion. The problem required students to use matrix multiplication to find the manufacturing costs and wholesale and retail prices of biscuits packaged in different combinations of flavours (see Appendix). The solution being explained has been previously generated on a graphics calculator, and is reconstructed by Paul for display via the overhead projector. The segment begins with another student interposing in a teacher's 'voice', "*Start when you're ready*".

12. P: To start the problem (reads out question (1)) Show the contents of the three proposed packages in a matrix with 3 rows and 4 columns. So I did that with packet name down the rows and the type of biscuit along the columns (Plays with the calculator to get up the screen he wants to show) and there it is, right there!

13. T: Sorry to interrupt! Can I show you another way of doing that? Just type SS1 on the screen, and go to home. Okay, then just clear, so you've got that there. SS1 and then 'enter'. (Students ask questions.)

14. T: You've got to make sure that the calculator knows what you're looking for. So, tell us about what you put in that row. Why is this so?

In this interchange Paul adopts a teacher 'voice' from time to time in displaying his work. "*There it is, right there.*" The teacher intervenes to demonstrate calculator operations, and portrays the calculator in a 'dumb' servant role. "*Make sure that the calculator knows what you're looking for.*" This provides a setting for interpreting the contribution of the calculator.

21. T: Where did SS3 come from?

22. P: Oh, that's just what I was about to say. This! Up here is the number of packets intended to be produced. Down here, will be the type of packets in the same order (inaudible) Yeah! Yeah, I wrote that out later (Teacher and students laugh). I just translated that. Used that T.

23. T: Transposed, Paul! I know what you mean Paul, but it's called Transpose. So, you used 'SS3 transpose'

Here the teacher elaborates a description from Paul who simply "*used that T*". This is not satisfactory from the teacher's viewpoint as a mathematical point remains undisclosed—compounded by the interpretation of 'T' as translate. "*I just translated that.*" The teacher introduces the correct meaning of 'T' as "*transpose*" where its meaning of interchanged rows and columns is evident from the calculator display. This 'voice' of precise mathematics is combined with a 'pedagogical voice' aware of the importance of emphasis: the term '*transpose*' is used three times in a single short sentence. We observe that the use of the term 'translated' by Paul, to describe the change between row and column display, may represent a 'voice trace' from previous mathematics—where 'translate' is associated with movement and changed appearance, as in graphical work.

48. P: So then, in this one, you have the frequency here, of packets produced. You know, the different types of packets. Okay, going back to what I was doing before. Question 4, I did matrices (1) times matrices (2) equal . . .

49. T: Do it, then! (Paul works it out on his calculator)

50. P: Equals that, which costs...which,...These are different types of packs here, and up here, is cost in cents. And I just converted that manually and made the matrices in dollars. Is there any way of

doing that? - tricky, you know! Anyway, I just did that by hand and then created a new one called SS5.

51. T: Okay, well! This is what you could have done, Paul! You could have gone um.... This is something you could show...Just push 'up' a couple of times, again, then 'enter'. So, now, enter that directly into the thing, so that saves you retyping a lot and divide by a hundred Okay, push 'diamond' across to equals. (Students gasp in awe, and Paul nods.) So, what we've done is actually introduce a new type of operation when we're dividing by a number. Can you tell me how that actually works? When you divide it by a number? What does it do? (Students mutter something) Divides what?

52. S's:Divides every number in the result?

53. T: Divides every number in the matrix by that number! And as we've discussed before (inaudible), just multiplying by 100, doing the inverse and that's called scalar multiplication. So, that's two types of multiplication you've seen now. Matrix multiplication which is what you've been doing with these products and scalar multiplication as well. And that's when you multiply and divide through by a constant and in this case, it would be useful to turn cents into dollars.

54. P: I didn't actually divide just the result in dollars. I just did the matrix. If I divided both, I'd get the wrong answer.

Paul rehearses matrix multiplication using the calculator in 'servant' mode and then combines 'hand' arithmetic with 'automatic' calculator processing to create a new matrix, noting an unresolved challenge with respect to the calculator. *"Is there any way of doing that? - tricky, you know!"* The inference is that the calculator *ought* to be able to do it. The teacher now seizes an opportunity to scaffold group learning of the new concept of 'scalar multiplication' in a ZPD enactment. He first achieves a dramatic impact by crashing through calculator operations to produce a graphic display—a significant move that achieves motivation by creating a 'learning gap'. In doing so the calculator is again assumed to be a reliable servant whose output can be accepted. In another sense the calculator is also accepted as an extension of the teacher's trusted expertise—there is no challenge to his button sequence that is incidental to the purpose. The mathematical concept to be learned, 'scalar multiplication', is now explained retrospectively with the benefit of the visible and dramatic end result—an approach not possible if each step must be generated successively by hand. *"So, that's two types of multiplication you've seen now. Matrix multiplication which is what you've been doing with these products and scalar multiplication as well."* Paul rounds off the exchange by relating his mixed mode calculation to the new procedure.

88. T: Why am I hearing so many discussions when Paul is waiting to go on. You should be up there!

89. P: Okay. This SS4 which is each packet named in order and the costs. So, I did SS6 [label for transpose of SS3] times SS4 to work out the amount of money that shops will pay, which is similar to what I got for question 6. Everyone happy with that? After doing, question 7 was pretty similar except I multiplied that by SS7 - this is recommended retail price. SS6 x SS7 gives you the calculation of the total amount that customers will pay for the biscuits. Did everyone have that? And that concludes my demonstration!

Here we hear the teacher's voice adopted also by Paul as he rounds off his demonstration. *"Everyone happy with that?"... "Did everyone have that?"* These queries, while teacher like in tone, do not present the solution as something predetermined and beyond question, but rather represent an invitation to students to submit further challenges so that consensus can be reached. Knowledge is not taken for granted in this classroom: students are expected to offer their work for public critique by peers in order to achieve a communally validated solution.

Student Perspectives

Paul was interviewed soon after the lesson to seek his views on the practice of students presenting their work via the overhead projector. He commented that *"those projector things are good because everyone can see what you're doing"*. Paul also rejected the suggestion that students may be placed under unfair pressure when asked to make their work public in this way, replying *"I think it's good that we should have to get up in front of the class sometimes because, or else you might just never do anything"*. His comments were echoed by other student presenters interviewed after subsequent lessons. For example, students recognised that there is a high degree of accountability associated with public sharing of knowledge. Justification also becomes more authentic when students have to convince their peers that their mathematical arguments are correct. Students agreed that watching a peer explain their work is helpful because it gives them alternative perspectives, the explanation is in language they understand, and they feel reassured when they see someone making the same mistakes they did.

CONCLUSION

While mapping the characteristics of a community of practice culture requires longitudinal study, snapshots reveal attributes in various stages of growth. In global terms we look for this growth developing through processes associated with the classroom culture application of the ZPD as described earlier. These are displayed in the quality and style of student and teacher dialogue *vis a vis* the inclusion of authentic mathematical argumentation such as conjecture, clarification, justification, and interrogation of anomalies. Evidence of such characteristics appear in various forms including 'voice traces' that portray developing expertise appropriated from a variety of sources from which learners have drawn in building their cumulative mathematical know-how. A contemporary interest is the way in which technological artefacts are utilised as resources for learning, and the 'voices' that display the role and status with which they are deployed by teachers and students.

In the lesson discussed we have pointed to several indicators of community of practice attributes. Student talk has reflected the 'teacher's voice' both in giving explanations, and in creating an atmosphere for serious discussion—indicative of a 'partnership' approach in which teacher and learners alternate roles for the purpose of enhancing individual and group learning. Dialogue was driven by the desire to solve but also to understand. This was indicated for example by Paul's asides, in which he nominated puzzles that he felt should be accounted for, and in other excerpts (not included) when other students challenged an interim result. With the teacher's encouragement solution steps were reversed (assisted by the calculator display) until the mistake, a simple transcription error, was located and rectified. Mathematics then, is not seen as a mystery subject, but one in which anomalies are to be resolved, and reasoning is logical, transparent and defensible. Finally some of the 'voices' gave insight into the role that technology was playing in the learning episode. In this segment the graphics calculator was assigned a 'servant' role and used both to obtain results quickly by the students, and for creating a motivational 'learning gap' by the teacher. Through transactions such as the above a classroom culture of inquiry has been developed to act as a ZPD for the cultivation and nurturing of mathematical maturity and expertise. Our interest is in following the potential of such approaches to learning.

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APPENDIX

In planning for Christmas the biscuit factory decided to release three different packets.
Holly Dream (HD): contents 8 chocolate, 12 thin sweet, 6 shortbread, and 4 rich cream.
Sleigh Ringers (SR): contents 12 chocolate, 15 shortbread, and 5 rich cream.
Reindeer Extras (RE): contents 14 chocolate, 16 thin sweet, 12 shortbread, and 8 rich cream.
Manufacturing costs (cents): chocolate 8, thin sweet 4, shortbread 7, rich cream 10.
Packets will be sold to shops for \$2.10, \$2.95, and \$3.70 respectively.
Recommended retail prices will be \$2.49, \$3.49, and \$4.49 respectively.
Numbers of packets to be manufactured (millions) will be: HD (2), SR (1.5), RE (1).

In your workbook answer these questions:

(1) Show the contents of the three proposed packets in a matrix with 3 rows and 4 columns.
(2) Show the manufacturing costs per biscuit in a column matrix.
(3) Show the numbers of each packet to be manufactured in a row matrix.
(4) Use the matrices in (1), (2), and (3) to find the cost to the manufacturer.
(5) Show the prices at which each packet will be sold to shops in a column matrix.
(6) Use your matrices in (3) and (5) to find the total amount the factory charges the shops.
(7) Calculate the total amount which consumers will pay for the biscuits.

Graphics calculator labelling of matrices used by student in dialogue.

$$SS1 = \begin{bmatrix} 8 & 12 & 6 & 4 \\ 12 & 0 & 15 & 5 \\ 14 & 16 & 12 & 8 \end{bmatrix},\ SS2 = \begin{bmatrix} 8 \\ 4 \\ 7 \\ 10 \end{bmatrix},\ SS3 = \begin{bmatrix} 2 \\ 1.5 \\ 1 \end{bmatrix},\ SS4 = \begin{bmatrix} 2.10 \\ 2.95 \\ 3.70 \end{bmatrix}$$

TEACHERS EXPLORING NUMERACY LEARNING & TEACHING IN THE EARLY YEARS OF SCHOOLING

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This paper discusses the effectiveness of the professional development model used by the Catholic Education Commission of Victoria Numeracy Strategy Project. The project used an action research methodology within a team structure to explore numeracy learning and teaching. Three case studies of early years teachers indicate that the teachers developed some important insights about numeracy learning and teaching in the early years of schooling and provide evidence of reported change in beliefs and practice.

INTRODUCTION

Numeracy Education is an important focus of education systems across Australia. The Federal Government National Goal *that every child leaving primary school will be numerate, and be able to read, write and spell at an appropriate level* (MCEETYA, March 1997) has provided impetus for developing strategies to improve numeracy learning and teaching, particularly in the early years of schooling. For example, The NSW Department of Education implemented the Count Me In Too Project in 1997, and the Victorian Department of Education has recently begun the Early Numeracy Research Project.

During Term 4 of 1998, the Catholic Education Commission of Victoria (CECV) undertook the Numeracy Strategy Project. The project gathered 50 teachers, six university academics and several Catholic Education Office (CEO) consultants from across Victoria for the purpose of gaining knowledge about numeracy learning and teaching in action research contexts. In so doing, the project aimed to provide professional development for teachers who might take on leadership roles in forthcoming years. It was also anticipated that, as a result of the project, the teachers would provide advice for the CECV in the development of a strategic plan for numeracy education in Victoria. The project participants formed six teams which each investigated a different aspect of numeracy learning and teaching. This paper reports on the Early Years Team who researched numeracy learning and teaching in the early years of schooling (K-4). In particular, the effectiveness of the model of professional development adopted by the project is considered through exploring two questions: Did the model result in teachers developing insights about learning and teaching numeracy in the early years of schooling? Was there any evidence that this approach to professional development may result in changing teachers' beliefs and practices?

AN APPROACH TO PROFESSIONAL DEVELOPMENT

The model of professional development designed for the Early Years Team was built on four principles: (1) for changes in beliefs and practices to occur, teachers must first recognise an aspect of their practice is problematic; (2) for professional growth to occur resulting in changing teachers' practice, knowledge and beliefs, teachers need opportunities to experiment and reflect; (3) teachers engaged in a process of change benefit from the support of colleagues and mentors; (4) professional development needs to be driven by teachers' individual concerns, interests and the realities of their classrooms.

In light of these four principles, an action research methodology within a team structure model of professional development was developed. The team approach was adopted to enable teachers to explore issues of mutual interest and concern, to support each through a process of change and to provide opportunities for collective reflection and planning. In a similar way to the Maths in Schools approach (Montgomery, 1995), an academic with expertise in the research area joined the team as a research partner and mentor.

Clarke & Peter (1995) argue the importance of teacher experimentation and reflection in enabling professional growth to occur. The project thus adopted an action research methodology, using the cycle of plan, act, observe, reflect. This methodology provided opportunities for teachers to experiment and reflect on an aspect of numeracy education of individual interest and concern in the reality of their own classroom. Sparrow & McIntosh (1998) highlight the significance of professional development being driven by teachers' concerns, interests and the realities of their classrooms. They argue that teachers must be empowered to identify and meet their own needs for professional development, that they must have ownership of the research and the decision making and that professional development must be relevant to the experience of the teacher and their teaching situation.

In order to assist the teachers to recognise an aspect of their practice that may be problematic and in need of change, the teachers were each asked to identify and write about a dilemma they had faced when teaching number to young children. Such case writing, as proposed by Judith Shulman, " enables a teacher to describe a critical incident or dilemma encountered in personal practice. Collaborative reflection on the writing enables a teacher, and colleagues who read the case, to understand and interpret the described practice and thus to propose action for change or improvement" (Western Melbourne Roundtable Innovative Links Project, 1997, p 29). Cobb, Wood and Yackel (1990) conclude from their research that it is the problems and surprises which teachers encounter in their own practice that give rise to opportunities to re-organise beliefs. Therefore, to enable change, teachers must first perceive their current practice as problematic, and then search for new beliefs and practices to overcome the problems. Identifying and reflecting upon a dilemma assists this process.

THE EARLY YEARS RESEARCH TEAM

The Early Years Research Team formed as part of the Numeracy Strategy Project comprised nine teachers drawn from five schools across Victoria, an academic and a CEO Mathematics Consultant. Two teachers belonged to a Melbourne school, while the others taught in four country schools, three of which were situated more than 2 hours from Melbourne.

The team met initially for two days to: explore issues of mutual concern; to reflect on a problematic aspect of their practice identified through writing a dilemma faced when teaching number to young children; to develop a team research question and individual research goals relevant to their particular teaching situation; and to participate in professional development activities related to number learning in the early years of school. The team considered the stages in children's early number learning, strategies for assessing children's understanding of number, the role of visualisation in early number learning, the use of ten frames and other activities and games for supporting early number learning. The Early Years Team developed the following question to provide a framework for their collective research about numeracy learning and teaching in the early years of schooling:

How can we effectively support young children's development of number concepts?

During the six-week period in which the teachers researched in their classroom, either the academic or consultant visited the teachers in order to discuss their research and future courses of action. At the conclusion of the research period, the team met in Melbourne for a further two days. During this time, the team members shared the results of their individual research, discussed the findings, reflected on the effectiveness of the learning that took place in their classroom as a result of the strategies and activities that were implemented, identified any key insights gained about children's early number learning, and developed a set of recommendations for effectively supporting young children's development of number concepts. The teachers then produced a written report about their individual research projects.

CASE STUDIES OF THREE TEACHERS

The following descriptions outline the experience of three teachers in the Early Years Team. Pseudonyms for each person have been used for the purpose of this report. The case studies describe the dilemma that lead to each teacher's research goal, a brief description of their action during the research and any insights identified about learning and teaching numeracy in the early years. Comment is also made about any evidence of change in the teachers' beliefs or practice.

Geraldine

Geraldine taught a multi-aged K/1 class in a rural city in the Western District of Victoria. She has more than 25 years teaching experience, is a *Numeracy for All* tutor and a member of a district numeracy team. Geraldine was very enthusiastic about mathematics education.

Geraldine described her dilemma in the following way:

> In teaching young children number, some regularly confuse the digits and write 16 for 61, even though they may say sixteen. Other children have no difficulty at all and readily transfer the concrete work to the abstract. I was interested initially to investigate children's place value understanding, particularly with 'teen' numbers, in my prep/one class. In thinking about this, it occurred to me that in preparing learning activities, we may place limits on our expectations of how far children can go with number. I found myself asking: How far can my children go with number? What is their related understanding of place value? If I take away the limits of my expectations, will I give children the opportunity to develop their understanding of number, in a way that may not have been possible before? My dilemma thus broadened considerably to become one associated with young children and large numbers. I was now asking, "What are the numbers young children can deal with?"

Having reflected upon this dilemma, Geraldine developed the following action research goals: to investigate how exploring large numbers contributes to children's development of number concepts; and to develop and consolidate place value understanding, especially teen numbers.

Geraldine began by assessing the smallest and largest numbers known by her students. She was surprised that many children knew about very large numbers. She discovered that although the children knew large numbers, few children could correctly write numbers beyond 99. Their attempts were linked to the sound of the number, rather than the place value conventions of our numeration system. Geraldine then began to explore large numbers with her class using house numbers and Outer Space as a context. Calculators were also used. Eventually, Geraldine introduced place value work using pop sticks, Multibase Arithmetic Blocks and numeral cards. By the end of the project, Geraldine was amazed with the Year 1 children's ability to write and read 3 digit numbers. She was also pleased that several Prep children were writing and reading two and three digit numbers. In summary Geraldine wrote:

> It is important that we don't limit our expectations. I was constantly amazed and delighted by the differing abilities the children showed. They could do things I wouldn't have anticipated. They all grew in their understanding. We may need to consider this more seriously in determining number curriculum for children in the early years. Children can respond to open-ended, co-operative activities, some place value work and most importantly, the opportunity to explore larger numbers.

It is evident that Geraldine now believes that she should not limit her class to the exploration of two digit numbers. She found that many children knew about and were interested in much larger numbers. This gave her the opportunity to explore place value ideas in a much richer way resulting in the children learning about the conventions of reading and writing two and three digit numbers. As a result of this project, Geraldine now believes she is a more effective teacher:

> In all I feel that this project has been very valuable. It has made me a better teacher through having a set focus to work on and by being specific in my questioning and understanding of how children learn. I certainly valued the input from the other teachers in my group and have a renewed zest for teaching mathematics next year.

Lisa

Lisa has more that ten years teaching experience. She taught a Year 3&4 class in a Central Victorian rural city. She is a *Numeracy for All* tutor and a member of a district numeracy team. Lisa described her teaching dilemma as follows:

> During the year I have observed some of the children in my classroom portray a negative attitude towards maths. I felt that these children had this attitude for one of two reasons. 1. due to a lack of self esteem. While a second group struggle in their understanding of maths problems. Some of these group members are interconnecting.

Lisa reported that her prime objective for this action research was to initially identify and then to focus on these children within the mathematical classroom environment. Her action research goal was: to develop in children the ability to apply their mathematical knowledge to real life situations.

Lisa used both open ended questions and the Pupil Perception of Effective Learning Environments in Mathematics (McDonough, 1995), to identify students with a negative attitude towards mathematics. These students became the 'target' group. She supported these children closely as they engaged in problem solving activities relating to two upcoming school events, the school sports and the school fete. The problem solving activities usually related to money and measurement. Lisa reported that by the end of the project the confidence of the target group had increased, they began to spontaneously make connections between mathematical ideas and were receiving encouragement from their peers. She gained some interesting insights into the way in which children believe they most effectively learn maths:

> A significant proportion of the children felt they learnt a new concept more effectively if they were provided with the opportunity to manipulate some kind of materials during the initial teaching time. While in contrast to this the children who have developed more sophisticated problem solving skills are able to tackle the problem with their head, pen and paper.

Lisa reported that her practice had changed. She was now convinced about the importance of spending time talking to children and listening to their interaction. She concluded:

> I found this (experience) to be very rewarding. I especially discovered that I obtained so much information about children's maths learning by simply talking to them and listening to their interaction with peers.

Kathleen

Kathleen taught a Year K/1 class in a Northern Victorian rural city. She has more than 25 years teaching experience and described herself as nearing retirement. She was a member of a district numeracy team, but regretted the time this leadership role took away from her classroom teaching.

Kathleen described her dilemma in the following way:

> After many years of teaching, I have always had considerable difficulty in achieving satisfactory results in the teaching of automatic subtraction facts. To do this I have always taught the relationship between addition and subtraction. In doing this I was aware that my current children were capable of all addition strategies, e.g. 9+6=15. To establish where the children were currently "at" I wrote 15-6= on the board and asked the children for the answer. I then asked 'how' they came to that answer. Some responses were "I counted backwards," "10-6=4 therefore 9-6=3." However, only three children were able to 'think' this way. Others said they would use their fingers and counters.

As a result of reflecting on her dilemma, Kathleen developed the following action research goal: to develop ways of ensuring a complete understanding of the inter relationship between addition & subtraction in year 1 children.

As part of her action research, Kathleen introduced a new teaching strategy. She provided the children with *ten frames* and counters and used visualisation as a means of developing an understanding of number. The children used the *ten frames* to explore the different combinations for a given number. Kathleen also encouraged children to use the *ten frames* to find missing addends in equations. Eventually, the children began to say, "I don't need those frames any more," and were able to satisfactorily find the missing numbers. Kathleen also used games such as dominoes and snap with cards displaying different combinations of a given number to assist children learn number combinations. As a result of her research, Kathleen discovered:

> that the children need a strong understanding of numbers...they need to move from concrete objects to the visual and finally automatic recall....(Previously) I tried to move them to automatic recall without a thorough grounding in the visual aspect.

Kathleen also concluded that questioning individuals about 'how' or 'what' was an extremely important aspect of assessment. This was now an essential aspect of her practice.

The three case studies indicate that these teachers developed some insights about numeracy learning and teaching in the early years of schooling. They also provide evidence of a reported change in beliefs and practice on the part of the three teachers. It was apparent that the teachers' individual research goals related to an aspect of their practice that was shown by their teaching dilemmas to be problematic. Thus, the teachers clearly had ownership of the research and decision making throughout the project. It was also clear that the teachers valued the opportunity to discuss possible actions with the academic and other members of their team. Geraldine explained, "I certainly valued the input from the other teachers in my group." The actions that the teachers implemented related directly to their research goals and were often closely related to activities discussed during the initial professional development phase.

KEY INSIGHTS GAINED ABOUT CHILDREN'S EARLY NUMBER LEARNING

One important aspect of this professional development project was the opportunity and expectation that the teachers reflect upon and discuss the insights they had gained about children's number learning. Therefore. the teachers and academic in the team gathered together for one day to reflect upon and discuss the outcomes of their research. The interest and enthusiasm for what had been learned was very high and the teachers were very keen to discuss the implications of their insights and suggest further causes of action.

Table 1 describes the insights the teachers gained about aspects of children's early number learning considered during the project. It is clear that this model of professional development resulted in all nine teachers developing insights about learning and teaching numeracy in the early years of schooling. The insights are a comprehensive list. They relate to teacher expectations, affective learning, assessment, teaching strategies, curriculum, learning activities and professional development. The teachers were also able to use their insights to form a useful set of recommendations about learning and teaching numeracy in the early years of schooling. This was a stated aim of the Numeracy Strategy Project.

Table 1
Key Insights Gained About Children's Early Number Learning

Aspect of Teaching & Learning	Insights Gained About Children's Early Number Learning
Teacher Expectations	It is important not to limit the children's opportunities to develop number concepts through low teacher expectations. The children proved over and over again that they could do a lot more with number than we expected. Limitations should not be placed on the number range with which children are working.
Affective Learning	It is important that students develop a positive concept of themselves as mathematicians in the early years. This enables students to have the confidence to take risks as part of learning.
Assessment	It is important for teachers to have specific knowledge about children's understanding of number in order to support their learning effectively. In order to achieve this, teachers must spend time talking to children about the strategies they are using to solve problems. Written work samples do not provide the necessary information. It is the conversations with children that really inform teachers. The stages in children's development of arithmetic strategies provide a useful framework for considering children's arithmetic strategies.
Teaching Strategies	It is important to make the aspects of mathematics being explored explicit for students so that their attention is focused on important mathematical ideas. It is important for teachers to highlight the important mathematical ideas discussed by students during sharing times.
	The use of small groups in mathematics classes helps children develop confidence and provides important opportunities to discuss mathematical ideas.
	It is important to include opportunities for students to reflect on the mathematical ideas they have been exploring. An important question is, "tell me what you have learnt".
	Children like to use manipulatives when exploring new mathematical ideas. They find this very helpful. These may be both physical materials and visual images. It is essential to provide these opportunities.
	Visual imagery is an important aspect of number learning. *Ten frames* proved very helpful.
Curriculum	The development of number sense must be a major focus of early years maths programs.
	There needs to be a major focus on the development of number combinations (such as all the number combinations for 8) in the early years of schooling. This is important for developing number sense and powerful flexible arithmetic strategies for problem solving. The project team trialled a number of activities that were found to be successful in this area.
	There is a need to focus on backwards counting in the early years. Children have particular difficulties with counting backwards from numbers greater than 20 and less than 100. Backwards counting sequences bridging the tens prove particularly difficult.
	It is important for children to have the opportunity to explore the meaning of mathematical terms and understand the application of words such as total and difference in a mathematical context. Children need opportunities to make these connections for themselves.
Learning Activities	It is important that mathematics activities are relevant and purposeful by being linked to community events. This helps children become more confident and motivated to learn.
	Problem solving activities support the development of children's number sense.
	It is vital that children have the opportunity to use oral language to support their number learning. Students need to constantly discuss their mathematical ideas and explorations with each other and the teacher.

	Games were an important strategy for developing children's understanding of number. It is important to use games in a focused way.
	It is important to encourage children to estimate and predict when solving problems and then confirm their predictions.
Professional Development	It is essential for teachers to have the opportunity to engage in professional discussion about mathematics. This has proved to benefit our teaching enormously.

CONCLUSION

The model of professional development used by the Early Years Team as part of the CECV Numeracy Strategy proved to be effective. Indeed, the experience was personally rewarding for the teachers involved. At the conclusion of the project, the teachers reported that that they were highly motivated about mathematics teaching, had gained many insights about effective learning and teaching and in some cases felt professionally renewed by the opportunity to research and engage in professional discussion about their experiences and findings.

The project resulted in all nine teachers developing insights about learning and teaching numeracy in the early years of schooling. They were able to clearly articulate their findings and the implications of these findings for learning and teaching numeracy. They also developed a set of recommendations for effective mathematics learning and teaching in the early years of schooling that has potential relevance beyond these teachers' personal experience. The case studies of three teachers provide evidence that this approach to professional development may also result in changing teachers' beliefs and practices.

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UNDERSTANDING MATHEMATICAL TEXT THROUGH PEER EXPLANATIONS

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This paper reports on research that investigated processes through which students come to understand mathematical text. Transcripts from a Year 11 lesson are analysed to illustrate three contexts in which students interrogated mathematical text: reading assigned by the teacher, spontaneous reading, and reading peer-produced text. Although current moves for mathematics education reform discourage over-reliance on textbooks as a source of knowledge and authority, this study demonstrates that reading as a social practice can stimulate students' critical engagement with mathematical ideas.

Current interest in the role of language and communication in mathematics learning is reflected in curriculum documents such as the National Statement on Mathematics for Australian Schools (Australian Education Council, 1991) and the NCTM Standards (e.g. National Council of Teachers of Mathematics, 1991) These moves for curriculum reform are supported by recent research into the potential benefits of student discussion for developing mathematical understanding (e.g. Goos, Galbraith & Renshaw, 1996). However, research in this area has largely dealt with problem solving contexts, and has given scant attention to the possibilities for discussion that may arise when students try to make sense of explanations or examples in mathematical text. In fact, there has been little research on the use of mathematics textbooks in Australian classrooms, possibly because students so rarely read their textbooks for instructional purposes. Instead, students tend to use mathematics textbooks as a source of exercises, while it is the teacher who reads the textbook and then transmits the information to the class (Shield, 1991).

One explanation for this pattern of textbook use may stem from the differences between the language of mathematics and students' everyday language (MacGregor, 1989). The vocabulary, symbolism, syntax and concept density of mathematical text have all been identified as factors causing comprehension problems for students. For teachers, the solution to this problem does not lie in protecting students from difficult text by reading it for them and transmitting the information it contains. Instead, students need to gain experience in interpreting the explanations and examples in their textbooks—not only to help them come to terms with the rigorous formality of mathematical language, but also to encourage them to act as reviewers and critics of mathematical ideas produced by other people. The role of the text then changes, from that of a source of knowledge unchallenged in its authority, to a participant in students' learning conversations.

Lemke (1989) has described the process of using spoken language to make sense of instructional text as "making text talk". Students can make the text their own by talking their way to comprehension, elaborating and commenting on the text in multiple verbalisations which build connections between their own prior knowledge and language, and the formal language of textbooks. Since this process helps students become aware of their understanding, mediation of the text through peer or teacher-student discussion also has a metacognitive function.

This paper considers the question of how students come to understand mathematical text. From a metacognitive perspective, the assessment and development of understanding is seen in students' talk as they interrogate the text with peers, and in whole class discussion led by the teacher. However, the analysis also shows understanding in the making by identifying ways in which mathematical texts enter into different social contexts within the classroom. Sociocultural theories of learning view texts—together with other material and semiotic resources such as calculating devices, symbol systems, structures of reasoning and forms of discourse—as cultural tools which re-organise cognitive processes through their integration into human activity (Resnick, Pontecorvo & Säljö, 1997). From this perspective, then, understanding is scaffolded by social practices in which learners, teacher, and text are participants.

THE CLASSROOM STUDY

The research reported in this paper forms part of a larger study which investigated patterns of social interactions associated with metacognitive activity in senior secondary school classrooms (see Goos, 1998). While the study was carried out over a period of three years and involved eight teachers and their classes, this paper draws on data from a single Year 11 class in the third year of the study. Initial observations of this classroom revealed that the teacher regularly allowed time for students to study worked examples so that they would learn to find their way independently through mathematical text. The examples also introduced students to the formal reasoning involved in applying new concepts.

Subsequent observation identified three situations in which the students read and interpreted mathematical text: assigned reading, spontaneous reading, and reading peer-produced text. The teacher frequently *assigned* reading during lessons, allowing up to ten minutes for students to work through a piece of text themselves before reconvening for a whole class discussion. He did not expect the students to read in silence, but encouraged them to talk to peers to explicate the arguments and examples in the text. The teacher led the ensuing discussion by asking questions, such as *What happened there? So what's coming out of here? Is that right?*, in order to elicit detailed explanations of the text.

The second situation in which students read and interpreted mathematical text arose when they recognised their own lack of understanding of an aspect of the mathematics it presented, and joined with peers *spontaneously* to interrogate an example. Finally, there were occasions when students looked beyond the authority of the textbook and recognised the expertise of peers in producing exemplary texts, in the form of proofs or solutions to problems. *Student-produced texts* were interrogated for the same purpose as conventional textbooks—to understand someone else's thinking about a problem or mathematical idea.

An example of a lesson featuring spontaneous reading is presented next. The lesson comes from a unit of work which introduced students to fractals and chaos theory through the study of iterative processes. The unit was presented via a self-paced, teacher-prepared booklet, complete with examples and tasks, and students were expected to work together with minimal assistance from the teacher. All lessons in this three week program were videotaped, and portions of videotape were later transcribed for analysis.

Data Analysis and Coding

The analysis of the lesson transcript aims to identify features of students' collaborative metacognitive activity as they read mathematical text. The metacognitive function of students' dialogue is revealed through comprehension monitoring statements and explanations (Chi, Bassok, Lewis, Reimann & Glaser, 1989). Monitoring statements indicate that the students either understood, or failed to understand, the material presented in the text (noted in the transcript by the symbols *CM+* and *CM–*). Explanations are classified into one of four categories: (1) refine or expand the conditions of an action, (2) explicate or infer additional consequences of an action, (3) impose a goal or purpose for an action, and (4) give meaning to a set of quantitative expressions. (Only the third and fourth categories of explanation are illustrated in the sample transcript, annotated as *Exp-goal* and *Exp-quant* respectively.) Collaborative interaction is identified by sequences of conversational Moves which distribute a sentence between speakers, whether symmetrically, in the form of *collaborative completions*, or asymmetrically, as *instalment contributions* and partner *acknowledgments* (Clarke & Schaefer, 1989).

The Cantor Set

After the class had spent several lessons working on a set of activities related to the Koch Snowflake, the teacher asked students to read through the next example of a fractal, the Cantor Set. The text they were to read is reproduced in Figure 1. Note the error marked with a * : the common ratio is $r={}^{2}/_{3}$, not ${}^{1}/_{3}$. This error was detected by Alex and Dylan, the students who are the subject of the present analysis.

When the teacher reconvened the class, he asked questions to elicit students' elaborations of the table showing Levels and Lengths of Section Removed, and then moved rapidly through the subsequent part of the example in which the sum of the lengths removed is found (Lines 1 to 4 in Figure 1). (T, A, and D identify the teacher, Alex and Dylan; R and L refer to other students in the class.)

1. T: OK, finding the sum of these terms because we are adding up these bits, are we not? Finding the total space removed? We can see that's a third plus two times a third squared, plus four times a third ... that's what we found? (Students murmur assent) OK, what have I done on the next line? Just turned that into powers of two. Next line's taken a third outside of it ... Yep? (Students focus intently on the text, nod)
2. R: Factorised it.
3. T: With a bit of sleight of hand we get two, we get a third, sorry we get one, two thirds, two thirds squared, two thirds to the third, two thirds to the *n* minus one—what does that represent? Sum of ...?
4. L: A GP.
5. A: (softly) Ooh I'd better learn up on that.
6. T: We want to sum that to infinity?
7. D: (softly, to Alex) What?
8. A: (softly, to Dylan) All this. I'd better go through that.
9. T: We want to sum that to infinity, find out what it adds up to totally, yep?
10. D: (hesitantly, to himself) Yeaahh.

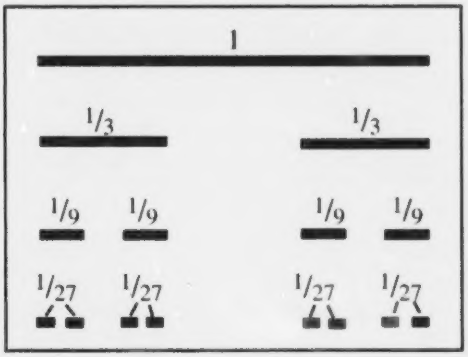
Alex and Dylan's uncertainty is clearly captured in their whispered asides to each other: they knew they had not understood the teacher's *sleight of hand* in moving through the steps labelled as Lines 1 to 4 in the example, since they had not reached this point in their initial reading. The teacher then asked the students to test their understanding of the construction of this fractal by finding how much space is removed from the Middle *Fifths* Cantor Set.

Almost immediately, Alex recognised that the answer would be the same as that for the original example, and then suggested—incorrectly—that they need only replace all instances of ${}^{1}/_{3}$ in the Middle Thirds example with ${}^{1}/_{5}$. Nevertheless, Dylan's uncertainty prompted him to re-examine the Middle Thirds example when he reached the point of reproducing Lines 1 to 4 in Figure 1. The analysis begins with Dylan's announcement that he wanted to go through the part of the example which applied the S_∞ formula, indicating a lack of understanding that he wished to remedy:

22. D: (makes a decision) I'm going to go through this thing (the example) and see *why*. *(CM–)*

Figure 1.
Middle Thirds Cantor Set

One of the most challenging fractals, in the sense of a common understanding of geometry, is the Cantor Set, yet it is the simplest. It is constructed by starting with an interval of length 1 and removing the middle third, leaving the two remaining end intervals. As you might have guessed, this is the beginning of an iterative process, so the next step is to remove the middle third of the two remaining intervals. This can of course be repeated infinitely many times. The full name of this construction is the Cantor middle-thirds set, which is represented diagrammatically below:



Each time the process of removing the middle third is carried out the amount of space between elements of the set increases. It might be interesting to find out the total amount of space removed as it would appear that there might be nothing left after the final iteration.

After the first iteration $\frac{1}{3}$ of the interval has been removed. After the second iteration 2 sections of length $(\frac{1}{3})^2$ have been removed and after the third interval 4 sections of length $(\frac{1}{3})^3$ are deleted. This gives us the following pattern:

Level	Length of Section Removed
0	0
1	$\frac{1}{3}$
2	$2\text{x}(\frac{1}{3})^2$
3	$4\text{x}(\frac{1}{3})^3$
n	$2n\text{-}1\text{x}(\frac{1}{3})^n$

Finding the sum of these terms:

$$S_\infty = \tfrac{1}{3} + 2(\tfrac{1}{3})^2 + 4(\tfrac{1}{3})^3 + 8(\tfrac{1}{3})^4 + \ldots\ldots \qquad \textit{Line 1}$$

$$= \tfrac{1}{3} + 2^1(\tfrac{1}{3})^2 + 2^2(\tfrac{1}{3})^3 + 2^3(\tfrac{1}{3})^4 + \ldots\ldots + 2^{n-1}(\tfrac{1}{3})^n \qquad \textit{Line 2}$$

$$= \tfrac{1}{3}[1 + 2^1(\tfrac{1}{3})^1 + 2^2(\tfrac{1}{3})^2 + 2^3(\tfrac{1}{3})^3 + \ldots\ldots + 2^{n-1}(\tfrac{1}{3})^{n-1}] \qquad \textit{Line 3}$$

$$= \tfrac{1}{3}[1 + (\tfrac{2}{3})^1 + (\tfrac{2}{3})^2 + (\tfrac{2}{3})^3 + \ldots\ldots + (\tfrac{2}{3})^{n-1}] \qquad \textit{Line 4}$$

The section in the large set of brackets represents a converging GP with a = 1 and $r = \frac{1}{3}$ *. Using the formula for the sum to infinity of a converging GP:

$$S_\infty = {}^{a}/_{(1-r)} = {}^{1}/_{(1-{}^{2}/_{3})} = {}^{1}/_{({}^{1}/_{3})} = 3$$

Thus the sum of all the lengths removed is 1/3 x 3 = 1, which is the length of the original section. This is quite a remarkable occurrence: no wonder it is said that the Cantor Set is made up of dust. In fact, the only points remaining are the end points of each interval.

The remainder of the transcript is divided into segments corresponding to the boys' stepwise interrogation of Lines 1 to 4 in Figure 1.

Segment 1—Form Powers of 2 (Lines 1 and 2)

The only element of this segment of transcript which dealt substantively with the example is found in Move 30, where Dylan recognised that Line 1 was obtained by writing out the sum of the lengths of each section removed.

27. A: (Leaning earnestly over his booklet) What's this bit up here? Sum to infinity. Maybe that bit'd help. That little (thumps place on page where the formula appears) S squiggle. *(CM–)*
28. D: S infinity.
29. A: Yeah.
30. D: (gesturing to table of Levels and Lengths Removed) So it's all these added up. So a third ... plus two four eight etcetera— *(Exp-quant)*
31. A: (simultaneously)—plus ... OK, yeah, etcetera. *(CM+)*
32. D: A third ... (pointing to his page) these, now this is, they've got term numbers here now. Two to power one, two to power two ... two to power three, yeah that's making sense. *(CM+)*

Both boys confirmed their understanding of Lines 1 and 2 (in which 2, 4, 8 are simply written as $2^1, 2^2, 2^3$), by commenting that the example was now *making sense*.

Segment 2—Factorisation (Line 2 → Line 3)

This segment of the transcript again begins with a monitoring statement indicating comprehension failure, *what's this one here?*, initiating a series of explanations which gave meaning to the connection between these two lines of working.

37. A: So this one, what's this one here? *(CM–)*
38. D: Now they've bunged them all back, this one.
39. A: But why is it, from there to there, what happens here? *(CM–)*
40. D: A third, one. Why have they got *one*? Is that ...? One plus ... A third ... that's because you— *(CM–)*
41. A: Ahh! They've brought the, they've moved the powers back one (writing something on the example). *(Exp-quant)*
42. D: (looking at example) Huh? *(CM–)*
43. A: (taps Dylan's shoulder) Look. (Dylan looks at Alex's copy of the example) The common, common thingo, is a third, right? *(Exp-quant)*
44. D: Yeah.
45. A: So they've put that outside the brackets. So they've just done that, made that one (writes on the example), made that two. See that? They've got the third there— *(Exp-quant)*
46. D: Yeah.
47. A: —times one, so that is now that third there. (glances briefly at Dylan) *(Exp-quant)*
48. D: That's that (points to example in Alex's booklet) ...
49. A: Plus, that's a third, that now means a third, plus *two* times a third, *squared* ... *(Exp-quant)*
50. D: Ahh. *(CM+)*
51. A: Because it's a third *times* that. (Dylan nods) Right? And so on until you get, plus a third times ... (expression of sudden understanding on his face) yeah see, see now it's to the *n* minus one at the top there. *(Exp-quant)*
52. D: Yeah. (now looking at his own copy of example) *(CM+)*
53. A: Right. That's making sense. *(CM+)*
54. D: Sort of! *(CM–)*

Alex signalled that he understood how Line 3 was obtained by explaining that *they've moved the powers back one*. However, this cryptic statement was not a clear enough explanation for Dylan *(Huh?)*. Alex then delivered an extended elaboration of his initial explanation, as instalment contributions in Moves 43, 45, 47, 49 and 51. Two issues needed clarification: why the powers of $^1/_3$ had been reduced by one, and the origin of the 1 inside the large bracket. Both were explained by Alex pointing out that the common factor of $^1/_3$ (*common thingo*) had been taken outside the brackets. Despite acknowledging each instalment, and even indicating some understanding, Dylan did not share Alex's confidence that the example was now making sense (Move 54).

A striking feature of this segment of the transcript is the way in which Alex directed Dylan's attention to salient features of the example, and checked Dylan's understanding of each element of the explanation as it was delivered. For example, Alex ensured that Dylan was ready to pay attention by tapping him on the shoulder and indicating that they should both look at Alex's copy of the example in the Chaos booklet. He also glanced at Dylan to search for signs of understanding, and maintained their shared focus by punctuating his explanation with questions such as *Right?*, and *See that?*.

Segment 3—Express as a Geometric Series (Line 3 → Line 4)

Satisfied that he could justify the first three lines of working, Alex moved on to Line 4, in which the index law $a^n b^n = (ab)^n$ was applied. In this segment of the dialogue, the boys made a significant conceptual advance by recognising that the goal was to obtain an expression in the form of a geometric series.

55. A: (still staring at his booklet, Line 4) And then all of a sudden they've decided they've got two's as a common factor in there ... *(CM–)*
56. D: Yeah, well that's right.
57. A: And the power— Oh they're going so they get the powers the same! So you can put two in there. *(Exp-goal)* (Pause, Dylan looking at his own booklet) See that? (Alex indicates his copy of the example, then points to the place in Dylan's) See that? I mean they put that—
58. D: They get each, denominator— *(Exp-goal)*
59. A: They did this section here so, yeah, this power there— *(Exp-goal)*
60. D: That's not a power!
61. A: (still looking at own booklet) —yeah, so that power there is the same as that there. Three three, two two.
62. D: So they [inaudible] that—
63. A: So they can put the two into the, thing inside.
66. D: (to himself, looking at own booklet) Two thirds, two thirds, two thirds.
67. A: So two thirds to the *n* minus one.

Again it was Alex who initiated the explaining process, by identifying the reason for the algebraic manipulations of Lines 1 to 4: *they're going so they get the powers the same*. Once more, too, Dylan hesitated, and did not respond to Alex's repeated queries of *See that?* Perhaps sensing his partner's uncertainty, Alex started to elaborate further; but Dylan immediately joined in and the students constructed the justification seen in Moves 57 to 63 as a series of collaboratively completed sentences.

Segment 4—Error Detection

While the final segment of the transcript does not deal explicitly with the steps in Lines 1 to 4, it does reveal the outcome of the boys' purposeful interrogation of this section of the example. Dylan continued by reading aloud from the text.

68. D: (reading) "The section in the large set of brackets represents a converging GP with *a* is equal to one and *r* is equal to a third."
69. A: But isn't *r* equal to *two* thirds? *(CM–)* (Pause) Yeah it is (confident, taps Dylan's shoulder), they've just done it wrong because you look down there ... look ... it says two thirds there (showing Dylan the place in a subsequent step), so it is two thirds.
70. D: Ah. (turns to teacher, sitting behind) Mr G, is this a mistake? (Holds up booklet for teacher to read)
71. A: (to teacher) Is that meant to be two thirds?

72. T: (reads example) That should be two thirds.
73. A: (pleased, vindicated) Right, good! OK! Sum to infinity thing, OK. *(CM+)*

Alex's query, *But isn't r equal to two thirds?*, expressed his incomprehension of the text as it stood, and brought to light the error which had remained unnoticed by all until now. Although the boys called on the teacher to verify their claim, they did so with a degree of confidence that indicates a willingness to actively question, rather than passively accept, the mathematical ideas presented in the text.

Despite their success in making sense of these four lines of the Middle Thirds Cantor Set example, Alex and Dylan persisted with their over-generalised strategy of replacing $^1/_5$ with $^1/_3$ for attacking the Middle Fifths problem, as they still overlooked the need to draw a fresh diagram and construct from first principles a table showing how much space was removed with each iteration. When Dylan completed his calculations and announced that his answer was $^1/_3$, both boys were puzzled at the apparent contradiction between the calculated and expected results. However, this impasse was only recognised as the lesson ended, and they had to wait until the following lesson for its resolution.

Postscript

In the next lesson when the teacher asked for a volunteer to outline the solution process for the Middle Fifths problem, it became clear that many students had made the same error as Alex and Dylan. Rhys was one of the few students who had produced the correct solution, and he offered a detailed, but rapid, justification of the steps which led to the pattern $^1/_5 \times 2^{n-1} \times (^2/_5)^{n-1}$ for the length of section removed. As soon as the teacher ended this lesson phase and instructed the class to continue with a new task, Alex and Dylan moved to Rhys's desk and began examining his solution, which had now taken on the authoritative status of a worked example. Unlike the situation with textbook examples, the author himself (Rhys) was available to make his own text talk, enabling the readers to compare their interpretation with the author's intentions. Alex and Dylan eventually recognised the cause of the different patterns for the Middle Thirds and Middle Fifths Sets; however, the most convincing evidence of their understanding appeared when they went beyond these specific examples and attempted to find a general form for the algebraic patterns that emerge from the construction of any Cantor Set.

DISCUSSION AND CONCLUSIONS

From the teacher's perspective, the lessons in the unit of work on Chaos Theory followed a repeated pattern of assigned reading followed by whole class discussion. However, the students had their own agendas which were organised around their desire to understand the mathematics presented in the text, and they frequently delayed complying with the teacher's instructions to start work on a new task until they were satisfied that they understood the ideas introduced in the preceding section. As well as initiating this spontaneous reading, students also interrogated peer-produced text whose authority had been validated through whole class discussion.

How did the text come to be understood? From a metacognitive perspective, this question is answered by considering patterns of comprehension monitoring and collaborative explanations which re-interpreted an example and highlighted important information. Comprehension failure was usually followed by explanations which initiated the inferencing process necessary for restoring understanding, so that repeating cycles of the form *Comprehension failure (CM–)→ Explanation →Comprehension restored (CM+)* became apparent as students interrogated each step in an example. This activity proceeded through collaboratively completed sentences or explanations delivered in instalments with partner acknowledgments, punctuated by verbal and non-verbal cues to listen to what was about to be said.

Although the students' explanations helped them to understand the text they were reading, comprehension was not purely an "in-the-head" process—understanding was scaffolded by social interactions involving students, teacher, and the text itself as participants. In making the text talk, students initially made much use of informal language; for example, Alex at first verbalised the symbol S_∞ as *S squiggle*, while Dylan exclaimed *Now they've bunged them all back* when he noticed that the powers of $^1/_3$ had diminished by one. While the students' colloquial language served an important purpose in helping them make sense of the ideas in the text as they worked with their peers, during whole class discussion the teacher explicitly guided them towards more formal articulation of these ideas by insisting that every action taken in the worked examples was fully explicated.

Current moves for mathematics education reform promote student interaction and discussion and criticise over-dependence upon the textbook for explanations of concepts and procedures. Yet, as the analysis presented in this paper has demonstrated, mathematical text can mediate social interaction between students in a way that transforms their relationship with mathematical knowledge. Rather than treating it as a pre-packaged object containing facts to be absorbed, the students interrogated, critiqued and appropriated the text by speaking its patterns in their own language. Their actions show that mathematics texts may still have a place in inquiry based classrooms—not as a source of exercises, but as a stimulus for critical engagement with mathematical ideas.

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EMERGING THEMES IN STATISTICS EDUCATION

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This paper reports research in progress. It brings together some issues and themes emerging from the literature review (to date), preliminary interviews, personal reflections and earlier research. These will be extended and refined in further research and will be included in my thesis which focuses on the search for possible (shifting) paradigms within which statistics education (research and practice) is framed.

INTRODUCTION

This paper reports on the early stages of research in progress for a doctoral thesis which looks at statistics education from a cultural perspective. The definition of 'culture' that is used refers not only to ethnicity, but also to educational and workplace cultures. This paper begins by locating the study within my thesis. I then review and comment on literature related to Introductory Statistics courses – the focus is on the perspective taken by the statistics education community, specifically those concerned with service course statistics. The education perspective will be addressed later. Issues arising from this literature is discussed and combined with themes emerging from preliminary interviews, personal reflections and my earlier research in an attempt to identify possible constructs for a framework. A form of 'gap' analysis will then be used to identify possible directions for further research.

BACKGROUND OF THE THESIS

Statistics has traditionally been taught at university level within mathematics departments. With the changing profile of the statistics practitioner we see an increased prominence of introductory statistics courses within university departments, and new influences being exerted on the nature and form of the statistics curriculum. Acknowledgement of these influences suggests new approaches to the design and delivery of these courses may be appropriate. Various approaches have been reported within the literature. What appears to be missing is a holistic view that simultaneously addresses cognitive, affective and volitional issues. I believe that a discussion of the issues from a cultural perspective will lead to a more holistic approach to statistics curriculum.

The meaning of 'culture' that will be used in this study is consistent with the definition provided by Stockton (1995, p5):

> "I use it to describe an integrated complex of shared assumptions within a given society, manifesting itself in a distinctive pattern of behaviour accepted as 'normal', which is learned and transmitted by conventional symbols, adapting itself organically (in terms of the group's current values) to changed circumstances.'
> This description stresses:
> - the internal core of values, attitudes, goals, ways of thinking;
> - the external expression of these in behaviour and communication;
> - the dynamic for change in new settings of environment, economy and social structure."

My preliminary research (Gunn, 1996) related to this topic identified aspects of Australian indigenous culture that were relevant to the development of culturally appropriate pedagogy. The aspects were beliefs/religion, values, world view, view of knowledge and social structures/communication styles. These provide a basis for the analysis of culturally appropriate curriculum. My present thesis, as a natural extension of this research, is a consideration of the cultures that interact in the learning/teaching environment. Of specific importance are the cultures of the intended workplace, the perceptions that statistics practitioners and statistics lecturers have of the culture of statistics, and how these cultures are negotiated within the classroom.

The focus of the thesis is the introductory statistics curriculum (tertiary level) although it is envisaged that the findings will be generalizable. The tentative framework is based upon constructivism; enactivism; Jungian notions of the ***I***, the ***self*** and types of consciousness; and socio-cultural views that acknowledge the complex dynamic that links man and environment.

The study is qualitative and explorative. My preliminary ideas will be clarified and developed by a critical analysis of the literature pertaining to the cultures considered and current educational and psychological theories. Further understandings will come from interviews and case studies that, at this stage of the research, have only been piloted. The researcher serves as the instrument for the data collection. The research is conducted in the constructivist paradigm and this implies that conclusions reached in the research will be coloured by the experience that I bring to the project. For the past ten years I have been designing and delivering service courses in statistics (University level) for diverse cultural groups (ethnic cultures and workplace and student cultures) and in learning institutes with different educational philosophies. It is this experience that has led me to approach the research from a cultural perspective.

Some of the issues that I will address in my thesis are:

- The connections between the statistics curriculum and shifting social paradigms
- Influences of specific cultural groups on the perceived nature of statistics.
- Values, motivation, purpose, goals.
- Lecturers and students implicit and explicit assumptions.
- Different ways of seeing the world – ways of perceiving
- Different ways of thinking and knowing, different types of knowledge.
- The role of the emotions in learning and intuition, imagination, inspiration – what are they and how do they 'fit' with learning and views of knowledge.
- How and to what extent do these issues influence the approaches taken in tertiary statistics courses.

As mentioned earlier, the purpose of this paper is to identify emerging themes that may be relevant to the larger study. I hope to identify some of these issues in the course of this discussion. I will focus on information obtained from preliminary interviews and personal communications with statistics lecturers and consulting statisticians and will look at issues and themes that have already emerged from the literature review. Later I plan to conduct further in depth interviews and case studies as the thesis progresses and other issues become apparent.

ISSUES ARISING FROM THE LITERATURE REVIEW

I have grouped the approaches taken in the literature into three (partially) overlapping areas, based upon the perspectives taken and the issues that are addressed.

Content and Context Approach

The research that focuses on content and context issues addresses questions pertaining to the goals of the statistics curriculum. This implies that the researchers have given some consideration to the nature of statistics. In many instances (eg. Hogg, 1991; Vere-Jones, 1996) this extends into addressing the question of appropriate pedagogy for teaching statistics. In many, but not all cases, implicit assumptions are made about how students learn statistics – the decision about appropriate pedagogy is more often based upon perceptions of the types of skills that statisticians need and how the students will use statistics outside academia.

Teaching and Learning Theory Approach

The issue that is prominent in this approach is how students learn statistics and factors that affect their learning. Gal and Ginsberg (1994), Garfield (1995) and Gordon (1997) all agree that student's negative attitudes and beliefs towards statistics present a major barrier to learning. Garfield also comments on the general resistance of lecturers to shift from their existing beliefs and practices about what constitutes 'good pedagogy'. Included in this approach are discussions on student misconceptions about statistical concepts (Shaugnessy, Garfield, Greer; 1996) and appropriate pedagogy for the learning of statistics (Moore, 1997). In all these papers the prevailing educational theory appears to be constructivism. The social dimension to learning is explicitly acknowledged by some (for example Gordon, 1997) but not by all. This emphasis on constructivism (in its various forms) means that scant attention is given to the place of informal knowledge or the role of intuition, imagination and inspiration in the construction of knowledge.

Philosophical Approach

This approach in the research is gaining prominence as researchers move towards identifying the nature of statistics. The discussions revolve around what is statistics, what constitutes statistical thinking and who should teach statistics. (Hawkins 1996; Moore, 1997; Pfannkuch, 1999).

What is Missing in the Research

Although this review is not exhaustive it is representative. There does not appear to be any work in statistics on the different types of knowledge (such as separated and connected knowing (Belenky, Clinchy, Goldberger & Tarule; 1986)). Discussion of these questions is being undertaken by the mathematics educators but not yet by statistics educators at the tertiary level.

Furthermore, the cultural dimensions of learning and statistics do not appear to have been considered. Ethnomathematicians, amongst others, acknowledge that learning is a cultural phenomenon (Pinxten, 1994). Within the statistics education community there is mention of the need for a "*differentiated curriculum that caters to the needs of many groups*" (Shaughnessy et al., 1996). However the use of 'culture' and 'cultural constructs' as a means for bringing about this differentiation is not considered.

In general, the research is not yet connected. No firm philosophical foundation or theoretical framework has been proposed that connects the research into a 'big picture' view. This lack of explicit connections in the research may be due to the fact that statistics and education are only beginning to meld. This can be seen in the different perspectives taken in the research literature. On the one hand we find the statistician (working with implicit assumptions regarding aspects of the nature of statistics) and on the other, the educator's perspective (based on learning theories).

I propose that Statistics Education is about a balance between statistics and education, in the *context* of the individual, society, culture, and the dynamics of the whole system. This implies establishing the links between statistics and education, and humanist and socio-cultural theories. In order for these links to be explicit there needs to be an awareness of the (shifting) paradigm within which statistics education is framed. ***Paradigm*** is used here in a similar context to that employed by Capra (1996) when he coined the term '*social paradigms*', that is "a constellation of concepts, values, perceptions, and practices shared by a community, which forms a particular vision of reality that is the basis of the way the community organises itself." It is the search for this shifting paradigm that motivates the current research project.

One possible way to begin this search is to discuss the issues relevant to statistics education from a cultural perspective. This acknowledges learning as a cultural phenomena and knowledge as culturally specific (Begg et al, 1996 commenting on the work of Pinxten, 1994). It allows for the possibility that other cultural groups such as business management etc may be exerting an influence on the form of statistics and on the statistics curriculum.

THEMES EMERGING FROM THE INTERVIEWS AND MY REFLECTIONS

Some comments made in various communications have been broadly grouped on the basis of emerging themes. My observations and reflections are included to indicate the emerging framework. The source of the comments is identified as follows:

Background

M: [Ph D in Econometrics. Worked as a statistics lecturer in a university mathematics department for more than 20 years. Was involved in consulting work -both in-house and within the wider industry sector. Moved to industry where he was designing and delivering in-house statistics courses to chief executive officers. Returned to academia consulting and lecturing in statistics.]

M is of interest to the research because his comments document changes in his perceptions of statistics and statistics curriculum that appear to be due, in part, to his active engagement with new cultural groups. M also talks about how his changed perceptions led him to experiment with alternative approaches to management statistics curriculum.

Q: [Consulting statistician who lectures in university statistics. Has been involved in the introductory statistics course and is strongly motivated to understand how his consulting experience can be used in the teaching of statistics.]

Most of the comments from Q have come from private discussions and seminars that he has delivered. I hope to be able to interview Q in depth at a later stage in the research.

H: [Consulting statistician in agricultural sector, lectures in applied statistics at university.]

L: [Themes from the literature]

H and L are referred to in Table1. Comments in brackets are the researchers ideas.

Comments from M

On Influences that Brought about a Shift in Perceptions

M: Consulting helped but the recognition I think came out in industry that what we are teaching in University is totally irrelevant to what we actually do out in industry.

(This suggested a change in M's perception of what is statistics.)

M: Personal contacts – in particular the collaboration with a friend (with no formal academic training) who had been a trainer out in big companies.

(The influence of personalities already enculturated into the management culture seemed significant. It was also interesting to note M's comment that neither of these significant figures had any 'academic training'. That makes me wonder if he was also beginning to question 'what qualifies one to teach?)

On the Changing Roles of Statisticians

M: We (statisticians) don't mind if you crunch the numbers – in fact we developed software to make it easier for people out there to actually do their own technical work but our role now becomes one of facilitating the understanding of the concepts behind statistics. 'Think not crunch!'

(This indicated M's belief about how statisticians can respond to changes in society brought about by advances in technology. Also, about what is perceived to be important in the context of statistics education (a value statement)).

About a Particular Management Culture:

M: Managers are concerned with asking the BIG questions.

(Perception of the way managers 'see the world, or make sense of the world'?).

M: I think that managers are concerned with decision making, problem solving.

(Awareness of the type of thinking relevant to management).

Translating these Perceptions into the Curriculum Context

M: Tests of hypotheses is quite subtle and conceptually difficult. My belief is that estimation is the only thing that managers need to know about. They need to know what the error is in their estimates and so on. So they are going to make decisions based upon the actual results that they got .... not if its significantly better.

We need to teach managers an awareness of the possibilities – conceptually open the vistas. But we also need to make them aware that being able to 'crunch' is not the same as understanding the statistical concept. 'Think not crunch'.

But then realistically, that is the situation they are going to be in when they start work so it would be a much better training for them I think to get into that way of working. (elaborating on his earlier comments on group work and team projects).

Relevance is more important for managers than rigor.

(He advocated using the statistical language that is used by the public provided it is doing the same job as statistical jargon. The example quoted was the use of 'margin of error' rather than 'confidence intervals' - a language issue.)

About the Sub-cultures within a Company

M: Senior managers were much more interested in the BIG picture – the philosophy. Engineers or technicians were a bit more interested in some statistical methods. Shop floor workers were keen on a hands on sort of experience .... The shop floor workers in particular you couldn't stand up and lecture. You basically had to give them activities.

(Appears to be acknowledging the different philosophies, ways of thinking, ways of doing (with regard to statistics) of the sub-cultures within the company and how this influences the approach taken in the curriculum. This is consistent with later comments that M made regarding the content and context of the course.)

Personal Reflections

Many of M's observations regarding the culture of management are consistent with my own experiences with mature-aged middle management students. Of course, the similarities in observations may just be a classic case of 'finding what you expect'. However, the interviews were conducted as informal discussions, a sharing of thoughts and reflections. Therefore, most of the comments that I have reported on in this paper were not (intentionally) solicited opinions.

Comments from Q

Q: We need to accommodate for the changing 'youth culture'. (he went on to explain) this was reflected in the students preoccupation with assessment, unwillingness to think for themselves, the perception of statistics and its relevance to their life, and the general 'temperament of youth' which is a reflection of the times in which we live.

(type of knowledge that students have and shifting social paradigms?)

THE EMERGING FRAMEWORK

From the interviews, communications and reflections (to date), themes and issues are emerging that may be distilled into four interrelated and integrated constructs, namely world view, values, knowledge, and language and communication.

World View (W)

- Ways of seeing the world, of making sense of the world
- The relationship between SELF and the world, includes the sense of SELF
- Spirituality, beliefs.
- Perceptions of 'how things are'.

Values (V)

- What is considered to be important, to have value, to be significant.

Knowledge (K)

- Focuses on the issues of 'What is knowledge?', 'How do we come to know?', different types of knowledge and ways of knowing.

Language and Communication (C)

- Includes noticing, listening, body language, written language, spoken language, statistical language, and preferred ways of communicating.

To illustrate the distillation process I have taken a small sample of some of the more commonly recurring themes and identified the associated constructs. (see Table 1). The sources of the themes are also identified using the symbols M, Q, R, H and L.

These constructs are a modified version of those that were suggested in my earlier research. I originally identified Beliefs and Religion as a cultural construct of relevance (Gunn, 1996). However, I now believe that this aspect forms part of the larger construct of 'world view'.

The understanding of *'knowledge'* has been refined after my deliberations on women's ways of knowing (Belenky et al, 1986). I now believe it is appropriate to place a stronger emphasis upon different ways of knowing and the related issues of power and I include recognition of different types of knowledge.

These constructs have generally been affirmed by the literature, the most obvious being the importance of the construct of 'world view'. This can be seen in the relative prominence of the question 'What is statistical thinking?" in the more recent research on statistics education (for example, Moore, Pfannkuch, amongst others). Further confirmation has come from the preliminary interviews that have been reported upon in this paper and is continuing to be reinforced by interviews in progress.

IN CONCLUSION

Yet to be identified within this emerging framework are non-cognitive issues such as intuition, imagination and inspiration; the role of emotions (other than fear) in learning. There are also ideas about the dynamic relationship between self and the world, ways of being and doing, that need to be 'located'. Exploration of these ideas will expand the dimensions of the study and bring a greater depth to the emerging framework.

Table 1
Emerging Themes and Associated Constructs

Themes	Source	W	V	K	C
Statistics is about an understanding of variability.	L		√		
Statistical Thinking is about -					
understanding that any process is subject to variation	M		√		
.... and knowing how to react to it.	M	√			
We never end up with a definitive answer – only estimates; there is always a degree of uncertainty in our answer.	H	√			
Throughout the solution process a strong link is maintained with the real world problem.	H	√			
Maintaining connections between abstractions and the real world problem. Relevance of the problem solving cycle.	L	√		√	
Statistical Literacy	L	√			√
Statistical language cf everyday language	L				√
Co-operative learning strategies	L			√	
Group work, team work, projects	L, M, Q			√	√
The relationship between maths and statistics:					
• Statistics is/isn't mathematics.	L	√	√		
• The maths language may 'work' for some students when they are coming to an understanding – but then again, maybe it doesn't.	L	√		√	√
• "We talk a lot about giving students the 'big ideas' but some of the 'big ideas' in statistics aren't mathematical."	Q	√	√		
Context is important (situated learning theory)	L			√	

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MENTAL ADDITION AND SUBTRACTION STRATEGIES: TWO CASE STUDIES

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This paper tracks two children's mental strategies over a 5 year period (Years 2 to 6). Although the children were students in traditional classrooms, where teacher-taught algorithms may have conflicted with the children's spontaneous strategies, they continued to develop their own efficient strategies throughout the period of the longitudinal study. However, by Year 6, both children were also employing taught pen and paper algorithms which were less effective for mental calculations. Finally, some implications for teaching are discussed.

Ample evidence exists in the literature to suggest that children have difficulties learning computational algorithmic procedures by traditional transmission teaching methods (e.g., Olivier, Murray, & Human, 1990), and understanding what is happening when they do learn the procedures by the traditional methods (e.g., Resnick & Omanson, 1987). Contrary to this conclusion, there is evidence from research in reform classrooms that active involvement in mathematics learning enables children to understand computation, particularly where formal mathematical knowledge is built on informal knowledge (Carraher, Carraher, & Schliemann, 1987; Carroll, 1997; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter, & Fennema, 1997; Kamii, Lewis, & Livingston, 1993; Thompson, 1994).

Carroll (1997) and Kamii, Lewis, and Livingston (1993) documented the mental and written computational procedures invented by children who are active in their learning. They showed that children can produce a wide variety of efficient strategies that exhibit sound number understanding even though there was little direct teaching of algorithms. They also found that the active development of knowledge encouraged children to participate in the construction of problems and the explanation of solution strategies.

In Queensland, there are few reform classrooms and the traditional pen and paper algorithms are still taught out of context and in situations where children have little or no input into constructing problems and explaining solutions. As Cooper, Heirdsfield and Irons (1995 & 1996) reported, this has resulted in a tendency for Queensland children to use strategies for mental computation that reflect the procedures underlying the pen and paper algorithms regardless of their knowledge and ability to use more efficient strategies.

This paper reports on the progress of two children, Catherine and Adrien, in traditional classrooms over a period of five years. These children used a variety of creative strategies before traditional algorithms were taught to them.

THE STUDY

Subjects

Catherine and Adrien were two of 140 children chosen in Year 2 to participate in a large five year longitudinal Australian Research Council funded study into children's spontaneous mental strategies for addition and subtraction (Years 2 to 6). Children were originally chosen by the teachers to represent one third above average, one third average, and one third above average.

Catherine and Adrien attended different schools, Catherine attending a State Primary School and Adrien, a Catholic School, both being coeducational schools. Adrien changed schools in Year 5 and attended a Catholic boys' school. Although a focus of the study did not include classroom practice, anecdotal records indicated traditional teaching practices in both classrooms. Classroom discussion was rarely (if ever) heard. Children were only ever called upon to explain taught procedures. The teacher's voice was the predominant voice heard in the classrooms, as review of the videotapes testified. Blackboards were covered with many examples of algorithms, and rarely contained any contextual problems.

The children's ability levels were not revealed to the interviewer; however, their performance indicated that Catherine and Adrien were likely to belong to the "above average" group. Adrien's classroom teacher stated that he excelled in mathematics. However, Catherine's teachers stated that she was "not as good at maths as some make out".

Interview Procedures

The children were withdrawn from the classroom and the videotaped interviews were conducted in a separate room. The instrument was Piaget's clinical interview technique. The interview duration was limited to 30 minutes, to avoid the children's tiring. There were three addition and subtraction interviews in Year 2 and two interviews in each of Years 3 to 6. The children were presented with tasks, asked to solve them, and directed to explain their working.

Tasks

The tasks consisted of one, two and three-digit word problems and algorithmic exercises (see Cooper, Heirdsfield, & Irons, 1995 & 1996). The word problems were presented in number and picture form (the children listened as the interviewer said the problem). The addition word problems consisted of join addition; the subtraction word problems consisted of take away, missing addend, and difference. Algorithmic exercises were presented in both vertical and horizontal form. The numbers were chosen in the hope of eliciting wholistic strategies (c.f., traditional pen and paper algorithms); for example, 246 + 99 is more efficiently solved by employing the wholistic strategy of adding 100, then taking 1. Over the period of research, progressively more difficult examples were presented, as the children matured. There was an attempt to present the easier examples throughout all the interviews, but this was not always possible because of time constraints on the length of the interviews.

FINDINGS

In the literature, computational strategies have been classified (e.g., Beishuizen, 1993; Cooper, Heirdsfield, & Irons, 1996; Reys, Reys, Nohda, & Emori, 1995) into 5 major categories (see Table 1). It is the intention of this paper to report detailed descriptions of interesting strategies and not to collapse full and interesting descriptions into single terms. However, the categories are useful for comparing children and for describing strategy change across time.

Strategy Use Across the Interviews

In Table 2 the more interesting solutions for particular questions are reported for each child across the interviews. The last two interviews in Year 6 are not reported in the table, as no new strategies were used by the children, and one predominant strategy was being used by both children. This phenomenon is discussed later.

Table 1

Mental Strategies for Addition and Subtraction

Strategy		Example
Counting		**28+35**: 28, 29, 30, .. (count on by 1)
		52-24: 52, 51, 50, .. (count back by 1)
Separation	*right to left (u-1010)*	**28+35**: 8+5=13, 20+30=50,63
		52-24: 12-4=8, 40-20=20, 28 (subtractive) : 4+8=12, 20+20=40, 28 (additive)
	left to right (1010)	**28+35**: 20+30=50, 8+5=13, 63
		52-24: 40-20=20, 12-4=8, 28 (subtractive) : 20+20=40, 4+8=12, 28 (additive)
Aggregation	*right to left (u-N10)*	**28+35**: 28+5=33, 33+30=63
		52-24: 52- 4=48, 48-20=28 (subtractive) 24+8=32, 32+ 20=52, 28 (additive)
	left to right (N10)	**28+35**: 28+30=58, 58+5=63
		52-24: 52-20=32, 32-4=28 (subtractive) : 24+20=44, 44+8=52, 28 (additive)
Wholistic	*compensation*	**28+35**: 30+35=65, 65-2=63
		52-24: 52-30=22, 22+6=28(subtractive) :24+26=50, 50+2=52, 26+2=28 (additive)
	levelling	**28+35**: 30+33=63
		52-24: 58-30=28 (subtractive) : 22+28=50, 28 (additive)

One digit word problems were solved by recall or by using the *build to 10* strategy by Adrien e.g., [14-8=(14-4)-4], and Catherine [e.g., 8+5=(8+2)+3]. Catherine continued to use the strategy for more difficult calculations [e.g., 136+56=(130+50)+(6+4)+2], as did Adrien (e.g., 75-29 in Interview 1, and 136+56 in Interview 8, see Table 2).

Table 2
Addition and Subtraction Strategies for Word Problems and Algorithmic Exercises, used by Catherine and Adrien

Interview	Catherine's Strategies	Adrien's Strategies
1 (Term 1, Year 2)	**45-21**: (45-20)-1=25-1=24.	**45-21**: (45-20)-1=25-1=24.
		32+15: 32+10, 42+5=47.
		29+35: 25+35, 60+4=64.
		36+29: 36+9, 45+20=65.
		75-28: 75-20, 55-5, 50-3=47.
		96-39: 96-30, 66-9=57.
		65-38: 38+22=60, 5 more to 65. 22+5=27.
		95-26: 95-20, 55-6=49.
2 (middle Year 2)	**75-28**: 70-20, 50-8, 42+5=47.	
	96-39: 90-30,60-10, 50+7=57.	
3 (Term 4, Year 2)	**75-28**: 70-20,50-3(3 below)=47.	
4 (Term 2, Year 3)	**38+37**:30+30, (8+2)+5, 60+15=75.	**130-49**: 130-9, 121-40=81.
	136+56: (((130+50)+6)+4)+2.	**253-98**: 250-90,160-10, 150+5=155.
	100-59: 50+50=100, 50-9=41.	

Interview	Catherine's Strategies	Adrien's Strategies
5 (Term 4, Year 3)	**95-47**: 90-40, 50-2=48. **106-88**: (100-80)-2=20-2=18. **90-68**: 6+3=9(90), 30-10=20, 20+2=22. **165+217**: 200+100, 60+10, 5+7. **250-127**: 200-100, 50-20, -7.	**136+56**: 136+50, 186+6=192. **246+99**: (246+100)-1=345. **246+178**: 246+170, 416+8=424. **82-57**: 57+23=80,80+2=82, 23+2=25. **253-98**: (253-100)+2=155.
6 (Term 1, Year 4)	**106-88**: 80+20=100, 20-2=18. **253-98**: 98+2=100, to 253 is 153, 153+2=155. **100-59**: 50+50=100, 50-9=41. **96-49**: 5+4=9 (90), 50-3=47. **325-168**: ((300-100)-40)-3.	**106-88**: 106-6=100, 100-82=18. **253-98**: (253-100)+2=155. **246+99**: (246+100)-1=345. **246+178**: 240+170=410, 410+6=416, 416+8=424. **92-57**: 92-60, 32+3=35. **62-36**: 62-30, 32-6=26.
7 (Term 3, Year 4)	**106-88**:88+2=90, 90+10=100, 100+6=106, 2+10+6=18. **325-168**:3-1=2(200), 100, 12-6=6, 50, 15-8=7. **92-57**: 57+3=60, 60+30=90, 90+2=92, 3+30+2=35. **100-59**: (100-60)+1. **120-45**: (120-20)-25. **253-98**:98+2=100, 100+100+3=253.	**106-88**: 106-16, 90-2=88. 16+2=18. **325-168**: 300-100=200. 100-68=32 and 25+32=57, answer 157. **92-57**: 92-30=62, 62-5=57, 30+5=35. **62-36**: 60-36, 24+2=26. **400-337**: ((400-300)-30)-7=63. **250-127**: 250-130, 120+3=123.
8 (Term 1, Year 5)	**79+45**:7+3=10,10+1=11 (110)9+5=14, 110+14=124. **95-47**: 47+40=87, 87+8=95, 40+8=48. **106-88**: 88+20=108, 108-2=106, 20-2=18. **253-98**:98+2=100, 100+53=153, 153+100=253, 2+153=155. **246+99**: 299 goes to 300, 300+45=345. **400-337**: ((400-300)-30)-7=63. **325-168**: 168+40=208, 208+100=308,308+20-3=325, 40+100+17=157.	**79+45**: 79+21=100, 100+24=124. **95-47**: (95-45)-2=40-2=38. **106-88**: 100-88=12, 12+6=18. **253-98**: (250-98)+3. **246+178**: (250+180)-6=430-6=424. **136+56**: 135+55, 190+2=192. **92-57**: 57+30=87, 87+5=92, 30+5=35. **62-36**: 62-2, 60-36, 24+2=26. **250-127**: 250-130, 120+3=123.
9 (Term 3, Year 5)	**100-59**:60+40=100, 40+1=41. **95-47**: 47+40=87, 87+8=95, 40+8=48. **92-57**:57+30=87, (87+3)+2=92, 30+5=35. **120-45**:45+60=105, 105+15=120, 60+15=75. **106-88**: ((88+2)+10)+6=106, 2+10+6=18. **400-337**:(337+60)+3=400, 60+3=63. **253-98**:98+2=100, 100+153=253, 2+153=155. **246+99**: 299 goes to 300, 300+45=345.	**100-59**: (100-60)+1=40+1=41. **95-47**: 100-47, 53-5=48. **92-57**: 67, 77, 87, +5, so 35. **120-45**: (100-45)+20=55+20=75. **106-88**: (106-90)+2=16+2=18. **400-337**: (400-340)+3=60+3=63. **79+45**: 79+50, 129-5=124. **253-98**: 255-100=155. **250-127**: 250-130, 120+3=123

Strategy Descriptions

Catherine showed some understanding of negative numbers. In Interview 3, she solved 75-28 by, "70 take 20 is 50. 3 below 50 is 47 (5 take 8 is 3 below)"; again for 95-47 and 106-88 in Interview 5; for 96-49 in Interview 6, and interestingly for 325-168 again in Interview 6. Adrien tended to use the whole number in his calculations (e.g., for 29+35 in Interview 1, for 253-98 in Interview 5, through to 253-98 in Interview 9). Adrien did not employ a *wholistic* strategy for the solution of 253-98 in Interview 4, but his strategy was nevertheless advanced for a child in Year 3. Catherine also used the whole number in some of her calculations (e.g., 106-88 in Interview 5, 100-59 in Interview 7), but not as consistently as Adrien.

Catherine tended to progress through calculations, left to right, whether she separated the numbers into place values or left one number as a whole (e.g., separate into place values - *separation*: 136+56: 130+50, 180+6, 186+4, 190+2; leave one number as whole - *aggregation*: 95-47: 47+40=87, 87+8=95, 40+8=48). Adrien also tended to calculate in a left to right fashion (*separation* and *aggregation*), when not using a *wholistic* strategy (e.g., 32+15: 32+10=42, 42+5=47). Although the children had no difficulty employing a left to right strategy, even when regrouping was employed, it became glaringly evident that some of the teachers did not understand its use, as the following example illustrates. When one of the researchers was explaining the purpose of the research to some of the teachers of the children who participated in the interviews, when the left to right procedure was mentioned, one of the teachers interjected with, "but you can't do it that way, because it's wrong!" Not only did the children use that method, but they understood it and were accurate. Further, left to right procedures have been reported elsewhere in the literature (e.g., Madell, 1985).

It was interesting to observe the changes in strategies for particular questions over the interviews. An example of changing strategies can be seen in Adrien's solutions to 253-98 over Interviews 4, 5, 6 (5 and 6 the same), and 8 and 9. The strategy that Adrien employed was mostly *wholistic*, but with variations. It could be argued that the strategy Adrien employed in Interview 9 was the most advanced, as it required a great deal of understanding of number and operations, yet involved less burden on working memory. Other examples were solved more consistently, for example, Catherine's solutions for 253-98, which was solved additively (although there were slight variations).

It was also interesting to note that new strategies arose over time, and a particular strategy was used for more than one example in the one interview. An example of this is the additive, build up strategy Catherine used for subtraction examples (106-88, 400-337, and 253-98) in Interview 9. A different additive strategy was also used by Catherine in Interview 9 for the examples, 95-47, 120-45, and 92-57. Further, a similar strategy was used by both children for the same example in the same interview, for example, 45-21 in Interview 1. Also, both Adrien and Catherine used the same strategy to solve the same example (400-337), but in different interviews (Catherine in Interview 8 and Adrien in Interview 7).

It should be noted, however, that children did not use a consistent strategy for all questions of the same type in any one interview. For instance, even in Interview 1, Adrien did not use the same subtraction strategies to solve all the subtraction examples. Why he used different strategies remains a mystery. The same strategy could have been used for 75-28, 96-39, and 65-38, but this was not the case. Similarly, one would surmise that 29+35 and 36+29 (in Interview 1) would have been solved employing the same strategy, but this was certainly not the case (see Table 1). It appears that Adrien had a variety of strategies at his disposal, even at such an early age.

As the children progressed through the years, a strategy similar to the taught pen and paper algorithm was used by both children. By Interview 7, Catherine was solving 246+99 by, "9+6=15, carry the 1, 1+9+4=14, carry 1, 2+1=3, answer is 345." Yet previously in Interview 4, she solved the same example by, "(240+90)+(9+1)+5". The method used in Interview 4 would appear to require less load on working memory, but it was not her chosen method in the later interview. For algorithmic exercises in the first 6 interviews, Catherine separated the numbers into their place values and proceeded left to right. However, in Interview 7, she calculated right to left, the method used in the classroom. Adrien's solution method for 246+99 of adding 100, then subtracting 1 was consistent from the final interview in Year 3 to the final interview in Year 6. However, by Year 6, Adrien was employing the strategy similar to the taught pen and paper algorithm for other examples, particularly the algorithmic exercises. It appears, then, that although these two children were capable of employing efficient and effective strategies for computation (even before classroom instruction of pen and paper algorithms had taken place), by Year 6, both children were using the less efficient (i.e., mentally inefficient) written algorithm. Thus, there was an instructional effect.

When Adrien commenced at the new school in Year 5, all boys were pre tested in subject areas, and placed in streamed classes. It was interesting and a little perplexing to discover that Adrien was not placed in the top group. It is quite possible that Adrien's excellent number understanding did not carry through to other mathematical areas. However, a little more was revealed when another boy who had also come from another school and was now at the present one, was streamed into a far lower group, yet, he too had demonstrated good number understanding during interviews. He came to one interview in tears, because he had been marked wrong in a test, when he had copied an example incorrectly, but had managed to solve his example successfully. It appeared that the new school demanded correct copying and "correct" procedures.

Other Strategies

Apart from the variety of strategies employed by Catherine and Adrien, a large variety of strategies was also documented from the large study. As an example, Table 3 shows a variety of strategies employed for the example, 79+45 (word problem).

Table 3
Strategies Employed to Solve 79 + 45

79 + 45:

- ((79 + 21) + 4) + 20
- (79 + 50) - 5
- 80 + 44
- (80 + 40) + 4
- ((70 + 40) + 9) + 5
- (70 + 40) + (9 + 5); 110 + 14
- (70 + 40) + (10 + 4); 110 + 14
- (9 + 5) + (70 + 40); 14 + 110
- 9 + 5, carry 1, 7 + 4 + 1

These strategies cover all those mentioned in Table 1 - *separation*, *aggregation*, *wholistic*, and *mental image of pen and paper algorithm* (with the exception of *counting*). The strategies were employed at different times by different children. Further, the only strategy that was taught in any classroom was the last one.

CONCLUDING COMMENTS

Both Catherine and Adrien exhibited understanding of place value and effects of operations on numbers. It was also obvious that the two children thought about the numbers as wholes, rather than as single digits in columns. They also possessed a sound knowledge of number facts. Even when fact recall was not used, advanced facts strategies were employed.

for example, *through 10*, which was used by both children. One final observation is that the children had access to a variety of strategies both during interviews and over the interviews that were reported here, that is, they exhibited flexibility.

Although it is clear that some children are capable of formulating their own efficient and varied computational strategies, I am not asserting that all children should be able to do this at such an early stage. It is obvious many children struggle with number concepts for many years. I am also not asserting that all children should be taught mental strategies of the calibre that Adrien and Catherine were able to formulate. After all, both children formulated different strategies and at different times, and with differing levels of complexity. However, young children should be given the opportunity to develop understanding and be encouraged to create their own computational methods, a proposal supported by such researchers as Carroll (1997), Kamii, Lewis, and Livingston (1993), and Madell (1985).

The answer does not lie in teaching mental strategies. At present, formal written algorithms are taught, and yet, children are still far from successful (McIntosh, 1991; Carraher, Carraher, & Schliemann, 1987). Further, there is little point in teaching algorithms, as "it is hard to follow the reasoning of others. No wonder so many children ignore the best of explanations of why a particular algorithm works and just follow the rules" (Madell, 1985). Although the traditional pen and paper procedures are efficient for some complex examples that may require calculations to be completed on paper, they are hardly efficient for examples of the type, 246+99 and 253-98, where wholistic strategies employed mentally or with pen and paper are far more efficient. Further, very complex examples are better computed with the use of a ten dollar calculator.

The research reported in the beginning of this paper supports recommendations for change in the mathematics classrooms. The concluding comments of this paper will summarise some of the insight learnt from these studies.

Children should be encouraged to invent their own computational procedures, as they develop better understanding of the effects of operations on number, and place value. Further, children take responsibility for their own learning. Time should be spent on students' describing various solution strategies for problems, and these strategies should be valued. Children's discussions are useful for not only discovering their understandings, but also any misconceptions. Finally, pen and paper calculations of the type, 27+5, 32-3, 100-1, and 102-97 should be avoided. Such calculations are more efficiently completed mentally.

In conclusion, Catherine and Adrien were members of very traditional classrooms. Yet, they were able to avoid the trap of resorting to taught mechanical algorithms in their mental calculations. They often employed far more efficient strategies, ones that exhibited number sense. What would they have been able to do if they were members of one of the "reform" classrooms?

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Acknowledgement

The author wishes to thank her co-researcher Tom J Cooper for his support in developing this paper. The paper is based on research supported by ARC grant No A79030404.

THE DEVELOPMENT OF A FRAMEWORK OF GROWTH POINTS TO MONITOR STUDENTS' COMPREHENSION OF ALGEBRA IN GRADES 7-9

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This paper is proposed to stimulate discussion in the area of algebra. It reviews relevant literature in algebra and proposes a framework of growth points to inform future research and to enable monitoring algebraic development through the early years of secondary school through understanding rather than through outcomes. It is work in progress in preliminary stages with some areas still to complete.

The teaching and learning of Algebra and its place in the curriculum has been of interest and debate to curriculum writers for many years. Grimison (1995) describes its introduction to the syllabi of the British Empire in the late nineteenth century and subsequent lack of change through a large part of the twentieth century. Whatever the reasons for its introduction, and the rights and wrongs of it as appropriate curriculum for all, it is part of the curriculum in Australian schools and indeed in the rest of the world.

However, the conflicts about the teaching of algebra have continued. There is general agreement that there are cognitive difficulties in moving from arithmetic to algebra and the teaching and learning of algebra is a major area of concern in mathematics education. Algebra still occupies a large part of the syllabi in secondary schools.

Some changes were introduced into our syllabi in the nineteen sixties and more recently, following *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991), most States have developed new curriculum documents (e.g., Board of Studies, 1995). The National Statement provided a framework that divides algebra into the sub-strands expressing generality, functions and equations. The framework is an outcomes based structure that manages to give general outcomes showing broad development.

While this framework provides a useful scaffolding for curriculum development and reflects ideas of development, it does not effectively provide a means of monitoring children's understanding of algebra in grades 7-9, based on the considerable work of researchers in the area of cognition in algebra. What is needed is a framework built on the development of students' thinking rather than outcomes.

Researchers have understandably focussed on particular aspects of algebra such as the modelling used when solving word problems (e.g. Lemut & Greco, 1998; MacGregor & Stacey, 1993, 1998), the understanding of the equals sign (e.g. Kieran, 1981; Pillay, Wilss & Boulton-Lewis, 1998), the translation from tabular form to symbolic form (e.g. Redden, 1994; Ryan & Williams, 1998; Warren, 1998), the solution of linear equations (e.g. Linchevski & Herscovics, 1994; MacGregor & Stacey, 1995) and functions and graphs (e.g. Herscovics, 1989; Swan, 1988).

Algebra is not a simple domain. There are many different aspects associated with algebraic cognition. In algebra we operate with a range of mathematical sign systems such as graphs, algebra code and tables on a variety of mathematical objects (Filloy & Sutherland, 1996). Kaput (1989) suggests four sources of meaning in relation to mathematical sign systems:

- translation within a system such as symbolic manipulation to alter an equation;
- translation across systems such as from a graph to a table;
- translation between mathematical sign systems and non mathematical sign systems such as natural language;

- the process of abstraction which occurs as students evolve mathematical sign systems through consolidation, simplification, generalisation and reification of the actions, routines and representations which produced intermediate sign systems in the teaching sequences.

As the mathematical sign systems develop and the abstractions occur there is an interaction between many of the aspects of development. For example the understanding of the equals sign develops alongside understanding of aspects of linear equations in an interactive way rather than linearly. Within each area a hierarchical development can be seen. While Kaput's sources of meaning are useful to indicate the scope of development they are not directly hierarchical as developments occur in each of the first three translation areas concurrently. What is critical is that the development is moving always towards abstraction and recognition of the underlying structure. In order to monitor students' progress through algebra in the early years in particular, a framework is needed which highlights some of the growth points in development and shows the hierarchical structure while at the same time acknowledges the separate developments in different areas.

Many researchers have been concerned with the move from arithmetic to algebra and in particular the cognitive gap that exists between the two (e.g. Bednarz, Radford, Janvier & Lepage, 1992). Algebra is not simply an extension of the numerical domain and a question of symbolism. Rather it is a way to manipulate relations and based on analysis. Algebra uses the same words and symbols as arithmetic but the meaning is not always the same. The objects are essentially different. Consider 3+5 and d+5. The first can be written more simply as 8, and in the process the actual parts can no longer be seen. d+5 remains as is, with the two original parts still open to view. Many students try to combine these to $d5$ or $5d$ or even 9 in an attempt at closure. Acceptance of lack of closure (Collis, 1975) is an important growth point in algebraic development. Understanding this is important in the development of the necessary abstractions.

Table 1
Generalised Arithmetic Laws

Generalised arithmetic laws		Research Support
1	Basic arithmetical understanding of operations: shown by the application of the four basic operations to binary problems.A high degree of facility with decimals and fractions is not necessary, students should be able to operate with simple decimals and fractions	Some support: Cooper et al 1997, Boulton-Lewis et al, 1997, Pillay et al, 1998, Horne, 1994; Kieran,1989; Linchevski & Herscovics, 1994
2	Commutativity: shown in the understanding of the differences between 2+9 and 9+2, and 6÷3 and 3÷6. Many students can successfully perform simple operations but have not yet realised the differences in structure	
3	Order of operations: This extends the operations from level 1 to include more than one operation	
4	Distributive law: This requires more of a structural understanding than simply an arithmetic understanding and should include division and subtraction as well as addition and multiplication	

Table 2
Concept of "="

	Concept of "="	Research Support
1	Equals meaning "now do something" or "find the answer". For example students who write $10 + 3 = 13 - 2 = 11$ are thinking of the equals as an indicator for action or a punctuation mark	Some support: Kieran, 1981; Pillay et al, 1998.
2	Equals means one expression equals the other if you work them out (but in terms of numerical calculations) (Willingness to have expressions on both sides of the = but still seeing it as "if you work this out it equals that".)	
3	Equivalence of expressions (including the transitive and symmetric nature)	

The understanding of the equals sign and the understanding of operations and the laws of arithmetic (and algebra) are two aspects critical to further development in algebra. Often success in algebra is hampered by poor arithmetical skills and understanding (Horne, 1994; Kieran, 1989; Linchevski & Herscovics, 1994). The development of operations and rules associated with arithmetic, and the development of understanding of the equals sign from a sign that an action is needed to an understanding of equivalence with symmetric and transitive properties, occur concurrently with interaction between these separate but related areas and other areas such as the solution of equations. In the proposed framework, these areas have been indicated by *Generalised arithmetic laws* in Table 1 and *Concept of "="* in Table 2.

Table 3
Identity

	Identity	Research Support
1	Understanding $3x$ and $x + 2$ with x representing a particular unknown number. For example $3x = 6$ if $x = 2$	Support for some parts: Cooper et al, 1997; Collis, 1995.
2	Willingness to accept lack of closure	
3	Combination of like terms with idea that x could be any unknown number	
4	Ability to find equivalent expressions using a range of operations where x could be many numbers	
5	Ability to find equivalent expressions with the recognition that x is an object which may be representing other objects such as a number or an expression like $(a + 2)$	
*	Use of strategies to check equivalence	

* It is not clear how checking strategies fit in hierarchy

The understanding of the x in algebra has had an associated hierarchy of understanding initially proposed by Küchemann (1981). It is not necessary in development though for students to move through all of the inadequate concepts in this structure. The x of algebra can be considered an as yet unknown, a general number, or a variable – which can be symbolized and operated on as an object (as if it were a number) (Wheeler, 1996). These concepts of the x link to equations, identities, properties, and functions and relations respectively (Janvier, 1996) and are all necessary to a full understanding of algebra. These understandings develop in all three of Kaput's translations within a sign system, between sign systems and with natural language. For this reason rather than consider the development of understanding of the "x" as one area, it occurs across a number of areas. Operating within an algebraic symbolic sign system involves manipulation of the symbols. This includes combination of symbols, understanding of 3x, the idea of acceptance of lack of closure (Collis, 1975) and the understanding of equivalence of expressions. This area is shown in Table 3.

Another area operating within the symbolic sign system is the solution of equations where students manipulate the algebraic equation in order to find a solution. Classified here as Equations, and shown in Table 4, many researchers have been interested in this area which is closely linked also to the conception of the equals sign. A few researchers have developed hierarchies for linear equations that indicate the development and the cognitive complexities of the situations. Pillay, Wilss and Boulton-Lewis (1998) have developed a schema from arithmetic through pre-algebra to algebra and have based part of this on the cognitive load faced by the students. During the transition from arithmetic to algebra they suggest that the equals sign changes from meaning each side has the same value to equivalence. They also at the same time see equations moving from purely numeric to simple linear one variable equations to linear equations where there is more than one unknown or variable. Their structure recognises the increasing complexities of the tasks.

Table 4
Equations

	Equations	Research Support
1	Numerical understanding 1 step or two steps using box (Completion of fill in gaps in arithmetic (or use of box) in one step equation format.)	Some support: Linchevski & Herscovics, 1994; MacGregor & Stacey, 1995; Pillay et al, 1998
2	Solution of one step and 2 step equations using algebraic symbols with whole numbers by trial/knowledge of number facts	
3	Use of backtracking to solve 2 or more step equations with whole numbers and inverse ideas	
4	Solution of equations with non whole number solutions	
5	Use of (inverse operations/) balance to solve 2 step equations	
6	Simple equations with more than one occurrence of the variable on one side.	
7	Simple equations with the variable appearing on both sides	

One difficulty is that within a particular area growth points can be chosen to show an order. There will be other aspects of learning as well but the idea of these growth points is to choose key aspects of development. The growth in any area is from limited arithmetic and concrete understanding towards abstract and structural understanding. It is not clear for students when each of these areas begins and ends, nor is it always clear which growth points in one area should or do precede a growth point in another area.

There are other areas. When students translate between systems the ideas of functional development and variable are raised. Students' early development, particularly as it is now presented in many syllabi, includes pattern recognition in the translation from table form of a function or relation to symbolic form. There has been much research in this area. In particular in Australia, Redden (1994), Warren (1998) and Ryan & Williams (1998) have all independently investigated student's development of rules from tables. Early development seems to involve students seeing the pattern in terms of the previous term in the sequence rather than being dependent on the independent variable x. Ryan and Williams (1998) suggested that the tendency to introduce the table in an ordered fashion with x values being sequential tends to focus the attention on this aspect rather than the relationship between the variables. While there are minor differences in the research findings in this field, generally there is agreement that early stages include the addition of a constant and the consideration of the rule as an iterative rule with the x value being the number of the term. In the later stage students can develop an algebraic rule, and move between the two variables in the relationship comfortably. Garcia-Cruz and Martinón (1998) have developed a three level hierarchy for this type of problem based on the degree of action and generalisation from procedural activity through procedural understanding and local generalisation to conceptual understanding and global generalisation. While this structure indicates important directions of development, it has used fewer growth points than are indicated by the body of research available. This area has been classified as *Function – table* in Table 5.

Table 5
Function – Table

	Function - Table	Research Support
1	Finding next term in a table with sequential independent variable values but unable to give explanation	Some support: Redden, (1994); Ryan & Williams (1998); Warren (1998); Garcia-Cruz and Martinón (1998)
2	Explaining next term in terms of previous terms but not recognising the independent variable	
3	Ability to complete tables (linear expressions) with missing terms by working from previous terms	
4	Finding terms in a table not presented sequentially but based on dependent variable	
5	Ability to express a linear rule symbolically	
*	Use of non integer numbers in table	

* It is not clear where in this understanding students start to be able to use a range of number types.

Table 6.
Function – graphs

	Function – graphs	Research Support
1	Reading a value from a graph with labelled axes	Support for some parts: Swan, 1988; Herscovics, 1989 The order is not clear from the literature.
2	Reading a point from a graph and plotting points on a cartesian grid	
3	General interpretation of a graph quantity as getting less or getting more	
4	Recognition of table to graph and vice versa	
5	Plotting a linear rule expressed algebraically	
6	Recognition of the link between gradient (steepness) and the rule	
7	Recognition of the link between c and the cutting of the y-axis	
8	Recognition of vertical and horizontal lines	

Another between systems aspect involves the translation involving graphs, classified here as *Function - graphs* in Table 6. Much research on functions and graphs has been at a higher level and, in recent years, has involved the impact of the use of technology. Swan (1988) indicated students' lack of understanding of graphs both at the basic level of interpretation of points and at the more dynamic level of considering functions and graphs as showing variation. Nemirovsy (1996), in responding to some work on using a technological environment to teach functions, discusses the issue of a point-wise approach versus a variation approach. Children using motion detectors to develop function ideas show a relational understanding although researchers such as Kieran, Boileau and Garançon (1996) have argued for a discrete mathematical approach to function as a set of points rather than a continuous variation. While there are extra levels of complexity in the full understanding of graphs as continuous, containing an infinite number of points, as Nemirovsky notes, children show a relational understanding before their understanding of points is fully developed so one does not necessarily precede the other. More work is needed on cognition in this area.

In Australian Education Council (1991), equations and inequalities are considered in the one sub-strand as closely related. There is little research to indicate student's cognitive development in this area of inequalities, but perhaps it also should be included in a framework. This is indicated in Table 7.

Table 7
Other areas to be included

	Inequalities	Research support
1*	Recognition of greater than and less than symbols and correct use in arithmetic expressions.	
2*	Understands that there are many correct answers to an algebraically expressed inequality with one operation.	
*	Further stages are needed	
	Modelling	Research support
	This section is still needed and there have been many researchers working with modelling and problem solving and how students use algebraic formulations of the problem. This area is concerned with the translation from natural language and situation to algebraic representation.	Some of the many researchers include: MacGregor & Stacey, 1993, 1998; Lemut & Greco, 1998; Brito Lima and Da Rocha Falcão (1997)

* This is purely speculative

Problem solving and mathematical modelling are a large part of algebra and for many provide it with its *raison d'être*. This involves the translation from natural language into algebraic symbolisation, although many problems that are intended to require the use of algebra can be solved with no reference to algebraic approaches. There are many nuances in natural language and many different ways of presenting the same problem. MacGregor and Stacey (1998) found that one form of a question on unequal partitioning directed the solution to an arithmetic solution through one cognitive model, while another form of the same question led more easily to an algebraic formulation of the problem. There was actually no significant difference between the two forms of the questions in the success experienced by the students, or in the degree of algebraic formulation finally reached but rather a tendency for the form of the question to direct the cognitive model initially in a particular direction. Brito Lima and Da Rocha Falcão (1997), using different basic problem forms, also found that the form of the question did not lead to significantly different results or ways of approaching the problem. In earlier work MacGregor and Stacey (1993) reported on a study which looked at students' syntactical translation of language into symbolic form in the study of equations. Contrary to their expectations they found that students did not frequently use syntactical translation in the development of symbolic representation of equations. This translation from natural language in problem solving situations to algebraic form is a crucial part of algebraic understanding. A hierarchy in this area would be expected to move from simple syntactical translations to more complex modelling skills. The growth points here may be classified in terms of levels of complexity. This is also indicated in Table 7.

All of these proposed hierarchies could be expanded to include many more details and fine differences. However, the purpose of this is to provide a framework which focuses on understandings as students move from limited concepts and arithmetic and concrete understandings to broader concepts and abstractions. For this reason there has been an attempt to distill the key ideas in a framework that broadly covers the area but is simple to use.

CONCLUSION

The proposed structure is intended to focus on student's understanding rather than on their skill at reproducing taught methods. It should provide a means of monitoring students' development in moving towards the abstraction of algebra. It has also highlighted areas where there does not appear to be a clear understanding of the cognitive development.

Such a framework needs a considerable amount of research but could provide a useful structure for other researchers and for teachers in schools who are trying to assist students to learn algebra.

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INFLUENCES ON SECONDARY MATHEMATICS CURRICULUM IN VICTORIA

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The creation of the Victorian Curriculum and Assessment Board marked a significant departure from prior approaches to curriculum determination. Responsibility for the development of senior secondary mathematics curriculum was entrusted to a select group of mathematics educators. However, attempts to develop a new approach to curriculum were challenged by other groups within the community. This study examines the processes and influences that helped mould a particular curriculum.

BACKGROUND

Until the nineteen sixties control of the senior secondary mathematics curriculum in Victoria was essentially the preserve of examination boards, wherein mathematicians, mainly from the university but also from the school sector, primarily the registered schools, determined both the nature of the mathematics content to be taught and the ethos of the learning environment (Connell, 1962). In the nineteen seventies government introduced a new examination board which recognised the increasingly important role of education within society and the responsibilities of teachers and professional educators for the maintenance of that role. This board was to usher in the era of School Based Curriculum Development, and the focus of curriculum development shifted towards professional educators and teachers. The nineteen eighties saw the election of the first Labor government in thirty years, which ushered in an era of social reform. Education, in general, and curriculum, in particular, underwent considerable change (Clements, 1989).

INSTITUTIONAL CHANGE

The Victorian Curriculum and Assessment Board (VCAB) assumed responsibility for curriculum development in 1986. The major concern of VCAB was to develop a new curriculum model reflecting the recommendations of the Blackburn report (1985). A new certification, the Victorian Certificate of Education (VCE), was to replace the Higher School Certificate (HSC).

The VCE Mathematics Study was developed by the Mathematics Field of Study Committee (FOSC). The FOSC was comprised of a majority of teachers, and nominees of tertiary institutions, community and industry organisations. The Study Design did not specify a course description. Rather, it provided a set of specifications for the development of courses over the two years of the VCE.

In developing the Mathematics study the FOSC capitalised on the experience of previous curriculum developments. The interest in problem solving and modelling, the *Frameworks* project (Ministry of Education, 1988), the Cockcroft Report (1982), and the diversity of courses that emerged during the HSC were all factors in its deliberations. While the selection of the content areas of mathematics did not contain any major surprises, the focus on what were identified as 'key processes' in the learning and application of mathematics did provide the basis for a redefinition of mathematics education. Skills practice, standard applications of those skills, and problem solving, modelling and investigation were processes identified as common to all areas of mathematics, and the FOSC used them as a unifying principle for the Mathematics Study (VCE Mathematics, 1987).

MATHEMATICS STUDY DESIGN

There was a single Mathematics Study, consisting of four blocks of units, titled Space and Number, Reasoning and Data, Change and Approximations, and Extensions. Within each of these blocks the mathematical areas of study were classified as algebra, arithmetic, calculus, coordinate geometry, geometry, logic, probability, statistics and trigonometry. As was required by the Blackburn report, each of these blocks of units could be taken at either Year 11 or Year 12. Three work requirements formed the basis of the work to be completed in each unit; Projects, Problem Solving and Modelling, and Skills Practice and Standard Applications. The work requirements were one of the most significant features of the Mathematics study. Whereas the content requirements of the Mathematics study varied little from those of HSC, the work requirements represented a paradigm shift in terms of pedagogy.

The intention of the Study was that the work requirements should underline the approach to the mathematical content that was to be learned. The learning situation was envisaged as active rather than passive - students should engage in learning and applying skills rather than in receiving information from the teacher; students should learn to use mathematical knowledge and undertake mathematical investigations in new situations rather than be presented with known results; the mathematical knowledge should be presented in an applied context rather than being presented as purely abstract results. But what was also of significance is that the Study attempted to be specific with respect to the way the work requirements were to be interpreted. Detailed explanations were provided for teachers specifying the issues to be addressed in each work requirement and the teacher's responsibilities. The Mathematics Study was descriptive with respect to mathematics content, prescriptive with respect to work requirements. To some extent this anomaly arose from the dual purpose perceived of the work requirements. They were designed to assist in achieving the objectives of the study, that all students engage in concept development and communication of developing ideas, but they were also to be used in the assessment process. As an integral component of the study, the work requirements added 'diversity and choice', but as an assessment instrument they encouraged 'uniformity and convergence'.

ASSESSMENT FACTORS

The demands for the preservation of diversity and choice among mathematics educators were at odds with the requirements of uniformity and convergence of a centralised curriculum bureaucracy. In the final analysis, the bureaucratic policies prevailed (Ellerton & Clements, 1994).

The overarching nature of assessment was further highlighted by the Common Assessment Tasks (CATs). Under instructions from VCAB CATs were developed for units 3 and 4 of each sequence. For the Mathematics Study the FOSC decided to implement four CATs: 1) Investigative project; 2) Challenging problem; 3) Facts and skills task; and 4) Analysis task.. CATs 1 and 2 were conducted during the school year and were initially assessed by the teacher and verified by VCAB procedures. CATs 3 and 4 were conducted under examination conditions and assessed by VCAB examiners. Problems surfaced due to the fact that students spent an excessive amount of time on CATs 1 and 2. The response of VCAB was to limit the length of the report in CAT 2 to 1500 words (VCAB Bulletin, 1992, No. 61). This did little to help abate the widespread criticism that the new VCE was far too time-demanding on both student and teacher alike (FOSC minutes). In effect, the complexity of the assessment procedures distracted attention from the nature of the content to be taught and from the innovative processes by which it was intended to be taught. The focus should have been on how to integrate these two components to achieve real change in the classroom - rather it was distracted by the needs of competitive assessment.

The expertise of the professional administrator is in administration, and this expertise was clearly evident in the modus operandi of VCAB. The Mathematics FOSC was supported by five permanent VCAB staff - a FOSC Manager and four Project Officers. Each Project

Officer compiled a report on one of the CATs (FOSC minutes, MATH10/89/6a, MATH10/89/6b, MATH11/89/6a, MATH11/89/6b) that reported on issues such as task development, authentication, workload and time management, record keeping practices, advice to schools, timelines, complaints. There was *no* comment on content, and only incidental comment on teaching. In addition an extensive statistical armoury was brought into play: correlations, score distributions, graphical presentations of the percentages of grades awarded, crosstabulations, multiple regressions, ANOVA, enrolment numbers and the like. The encouragement of new teaching styles envisaged through the agency of new assessment criteria faded in the face of administrative demands.

The assessment procedures were the major concern for VCAB as a whole. Obviously there were genuine issues involved - authentication of individual student work, the integrity of the credential, grading for tertiary selection, to mention some. This engendered a standard response - referral of the problem to an external, expert committee. A major review of procedures, by the ACER, was commissioned. A summary of the generally favourable report appeared in 1990 (VCAB Board Report, 1990, No. 43). What this summary did underline however was the extensive administrative coordination necessary to ensure that assessment procedures are effective. The expertise within VCAB would be essential. The report however carried a word of caution.

> The current negotiations about the Mathematics curriculum highlight two relevant issues, both of which will need continuing attention across all study designs. First, they demonstrate the lack of finality of any curriculum document and the need for regular review in a context of genuine willingness to adapt and modify. Secondly, they highlight the problem of some interested parties failing to pay attention to developments while they are occurring and then intervening late and publicly in an attempt to arrest a process (VCAB Board Report, 1990, No. 43).

Apart from the apparent contradiction implied in the statement - if the process is dynamic surely developments are going to occur that will necessitate participants re-evaluating their support or otherwise - the report indicated that the Mathematics study was under contestation.

CHALLENGES TO THE MATHEMATICS CURRICULUM

In 1990, a group of academics from Monash, Melbourne, Latrobe and RMIT formed the Tertiary Mathematics Group (TMG) with the purpose of 'commenting about the VCE Mathematics study'. Year 12 of the new VCE had not commenced yet dissatisfaction with the Mathematics Study was beginning to emerge. This dissatisfaction principally arose from within the ranks of tertiary mathematicians who were concerned with the adequacy of the VCE as preparation for tertiary study.

The TMG highlighted a number of concerns: firstly, that the structure of the Mathematics Study led to the compartmentalisation of mathematics and the treatment of mathematics as a series of discrete topics; secondly, that students could enter tertiary study with quite diverse backgrounds, many lacking adequate mathematics for their chosen course of study; and thirdly, that assessment procedures did not cater for students with little intention of pursuing mathematics beyond Year 12, that students sought outside help in the completion of the projects, and that assessment should be 'content-based' rather than 'aptitude-based' (FOSC minutes). The TMG proposed modest changes to the Study to allow the more able mathematics students to select a greater number of advanced units. These criticisms were rejected by the FOSC. It was pointed out that:

> VCAB specifications for the development of study designs require that different studies, and different units within studies, are about different things. In other words, some divisions of mathematical content into reasonably separate parts is a requirement. It is difficult for others to appreciate this (FOSC minutes, 1990, Response to TMG Proposals).

The crux of the TMG criticism was that mathematics cannot be conceptualised in this way without damage to the nature of mathematics.

> ... mathematics is not a subject composed of a series of discrete topics. Rather it is a cumulative and sequential subject, with each level relying on many of the concepts taught at the previous level (Tertiary Mathematics Group on V.C.E. Mathematics, undated).

Guidelines for the development of VCE studies constrained the way in which the mathematics study was developed. The issue had become ideological. This was recognised by the chairperson of VCAB who, in arranging a meeting to discuss the above proposals, wrote to FOSC members in the following terms:

> The differences in *opinion* are associated with the kinds of interpretations and extrapolations individuals make from the different theories of learning. They are also associated with the different *opinions* on 'What knowledge is of most worth' for secondary students (Letter from Richard P. Tisher to FOSC member, 1 June 1990, emphasis mine).

Perhaps he should have asked 'Whose knowledge?', for a divergence in conceptions of the nature of mathematics was clearly emerging.

Despite the fending off of the TMG challenge the minutes of the FOSC indicate not all else was rosy. A number of letters from schools expressed concern with the effect of the Mathematics study on the performance of girls as well as other more general worries. At a meeting of the FOSC in May 1990 a list of concerns was tabled along with advice on how to answer them. The concerns included assessment, streaming, monitoring and record-keeping, authentication of student work, student workload, resources for problem solving, modelling and projects, expertise with problem solving and projects, and tertiary selection (FOSC minutes). The FOSC may have felt it had dealt with the concerns of the TMG but the feeling in the schools was not so confident. The final straw was the relatively poor performance of ex-VCE students compared to ex-HSC students on the Melbourne University diagnostic test in Mathematics administered in February 1991 (W. J. Ewens, letter to P. Stacey, 10 April 1991). VCAB was bound to act.

EXTERNAL REVIEW

VCAB required an annual report from each of the studies. VCAB decided, in 1991, to institute an additional external review of the Mathematics study (Stacey, 1991). Its terms of reference were:

> To investigate the teaching of Mathematics within the framework of a single certificate. The review would take account of the experience so far of the working of a single design for Mathematics and would consider, in particular:
>
> 1. The provision of an appropriate coverage of mathematics in Years 11 and 12, and
> 2. The capacity of the single Mathematics design to meet students' needs and differing aspirations (Mack, 1991).

The final report was delivered in 1992 and recommended substantial changes to the Mathematics study. Effectively the report recommended a sequential, hierarchical structure. The VCAB requirement to develop a multi-study structure, in which all studies were equally rigorous and were deemed of equal value and status, was abandoned. While recommending the retention of work requirements and of CATs, both mandated by VCAB, the review proposed a rationalisation of offerings that would effectively answer the criticisms proffered by the TMG. In so doing it reduced the complexity of the Mathematics study to a choice from two units offered at level 1 & 2, and from three units offered at level 3 & 4. In essence, the demands of the TMG had been met. University control over entrance requirements once again provided the justification for changes in content of senior secondary mathematics. However, the flexibility of the Mathematics Study was lost, and so, presumably, was its appeal to a broad spectrum of the senior school population.

The general disquiet with the VCE was not confined to the schools. Widespread discussions in the press, on radio and elsewhere concerning the excessive and disruptive workloads caused by the project CATs, mainly through the authentication procedure, had focused attention on the teacher 'overload' problem. The FOSC responded by eliminating CAT 2. The new study was introduced with revised structure and with three CATs, to commence in Year 11 in 1993. Further changes to assessment were to follow. While the original intention of the FOSC appeared to be to integrate the assessment process with the process of teaching and learning mathematics, the use of the assessment criteria for purely grading purposes soon undermined this aim. The FOSC became preoccupied with the validity of the various CATs as assessment instruments. The school-based CATs suffered most - claims of cheating, tutorial help, sales of solutions, together with more substantive arguments relating to the advantages enjoyed by children from more educationally advantaged homes created a climate of distrust. The FOSC responded. CAT 1 was modified from a Problem-solving task to a report of a problem-solving activity authenticated by a test set by the Board of Studies (BOS), which had replaced VCAB during 1993. In addition, in 1994, the BOS introduced the General Achievement Test (GAT) to monitor school achievement for each school-assessed CAT. The notion of outcomes had clearly extended into higher levels of schooling as the BOS endeavoured to construct administrative solutions to the problems of authenticity and of comparability. The GAT was used to provide a profile that could be applied to each school-assessed CAT. The problem of authentication had become an obsession.

CONCLUSION

How is the extraordinary preoccupation with authentication and assessment to be interpreted? The VCE Mathematics Study represented a new initiative by mathematics educators that fell on the sword of bureaucracy. It would appear that the role of Year 12 in selection for tertiary entry yet again assumed paramount importance. Inadvertently, the genuine concerns of university mathematicians provided the opportunity to frustrate comprehensive reform of the senior curriculum. While concern within the community was another factor, there is little doubt that the extraordinarily complex assessment arrangements were the major factor in the failure of this innovation (McCalman, 1999). One outcome of this whole process was the strengthening of the role of the professional administrator in the control of curriculum. Thereafter, the role of mathematicians and mathematics educators would seem to be one of expert adviser to bureaucracy, within the framework of terms of reference decided by the bureaucracy. The decision as to who constitutes the expert, and whether advice is accepted, now appears to rest with the bureaucracy.

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DEAF STUDENTS SOLVING OF ARITHMETIC WORD PROBLEMS

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There has been little research into the intersection of language and arithmetic performance of deaf students although previous research has shown that deaf and hearing-impaired students are delayed in their language acquisition and arithmetic performance. This paper examines the performance of deaf and hearing-impaired students in South-East Queensland in solving arithmetic word problems.

Although the poor performance of deaf student in comparison with their hearing peers has been documented in a number of achievement areas, particularly literacy (see (Hine, 1980; Wilson & Hyde, 1997) not a great deal is known about the mathematical knowledge and related conceptual development of deaf students (Wood, Wood Griffith & Howarth, 1986; Titus, 1995). Research examining deaf students' achievement in mathematics has chiefly concentrated on the study of their skills in operations and number. These studies have generally concluded that there is no significant cognitive basis for major differences in mathematical performance between deaf and hearing student and that achievement differences that are observed are the result of a combination of linguistic and experiential delays for the deaf students (Titus, 1995; Serrano Pau, 1995). Increasingly, the role of language in mathematic comprehension and application to problem solving is recognised – for both hearing and hearing impaired/deaf students.

Stone (1980) reported the difficulty that deaf and hearing-impaired students had when sequencing. The capacity to sequence requires students to place things in order so that there is a need for understanding "greater than" and "less than", but it was found that hearing-impaired students undertaking O levels (16+) and A levels (18+) experienced difficulty in writing things in order in the science laboratory and in following the steps of theorem. The apparent delay in critical components of mathematical thinking such as sequencing and conservation are not due to inferior cognitive abilities but rather a lack of experience and language (Rittenhouse & Kenyon, 1991).

Early studies (Furth, 1966) concluded that while deaf students' limited success with mathematics demonstrated that they were capable of logical thought, they evidenced slower rates of educational development when compared to their hearing peers. Hine (1970) in a large scale standardised test found that hearing-impaired students of 10 years were working at the level of 8.5 year old peers in mathematics and that by the age of 15, they were working at the level of 10.5 year olds in mechanical mathematics and 11 year olds in problem arithmetic. Suppes (1974) made similar claims, but with recognition that cognitive performance was comparable with hearing students when the task did not involve verbal skills. More recently, researchers agree that the most difficult problems for deaf students in mathematics are those problems that involve linguistic applications, again indicating that language is a key to mathematical performance.

Barham and Bishop (1991) have noted that the linguistic complexity of mathematical terminology which can be confusing for deaf students. Terms such as "tens" and "tenths" or "sixty" and "sixteen" are difficult for students using lip reading as they look very similar on the lips. Other difficulties can be centred on the unique ways in which words are used in mathematics which are very different from their non-school use. For example, the use of "up" in the non-school context and then "add up" in the mathematics classroom can be very confusing. Kidd and Lamb (1993) also comment on words with double meaning such as "interest" and "table". Such multiplicity of meaning is very confusing for deaf students. Furthermore, they argue that the technical vocabulary (such as "reciprocal", "premium") along with variations of a form ("year/yearly", "base/basic") can result in confusion. There are also numerous ways of expressing the same idea or concept so for deaf students where there may difficulty in understanding a concept one way, there is greater difficulty when the same concept is expressed in a different manner. Another area of difficulty is with logical connectives such as "if" and "because". These are rarely explicitly taught to hearing students, but rather they learn them through interactions. In contrast, hearing-impaired students do not have much opportunity to learn the meaning of such words, yet they are essential elements for and used in much of mathematics.

Extending beyond word structures, to more complex configurations, word problems of an everyday nature involving the use of linguistic forms applied to arithmetic concepts and strategies have been found to cause difficulties (Daniele, 1993; Serrano Pau, 1995; Wood et al., 1986). The exact relationship between language and mathematics for deaf students has not been clearly established with some researchers indicating that there is some relationship with the levels of literacy in English language instruction (Zwiebel & Allen, 1988), while other suggest it is related to the level of hearing loss (Wood et al., 1986).

Serrano Pau (1995) studied deaf students attempting change, compare and combine problems in order to examine their problem solving abilities in comparison with their reading comprehension levels. While he acknowledged that the relationship between reading comprehension and problem solving ability was important, he acknowledged that the ineffective problem solving strategies adopted by the students, which were largely based upon on the strategies taught by teachers, were also relevant in the students' poor performance. Luckner and McNeill (1994) extend this research on linguistic competence and propose three learner capabilities were relevant for deaf students:

a) intellectual skills – including a knowledge of the concepts and rule structures relating to the problem solving task;
b) organised information in the form of appropriate schemata to enable an understanding of the problem; and
c) cognitive strategies that allow the learner to select the relevant information and strategies necessary to the problem's solution.

Luckner and McNeill (1994) suggested that deaf students may have difficulties in relation to arithmetic problem solving and it was important to identify ways to assist deaf students to develop organisational and procedural skills if their arithmetic solving were to be improved.

In summary, researchers have claimed that while deaf students consistently show delays in comparison with their hearing peers in arithmetic problem-solving, there is no simple or direct relationship established between these delays and the students' linguistic and experiential deficits or their degree of hearing loss (Wood, et al, 1986). In general, teacher ratings have been demonstrated to be effective predicators of student communication and academic performance (Hyde & Power, 1996). While only a small number of researchers have examined deaf students performance on problem solving tasks, even fewer studies have considered in detail their performance on arithmetic word problems.

This research reports on a project investigating Queensland deaf students' performance on arithmetic word problems. As previous studies have not considered the strategies used by deaf students in solving arithmetic word problems, this project investigated the following questions:

1. How do deaf and hearing-impaired students compare with their hearing peers when solving arithmetic word problems?
2. Are there identifiable strategies used by deaf and hearing-impaired students when solving arithmetic word problems?

Using the research instrument developed by Lean, Clements and Del Campo (1990) and their data as a comparison base, this project investigated the outcomes and strategies used by Queensland deaf and hearing-impaired students when solving arithmetic word problems.

METHOD

Sample: All moderately through to profoundly deaf students in South-East Queensland were surveyed and a sample were interviewed. Students from both mainstream and Special Education settings were included in the study. The students had been ascertained to require special educational assistance because of their hearing loss according to Education Queensland Guidelines for Ascertainment of Hearing-Impaired students. All students had mastered basic number facts (ie in this context, basic addition and subtraction facts to 100) and had basic English competency skills as determined from school records and teacher judgements. Comparable numbers of boys and girls participated in the study and students did not other significant or uncorrected impairments. In some cases, the sample size for a particular grade level is low due to the number of students enrolled in any particular year level.

Table 1
Number of student from each grade level.

Gr	1	2	3	4	5	6	7	8	9	10	11	12
N	3	6	7	4	11	12	6	15	7	3	0	3

The Task: The arithmetic word problems were those categorised by Heller and Greeno (1978) and used in the Australian and Papua New Guinea (PNG) context with hearing students by (Lean et al., 1990). The three types of questions – "change", "combine" and "compare" – were used in the study. All questions involve additive or subtractive strategies with numbers where the sum is less than 10.

Method: The study involved two key phases. Phase One consisted of a survey instrument which students across South-East Queensland completed. The survey consisted of the 24 word problems developed by Lean et al (1990) and using a pencil-and-paper format. The implementation of the survey was supervised by a trained research assistant (mathematics and hearing-impaired trained teacher) and in conjunction with the classroom teacher. This phase provided base-line data to compare deaf and hearing impaired students' performance against that of the Lean et al study. From the data collected through the survey, target students were identified and up to 6 students each grade level were interviewed to provide more in-depth data on the strategies used to solve the word problems. Phase Two consisted of the development of case studies of the target students. Students were encouraged to use a "think aloud" strategy to solve the tasks. The interviews were video-recorded and were undertaken with a trained sign-language research assistant. Students were able to use their preference for signed English or oral communication to present their responses to the tasks. All interviews were transcribed and analysis undertaken of both the mathematical and non-mathematical language used in the task resolution. The responses offered provided indicators of the problem-solving strategies used, features of the language used, of cognitive strategies and of mathematical reasoning.

Results

Table 2

Percentage of correct responses for each item

		1	2	3	4	5	6	7	8	9	10	12	tot
1	Ann has 2 cents. Mary has 3 cents. Who has more cents, Ann or Mary?	100	100	100	75	100	100	83.3	93.3	100	100	100	96.1
2	Bill has 5 cars. Mark has 2 cars. Who has less cars, Bill or Mark?	66.6	83.3	85.7	75	90.9	91.6	66.6	86.6	85.6	100	100	85.7
3	Which number is more, 4 or 7?	66.6	100	100	100	100	100	100	93.3	100	100	100	97.4
4	Which number is less, 3 or 5?	33.3	100	85.7	75	90.9	83.3	100	100	100	100	100	90.9
5	Tania had 3 apples. Ben had twice as many as Tania. How many apple did Ben have?	0	0	14.3	0	9.1	25	33.3	60	42.9	100	66.6	31.2
6	My jug holds 6 litres of milk. It holds twice as much as Luke's jug. How much milk does Luke's jug hold?	0	0	14.3	0	0	0	16.6	0	0	100	33.3	7.8
7	Which number is 3 more than 4?	0	0	14.3	0	18.2	8.3	33.3	53.3	28.6	100	33.3	26
8	Which number is 2 less than 3?	0	0	28.6	0	18.2	25	66.6	66.6	28.6	100	66.6	36.4
9	5 is 1 more than which number?	0	16.6	0	0	18.2	0	16.6	20	14.3	66.6	33.3	14.3
10	6 is 2 less than which number?	0	16.6	14.3	0	0	0	0	6.7	0	66.6	33.3	7.8
11	Barbara had 2 eggs. Dan gave Barbara 1 more egg. How many eggs does Barbara have now?	66.6	66.6	85.7	75	100	66.6	100	93.3	85.7	100	66.6	84.4
12	Jack has 4 pens. Dianne took 3 of Jack pens. How many pens did Jack have then?	33.3	83.3	85.7	75	90.9	66.6	100	80	100	100	66.6	81.8
13	Jeff had 3 bananas. Carmel gave Jeff some more bananas. Jeff then had 5 bananas. How many bananas did Carmel give Jeff?	33.3	16.6	42.9	25	18.2	41.6	50	53.3	85.7	100	66.6	45.5
14	Anna had five books. Tom took some of Anna's books. Then Anna only had 2 books left. How many of Anna's books did Tom take?	33.3	50	71.4	75	54.5	66.6	66.6	80	100	100	66.6	70.1
15	Paul had some pencils. His father gave him 2 more pencils. Then he had 5 pencils. How many pencils did Paul have at the start?	0	0	28.6	25	36.4	58.3	50	60	85.7	100	66.6	48.1

		1	2	3	4	5	6	7	8	9	10	12	tot
16	Sally has some pictures. She lost 2 of her pictures. Then she had 3 pictures. How many pictures did she have at the start?	0	0	42.9	0	36.4	50	50	80	57.1	100	33.3	46.8
17	David has 2 dogs and Jim has 4 dogs. How many dogs do they have altogether?	66.6	83.3	100	100	90.9	100	100	100	100	100	100	96.1
18	Helen has 3 ribbons. Lyn also has some ribbons. Helen and Lyn have 7 ribbons altogether. How many ribbons does Lyn have?	0	0	0	25	27.3	25	66.6	66.6	85.7	100	66.6	41.6
19	John has 2 buckets. Eric has 6 buckets. How many more buckets than John does Eric have?	0	0	0	0	18.2	8.3	33.3	60	57.1	66.6	66.6	28.6
20	Nick has 2 cups. Sarah has 7 cups. How many cups less than Sarah does Nick have?	0	0	0	0	27.3	33.3	33.3	86.6	71.4	100	66.6	41.6
21	Gina has some boxes. Ken has 3 boxes. Gina has 1 more box than Ken. How many boxes does Gina have?	0	50	85.7	50	54.5	83.3	83.3	93.3	71.4	100	66.6	72.3
22	Jo has some dolls. Pat has 5 dolls. Jo has 2 dolls less than Pat. How many dolls does Jo have?	0	0	0	50	36.4	16.6	33.3	86.6	57.1	100	66.6	41.6
23	Bill has some trucks. Tina has 5 trucks. Tina has 2 trucks more than Bill. How many trucks does Bill have?	0	16.6	0	25	45.5	25	50	33.3	28.6	100	66.6	32.5
24	Sam has some marbles. Sarah has 6 marbles. Sarah has 2 marbles less than Sam. How many marbles does Sam have?	0	33.3	71.4	0	0	50	33.3	40	14.3	100	33.3	33.8

The overall could not be analysed for statistical significance due to the small numbers in each year level cohort. However, the trends confirm similar patterns of difficulty experienced for hearing and non-hearing when comparison is made with the Lean et al (1990) data. As noted by other authors researching aspects of development of numeracy, mathematical and arithmetic concepts for deaf and hearing-impaired students, the data in this project further confirms the delay in the capacity of deaf and hearing-impaired to respond appropriately to tasks such as those commonly found in mathematics classrooms.

DISCUSSION

For many students – hearing and hearing-impaired – there is a reliance on trigger words which offer students cues as to the meaning of the problem. For example, the trigger word "more" is often seen to represent an additive operation. In part, this is potentially due to the widespread practice of teachers to introduce the addition operation in the context of word problems relevant to the students' non-school experiences. Common practices consist of teachers posing questions such as "I had three apples, then I bought three more, how many did I have altogether?" Such tasks expose students to practices where they can extrapolate the meaning of "more" to imply addition. For students whose access to the lexical density and complexity of the mathematics register, such as deaf students, their capacity to decipher the very subtle use of language is restricted.

Two trigger words which appeared to be particularly difficult for deaf students were "twice" and "less". Items 5 and 6 used the word "twice" in the context of doubling and halving respectively. For many students below Grade 6, the term itself was difficult with a number of the students asking for clarification of meaning. Where those students were able to offer a correct response, only one student was able to justify the response thus raising questions as to the students' deep understanding of the concept. For some students who were able to articulate their reasoning, "twice" meant "double". In most cases, such a crude understanding would enable them to solve many word problems. However as was apparent in Item 6 where the inverse was the case, students needed to halve the amount in order to respond correctly. As indicated by the overall performance, this task appeared to be one of the most difficult for the students. Most frequently, students relied on their understanding of "twice" to mean "double" and would offer a response of 12. Unlike the other tasks where the problems require the manipulation of different sets, this task required a doubling (or halving) operation and it would appear that the complexity of the task is in the specificity of the operation/s associated with the use of "twice".

Other common trigger words in mathematics are the terms "more" and "less" which are often associated with the operations of addition and subtraction respectively. The patterns of responses confirmed that these patterns. Furthermore, as Walkerdine (1982) has noted, for working-class students, whose linguistic code is somewhat restricted in comparison with that of the middle-class and formal school codes, deaf students found the concept of "less" particularly difficult. At the decontextualised level (Items 3 and 4), where there the word problems are not embedded within a context, where the binary oppositional terms of "more" and "less" were used, there were fewer students able to respond correctly to Item 4 where the relational term "less" was integral to the item. Further through the tasks, students misconstrued items using the term "less" in a number of different ways. In the comparative tasks such as Item 8 where the task asks which number is 2 less than 3, some students perceived the use of "less" to connote subtraction and were able to respond correctly, in spite of not understanding the task. However, most students saw the task as a comparative one in which they had to identify which was the lesser in value of 2 or 3. The wording of Items 7 and 8 were interpreted similarly, whereby the students saw them as asking which was the greater (or lesser) of the two numbers, rather than as addition or subtraction equations. The signifier "than" seemed to be redundant wording as many of the students did not know the word nor its role in the sentence, resulting in the (mis)interpretation of the task as being "which is more, 3 or 4".

The use of prepositions has been found to cause difficulties for students whose first language is not English. McGregor (1991), working in the multicultural classroom, found that non-English speaking background students found the use of propositions particularly difficult. For example, she cites the example where the statements are made: The temperature fell by 10 degrees ....to 10 degrees, ...from 10 degrees, ... Rudner (1978, cited in Barhnam & Bishop,. 1991, p. 182)) noted a similar difficulty with deaf students who struggled with conditionals (if, when); comparatives (greater than, less than), negatives (without, not) and inferentials (should, could). Difficulties were noted with the use of "is" in Items 9 and 10 where interpreted the question as being 5 and 1 more is which number (as opposed to 5 is 1 more than which number).

Contextualised Word Problems

Where the problems are stated simply and in order of operation, there was a greater likelihood of the students being able to offer correct responses (such as Items 11 and 12). The actions involved in the change tasks (giving and taking) provided strong contextual cues for the students as to the appropriate operation. Difficulty arose when the indefinite "some" came into the word problems. Not only was the amount missing but it also produced confusion as to the appropriate action (operation) to be undertaken. Where the non-stated amount came at the first line in the task, the students experienced great difficulty in making sense of the task. In Items 15 and 16, the lack of a definitive amount confused the students as to the requirements of the task (find out how many were there at the beginning). The students searched for key words (such as "lost") to offer some cues as to what they might need to do. In the case of Item 16 where something was "lost" students took this as a cue for subtraction. Where the first line is ambiguous, the students ignored it and saw it as having no relevance to the task. They then proceeded to operate on the number given using what cues they could extract from in the task.

The comparative tasks in Items 19 and 20 proved to be difficult due to the structure of the final sentences, which required a comparison to be made and a difference noted. As with the decontextualised tasks, students were at risk of interpreting the task as one of which was the bigger or smaller quantity and naming that quantity. Alternatively, they would interpret the "more" or "less" in the wording to signify addition or subtraction.

CONCLUSION

The data collected confirmed previous studies where deaf students are somewhat delayed in their performance in mathematics. Furthermore, the data collected in this project support the notion that the specificity of language used in mathematics – or more specifically-arithmetic word problems, creates difficulties for deaf students.

Many of the deaf students relied on top-down approaches to comprehending the tasks, recognising that they would not understand everything that they had read. Relying on this "key word" reading strategy, they use their understanding of key words and this knowledge base to construct a meaning for the tasks. Such strategies often mean that signifiers such as "more" and "less" are interpreted to imply addition and subtraction. From the responses offered in these tasks, this appears to be a very common strategy used by these students. The reliance on the meaning of key words resulted in many incorrect responses. With these restricted understanding of semantics, deaf students are compelled to rely on interpreting fragments to make sense of that to which they do understand. The complex semantics and lexical density of the word problems along with the lack of redundant and supportive information found in such arithmetic word problems hinders the capacity of deaf students to make sense of the tasks, and hence their capacity to offer correct responses.

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MATHEMATICAL KNOWLEDGE IN A MATHEMATICS TEACHING EPISODE

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This paper addresses issues relevant to an analysis of mathematical knowledge in a mathematics teaching episode. Moving from a concern with what it terms the 'teaching paradox', the paper seeks to analyse an episode taken from a year 11 mathematics classroom. In this analysis three main concepts are used: knowledge base, knowledge artefact, and functional knowledge claim. The main finding of the analysis is that mathematical knowledge stands in a peripheral relation to mathematics teaching.

BACKGROUND

In their book *Situated learning: legitimate peripheral participation* (1991), Lave and Wenger make the observation that normal classroom practice revolves around a teaching curriculum rather than a learning curriculum. What this amounts to for mathematics education is that teachers do not teach 'mathematics' so much as teach 'mathematics teaching'. If true, this observation would seem to have paradoxical implications, such as, the more teachers teach mathematics, the less mathematics is taught, and so on. These kinds of points have, of course, also been made by others in the mathematics education literature, most notably Brousseau (1986) and Steinbring (1989), where they have been linked with studies investigating classroom interactions from micro-sociological perspectives. I will refer to this paradoxical relationship between 'mathematics teaching' and 'mathematics' as the 'teaching paradox' and will use this as a starting point for an investigation into mathematical knowledge in a mathematics teaching episode.

In seeking to better understand this paradox, I have turned initially to Shulman's now classic typology of teacher knowledge (1986, 1987). In this, teachers' knowledge is posited to consist of content knowledge, curriculum knowledge (general knowledge of how to teach), pedagogic content knowledge (the knowledge of how to teach particular topic areas of content knowledge), and other kinds of knowledge relating to the needs of students and characteristics of the context of teaching. Of these, possibly content knowledge and pedagogic content knowledge are of most direct interest to my study; they can be taken to roughly equate with 'mathematics' and 'mathematics teaching', respectively. For, holding both the disciplinary needs of mathematics and the learning needs of students clearly in view, expert teachers work through the curriculum topic by topic, making and implementing along the way reasoned judgements based on their experience, knowledge and belief of what good mathematics teaching amounts at each stage of progress.

Whilst Shulman's contribution has been enormously helpful in political debates surrounding the status of teacher knowledge and models for teacher education, when viewed from the perspective of the teaching paradox, however, it presents as strangely inadequate. For if the goal of good teaching is to advance mathematics learning, then according to the teaching paradox, the place of pedagogic content knowledge (mathematics teaching) surely ought to be less prominent. Instead, however, Shulman places this body of knowledge at the centre of teachers work and insists that expertise in this body of knowledge is at the heart of teaching expertise. In trying to explain this state of affairs, one option to be considered might be that the language used has become entangled, that precision has been lost, and that this have given rise to meanings contrary to intentions. Another option could be that teaching simply does involve paradoxical relationships, and that learning occurs precisely because of them, not despite them. Whether one or both or some other alternative is the case, I would suggest that developments in knowledge of teaching and learning are bound up, in part at least, with these matters.

In this paper, I want to explore an aspect of the teaching paradox. Using data obtained by video-taping a Year 11 mathematics class in operation, I will seek to demonstrate how Shulman's pedagogic content knowledge can overshadow mathematical knowledge in an actual teaching situation. My analysis will lead to a new model of relationships among the knowledge bases of the teacher and learner. The paper will conclude with a statement of implications for teaching and learning and further research.

Theoretical Framework

Consistent with cultural historical perspectives on learning initiated by Vygotsky (1978) and others, my study attempts to build an epistemological analysis of interactions observed, by analysing task performance. By a task, I mean, roughly, an activity with a definite purpose or object in view. Whilst analyses of this kind are possible using a variety of alternative perspectives (eg reflective practice (Schön, 1987); social constructivisim (Cobb, 1994); discourse theory (Walkerdine, 1988); sociology (Dowling, 1998)), the approach I am attempting is oriented towards synthesising activity theory concepts to cognition with matters arising within the philosophy of knowledge (see Fenstermacher, 1994; Kessels & Karthagen, 1996; Thomas, 1997). For such an attempt, I will make use of the following conceptual tools: knowledge base, knowledge artefact, and functional knowledge. Purposes these are put include: identifying, analysing and describing instances of knowledge; clarifying the language used about teaching and learning; exploring assumptions concerning these concepts. In constructing the 'tool-kit' (see Well (1996)), I have sought ideas which are shaped by the conceptual environment in which the problems are posed, for instance, (i) the concept of the social construction of knowledge, (ii) the activity dependent characteristics of knowledge, (iii) the generation of knowledge identity (that which enables it to be asserted that one piece of knowledge is an instance of another or is of the same kind as another), and (iv) the generation of knowledge *difference* (that which, in the language of Gaston Bachelard, induces epistemological rupture and projects the reconstitution of knowledge). I have set these out briefly in the following list.

Knowledge base: Following Jong & Ferguson-Hessler (1993), this term is used to describe a body of knowledge types (situational, declarative, procedural, strategic) and knowledge qualities (surface/deep, isolated/structured, automated, modality, general/domain specific) salient to task performance. Whatever knowledge, described by type and quality, is used in performing a task belongs to the knowledge base of that task. A teacher's knowledge base consists of all the required by the teacher to perform the tasks of teaching. Likewise a mathematics knowledge base, consists of all the mathematical knowledge required in order to perform mathematical tasks.

Knowledge artefacts: I have borrowed the term 'artefact', a "product of human art and workmanship" (*Shorter Oxford Dictionary*), from Vygotsky (1978), Leont'ev (1981) and Lave & Wenger (1991) who refer to artefacts as physical, linguistic and symbolic entities; and used it to coin the new term 'knowledge artefact'. The purpose of this term is to emphasise that an object is meaningless and literally useless unless knowledge, in the form of a context, is brought to it. For this reason, the same object could be any number of different knowledge artefacts, depending on the varieties of contexts brought (for example, think how many purposes we have for a blade, each having a place within their own sphere) and, conversely, different objects could be the same knowledge artefact (think of 'text' - it could be hard copy or on monitor, etc). These examples suggest that it can be important to be able to distinguish between an entity's object status and the status an entity has in advancing a purpose or task. It is also assumed that the following can be said of 'knowledge artefacts': (i) they are human products, (ii) they produced within the context of a set of socially/culturally constructed rules, (iii) they hold different values depending on custom, and (iv) they evolve over time, and therefore have a history represented as traces within the tasks they perform used.

Functional knowledge claim: Getting a task done not only requires the necessary knowledge artefacts and requisite knowledge base - it also requires that these actually be implemented. Now, on reflection, 'implementing' involves making a claim in at least two respects. In the first, to implement a course of action is to quit speculating about how things might be, and to declare how, for you, things are placed at the time. In this sense, getting a task done simultaneously involves making a claim. A second way in which 'implementing' a course of action involves making a claim, is as follows. To do something for a special purpose requires both that a proposal for action be formed (that is, that there be an object in view, tools to hand, and the means whereby the task can be performed and managed ready), and that action really be engaged. Because we actually never really know what our action will lead to, our expectation, commitment, values, and beliefs, and the cognitions surrounding them, are engaged, and these help us make selections and come to a view about the direction our action is taking us. This argument suggests that implementing pertinent knowledge involves making, entertaining and testing commitment, having and creating expectations, and so on. It also suggests that claims which achieve the purpose for the sake of which they were originally made, have the strongest grounds for being declared valid. Other validity grounds are also possible - for instance, where the grounds for expectations and commitment associated with an action are well argued or praiseworthy, etc. In order to facilitate discussion of these collateral events in the life of task performance, I have coined the term 'functional knowledge claim'.

METHODOLOGICAL ISSUES

As indicated previously, data analysed in this study was obtained by video-taping a Year 11 mathematics lesson in progress, devoted to the topic of demonstrating that the argument of the product of complex numbers is the sum of arguments of the numbers involved. My method of analysis was as set out in the following. Firstly, I analysed the content of the transcript and partitioned the material into sections and subsections according to the subject matter involved. These will be reported in detail in a subsequent publication. Secondly, one subsections drawn from this analysis was chosen for analysis in this paper. Thirdly, each of these segments were analysed using the tools referred above. In particular, within respect to each transcript segment, knowledge bases and knowledge artefacts were identified. These steps involved making significant assumptions about the validity of inferences made. A full discussion of these, beyond the scope of the present work, is needed. Fourthly, data were analysed using the conceptual tools previously discussed, and this lead to the development of a new model of teaching and learning interactions. Analyses reported below represent work in relation to the third and fourth stages indicated above.

ANALYSIS

A useful start to the data analysis will be to note that for any two complex numbers, the argument of their product is equal to the sum of their arguments. In the terms of my theoretical perspective, this mathematical fact is an artefact within a mathematics knowledge base. Figure 1, provides a demonstrating of this artefactual status.

Figure 1
Mathematical Demonstration that the Argument of the Product of Complex Numbers is the Sum of their Arguments.

Given $z_1 = r_1 \operatorname{cis}\theta_1$ and $z_2 = r_2 \operatorname{cis}\theta_2$,

$$\begin{aligned}
\arg(z_1 z_2) &= \arg(r_1 \operatorname{cis}\theta_1\, r_2 \operatorname{cis}\theta_2) && 1.1\\
&= \arg[r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)], \text{ by definition} && 1.2\\
&= \arg\{r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\} && 1.3\\
&= \arg\{r_1 r_2[(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]\}, \text{ using the relevant trignometic identities} && 1.4\\
&= \arg[r_1 r_2 \operatorname{cis}(\theta_1+\theta_2)] && 1.5\\
&= \theta_1 + \theta_2 && 1.6\\
&= \arg z_1 + \arg z_2 && 1.7
\end{aligned}$$

Thus, $\arg(z_1 z_2) = \arg z_1 + \arg z_2$, as required.

In contrast to this, Figure 2 presents the teacher's blackboard script for that part of the lesson relating to the topic of Figure 1.

Figure 2
Transcript of Teacher's Blackboard Script

$$\begin{aligned}
&\arg(z_1 z_2) && 2.1\\
z_1 &= r_1 cis\theta_1 && 2.2\\
z_2 &= r_2 cis\theta_2 && 2.3\\
z_1 z_2 &= r_1 cis\theta_1 \times r_2 cis\theta_2 && 2.4\\
&= r_1 r_2 cis\theta_1 cis\theta_2 && 2.5\\
&= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) && 2.6\\
&= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] && 2.7\\
&= r_1 r_2[(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] && 2.8\\
&= r_1 r_2 cis(\theta_1+\theta_2) && 2.9\\
\\
\arg(z_1 z_2) &= \theta_1 + \theta_2 && 2.10\\
&= \arg z_1 + \arg z_2 && 2.11
\end{aligned}$$

The transcript following refers to Figure 2, and provides a record of interactions among teacher and students.

T: Excellent. Well done. Well, one person has made the next jump. Good. What do you think is the next stage? *[teacher talking to student]* That one plus that minus that one there. You put them together. Now anyone else get the next step? We've got three people so far. Peter? Hmmm *[continues speaking to individuals]* OK Let's just make the next quantum step. Look at the first relationship. What is it? Who sees what it is, there's quite a few people I can see who've written things down. Steve, what is it, this first thing up here? *[points to 2.7, Figure 2]* It's written on the board, hint hint hint.

S: *[student inaudible]*

T: *[writing 2.8]* Instead of $\theta_1 + \theta_2$ what is it? $\cos(\theta_1 + \theta_2)$. And what's the other thing there? Now most people got to there *[2.7]*. Remember when you're dealing with complex numbers you need to put things together. So you need to put your real parts together and your imaginary parts together. I think that's where a lot of people couldn't see this next jump. OK, now, you got there. Pens down, and let's just think about what this means. You've got z_1z_2, is equal to that. *[referring to 2.7]*. Right, any worries? Right, this *[2.8]* is in trig form. What's the first bit? *[circles r_1r_2]*

S: Modulus

T: Modulus. So in other words we've got two complex numbers we multiply together, and we know that the modulus will be multiplied together, the two moduluses. Alright, what about this thing here? Let's just rewrite that in the other form, we can write that as $\operatorname{cis}(\theta_1+\theta_2)$. Alright. Now remember what our original, what I originally asked you. I said predict what $\arg z_1z_2$ *[writing 2.10]* is going to be. Now remember your arg is an angle, and really what we come up with, well perhaps you can tell me. What have we got there? The argument of the product is equal to what?

S: The sum of the angles.

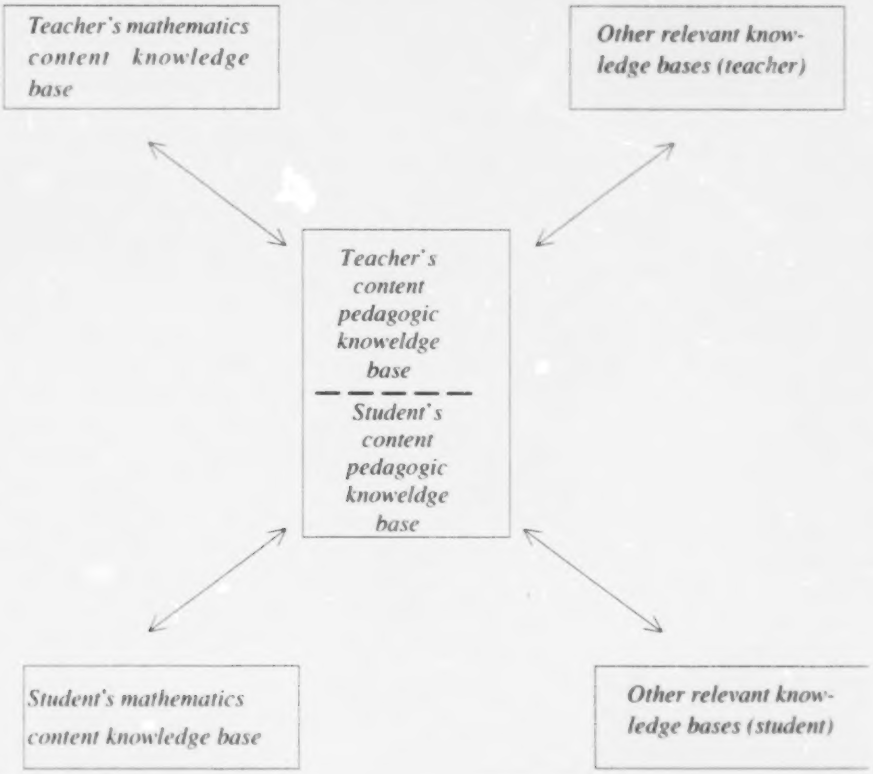
T: The sum of the angles. The arg of this thing here *[2.9]* is $\theta_1 + \theta_2$. OK. So we've got the arg $z_1 z_2 = \theta_1 + \theta_2$. *[2.10]* Right? Now, when we've got $\theta_1 + \theta_2$ what's θ_1 equal to? Come back up to here *[points to 2.2]*, what's θ_1 equal to? It's the arg of z_1 *[writes 2.11]*. And what's θ_2? argz_2. So was our original hypothesis correct?

S: No... *[inaudible]*

T: No, because we predicted - you predicted or someone predicted - who do we blame? Steve? Well he was the only one willing to have a go, predicted that the arg of the product was going to be the product of the arg, which is the normal thing we've had with practically everything we've ever done. But notice here that the argument of two complex numbers is equal to the sum of the individual arguments. OK, so when you look at that it's not what you'd obviously think *[teacher rubbing hands]*. It's going to be something that's slightly different *[teacher looking at students; pause]*. More than slightly different *[teacher walking back to the board]*, dramatically different. We're going from the arg of a product is the sum of the individual parts. Now let's just sum up that before we do anything more with it. So pens down and let's just think for a minute, rather than copying it *[?]* dramatically, because remember what we're after is trying to get some idea of how we go about proving these things. And looking at it. Now, we did a couple of different things today if you look at it. The first thing, what we did was, we developed a relationship. Now the reason we developed that relationship was that we had to use it. I mean, we could have got to there, I could have said to you, oh yeah great, that thing there *[points to 2.7]* is $\cos(\theta_1 + \theta_2)$, but you wouldn't have believed me would you? No you don't believe me in anything do you. OK, so I could have told you that, but the first part of the lesson what we did was we developed this relationship. That when we've got a sum of an angle and we take the cos of that or the sin of that and even we can go ahead and work out the tan of those things, then we can relate it back to the cos and sin of the individual angles. Alright? The second thing that we did was we said righto, let's go now to our arg$z_1 z_2$ *[2.10]*, and what we did, notice we started off with $z_1 z_2$ *[points to]* and we used our trigonometric form and we've gone through and developed something like that. We used our expansion idea *[refers to 2.7]*, we've come back to $r_1 r_2$, and put it all together, and we've gone from taking the separate parts and we've put it into $\theta_1 + \theta_2$ *[points to 2.8]*. So we've ended up with this relationship here *[draws blocks around 2.10 and 2.11]*.

As a comparison with Figure 1 will show, many knowledge artefacts contained in these episodes correspond with or are consistent with the mathematics content knowledge artefact exhibited in Figure 1. This finding is, of course, expected. However, the transcript reveals other kinds of knowledge artefacts. For example, in lines 10-12, the teacher says "Remember when you're dealing with complex numbers you need to put things together. So you need to put your real parts together and your imaginary parts together." Although this statement appears to address the management of complex numbers, it does so *as if* complex numbers were concrete and the management issue concerning them revolved around a sorting operation. As neither of these is the case, it is concluded that this statement is not a mathematics knowledge artefact. However, the statement is certainly relevant to complex numbers in the sense that given the right code it can be translated into a functional knowledge claim capable of generating mathematics knowledge artefacts; that students have access to this code appears assumed in the transcript, for in the next sentence the teacher adds "I think that's where a lot of people couldn't see this next jump". This suggests that *both* teacher and students are performing tasks which at this point draw on a knowledge base peripheral to the mathematics content knowledge bases of each. From the teacher's perspective, this knowledge base involves performing tasks relevant to both mathematics and teaching, 'mathematics teaching', and so, can be designated mathematics pedagogic content knowledge; from the student's perspective, the knowledge base in question also involves the performance of tasks relevant both to mathematics and teaching (for instance, knowing how to reconstitute a teaching artefact as a mathematical one, see above), and so also seems to be a kind of mathematics pedagogic content knowledge. Figure 3 summarises this analysis.

Figure 3
Proposed Model Indicating Relations among Teacher and Student Knowledge Bases


Teacher's mathematics content knowledge base
Other relevant knowledge bases (teacher)
Teacher's content pedagogic knoweldge base
Student's content pedagogic knoweldge base
Student's mathematics content knowledge base
Other relevant knowledge bases (student)

This episode also suggests that the teacher's ways of teaching, her functional knowledge claims within her pedagogic content knowledge base, are built around the dramatisation or staging of her mathematical knowledge base. For instance, in the lines 9-12 the teacher digresses to describe a set of events and imperatives which are drawn together in unity and lead to crisis, namely, "people couldn't see this next jump". This crisis could have been avoided, she states, had "people" only remembered to that when "dealing with complex numbers you need [sic] to put things together". This examples also shows that in her dramatisations of mathematical knowledge, mathematical operations are sequestered (Lave and Wenger, 1991) or bracketed within the roles of observer, critic, coach and commentator. Later in the episode, students are reminded that earlier in the lesson they were asked to "predict what argz_1z_2 *[writing 2.10]* is going to be" (lines 19-20). Then in lines 21-22 a conclusion is reached that the argument of the products is equal to the sum of the arguments. Next, in line 26, the teacher brings to the drama to a crisis with the question "So was our original hypothesis correct?" Students answer "no" and the teacher responds "No, because we predicted - you predicted or someone predicted - who do we blame? Steve?" (line 28). As with the previous example, the crisis around which this drama turns possesses an almost moral quality - something on which the concept of "blame" can be attached, even in jest. A second crisis ensues in lines 32-34. Here the transcript shows that the teacher uses a complex set of words, speech patterns, and hand, body and facial gestures to indicate that the outcome of this result is "dramatically different" (line 34) from "what you'd obviously think" (line 32). The teacher's choice of the second person pronoun is also interesting here, and perhaps gives a clue why she is choosing to be so exaggerated in her expressions at this point of the lesson. In her range of pronoun usages 'I' and 'we' are invariably used to indicate her, 'you' to indicate a student or students. Further it is almost always the case that a misunderstanding or misconception is attributed to 'you' in her discourse. This could mean that when she says "it's not what you'd obviously think [*teacher rubbing hands*]" (line 32), she is really saying "it's obvious that you'd fall into this trap". If so, then this provides an example of the teacher playfully teasing her students by setting up a contrasting relationship between an inner meaning and an outer meaning. This example of irony, echoes my previous observation, that mathematical knowledge is sequestered in the course of the lesson, it is bracketed by the need to foreground the dramatisation of mathematical knowledge (alias pedagogic content knowledge) and encourage 'in role' student behaviours. Such behaviours are seen when she says "Well, one person has made the next jump. Good. What do you think is the next stage? *[teacher talking to student]*"(lines 1-2), "No, because we predicted - you predicted or someone predicted - who do we blame? Steve?" (line 28). These analyses further illustrate Figure 3, by demonstrating the richness of the non-mathematical life of 'mathematics teaching' and indicating that both teacher and students share functional knowledge claims within the 'mathematics teaching' knowledge base. Mathematical knowledge is shown to stand in a peripheral relation to mathematics teaching.

This episode also provides information concerning the constructed nature of the shared teaching knowledge base. The scripted nature of her teacherly interactions becomes more clear. For instance: in lines 1-12 the teacher draws students into an active engagement with mathematical manipulations; in line 13 she prompts active reflection on the artefact these manipulations generate (*2.7*); then in lines 14-18 more manipulations follow, making use of an identity previously obtained; in lines 19-21 active reflection in which students are prompted to come to a view about the significance of their recent work for a task previously encountered, is again prompted; more related mathematical manipulations follow in lines 21-25; and finally, in lines 26-51, the teacher generates a (meta)narrative consisting of a commentary on the overarching mathematical strategies followed in the lesson up to this time, ideas that were used and tasks performed, and thereby reconstructs the context of the preceding work. This analysis indicates that her pattern of progress from manipulations to reflection is frequently repeated, and that students participate in these manoeuvres as the drama unfolds. This seems to point to the scripted nature of her performance and illustrates that the organisation of her exposition is purposeful and strategic, and that the logic of the script she follows allows her to produce, manipulate and regulate mathematics related knowledge artefacts for the purpose of teaching.

Note that the model presented in this figure indicates relationships among a broader range of knowledge bases than those discussed above (namely, those in shaded blocks). These refer, of course, to other components of teacher knowledge, for example, general pedagogic knowledge, curriculum knowledge, and so on (Shulman, 1986, 1987; Sternberg & Horvath, 1995; etc); and also student knowledge, attributions and beliefs relating to themselves as learners and mathematics as a body of knowledge, the various contexts of the school institution, their place within the social matrix of gender, culture and language values, and so on (Brown, 1997). Without referring to these matters in any detail, it will be recalled that the preceding analysis touched on the role of irony in building mathematics teaching knowledge claims. I would speculate that this, together with the good humour implicit in it, expresses the expectations and commitment of the teacher as she makes and implements her functional knowledge claims within the lesson.

CONCLUSIONS

In this paper I have attempted a detailed epistemological analysis of a lesson episode in order to see how mathematical knowledge is handled and how it stands in relation to teaching knowledge. Notwithstanding the obstacles faced (including unclear and uncertain terminology, conceptual confusion, methodological constraints - each of these suggesting areas for further study), evidence has been provided that mathematical knowledge stands in a peripheral relation to the lesson studied. The paper concludes by raising the question whether mathematics learning takes place because of this peripherality or despite it.

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Acknowledgement

I wish to acknowledge the contribution of my friend and colleague, Dr Falk Seeger, Institut für Didaktik der Mathematik, Universität Bielefeld, Germany to my analysis of the transcript reported in this paper.

LEARNING STYLES IN SECONDARY MATHEMATICS CLASSROOMS

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A small rural school introduced single-sex classes in an attempt to increase the participation rates of girls in highest level mathematics classes at the senior level. The study revealed differences in the learning styles of the students. This led to the adoption of a new construct to evaluate the single-sex classes. This construct offers suggestions in the way mathematics classes could be conducted in high schools to assist students with non-traditional learning preferences.

BACKGROUND

The impetus for the study was the lack of participation of girls in senior mathematics classes which was of concern to the mathematics staff. Single-sex classes were thought to be a viable intervention strategy to assist the girls and to bring the issue to the attention of other staff, parents and the community. The school, upon which this study was based, is situated in a mostly cereal grain farming community in Western Victoria. Two similar nearby towns (in terms of demography and population) and a larger regional center acted as feeders for the school. The area is very conservative in its thinking, both politically and socially, and issues such as gender equity and affirmative action needed to be handled delicately. The science teachers at the school had been part of an extended science Professional Development program in 1994 to investigate ways to make the science curriculum more accessible for girls. At an inservice day in that year, attended by the mathematics coordinator (also a science teacher), the discussion focused on single-sex mathematics and science classes as a possible intervention strategy to encourage more girls to participate in senior classes.

When this idea was first proposed there was general opposition to single-sex mathematics classes by other staff. Their main objection centered on the idea that in single-sex classes girls would develop skills in a comfortable environment where they wouldn't need to compete with boys for the teacher's attention. The contention was that this would make the girls weak and unable to cope or relate with boys in their other classes. However, the school had been conducting single-sex Physical Education classes for about 10 years. These were defended staunchly by the staff and yet single-sex mathematics classes were viewed negatively. So it was in the face of firm opposition that in 1995 the single-sex mathematics classes started on a week by week trial basis in Year 8. This opposition prompted the author to conduct research for a Master of Education degree on the introduction of these classes, data was collected from the Year 8 classes in 1996 (see Author, 1998a). There were 3 class groups in the study, at Year 8 level, one mixed-sex class (MS, n=23), one single-sex girls class (SSG, n=22) and one single-sex boys class (SSB, n=21).

THEORETICAL FRAMEWORK AND PREVIOUS RESEARCH

This study was based on similar studies (Fuller, 1995; Morrow, 1991; Rowe, 1988; Rowe, 1990) which investigated differences in self-confidence in mathematics and mathematical achievement as possible causes for the lack of participation of girls in senior mathematics classes. The framework predicted that an increase in self-confidence in mathematics would lead to increased participation for girls in senior classes. The study included a mathematics attitude questionnaire (an adaptation of the Fennema-Sherman Mathematics Attitude Scales, Fennema & Sherman, 1976) and a set of parallel mathematics achievement tests (PATMATHS Test 2A and 2B, ACER, 1984). [1] Field notes were collected from discussions between the researcher and students, teachers and parents. No formal interviews were conducted, though some notes were collected from conversations had at Parent-teacher interviews and from mathematics faculty meetings.

As the research was being conducted it emerged that the initial construct was not a particularly useful lens in which to view the outcomes and findings. This was partly due to the fact that no increase in self-confidence was measured for any of the classes, while there were significant increases in mathematical achievement for the single-sex classes. Further discussions with the classroom teachers revealed particular behavioral traits well worth investigating. There also appeared to be serious conflict with the results of the attitudinal survey and discussions with the students, particularly the girls. While the survey results found a slight decrease in the self-confidence of girls, their general behavior and interest in mathematics demonstrated higher levels of self-confidence. Anecdotally, students wanted to talk about the positive experiences they were having in the single-sex classes, particularly the girls. In these discussions they were confident to talk about and discuss their mathematics and the events that happened in class. Concerns arose over the reliability of the scale to measure the level of self-confidence compared to the behaviour and comments of the students. The students' perceptions of themselves may not be consistent across all of their experiences, thus answers on the survey may reflect their long term attitudes towards mathematics, even though they enjoyed their experience and felt more confident that year. While they enjoyed mathematics in that year, their longer-term perceptions of mathematics study and their longer term intentions to study it may not have changed.

The work by Gilligan (1982), Belenky, Clinchy, Goldberger, & Tarule, (1986), offered a useful framework for analyzing these issues and offered a different lens through which to view the research data. Building on the moral reasoning proposed by Gilligan (1982), Bleneky et al (1986) suggest a framework for not only the way people reason, but the way they build knowledge and construct understanding. These authors proposed that people may have two distinct reasoning and learning styles at the procedural knowledge level based upon different perspectives and rules. *Separate Knowers* develop their knowledge separately from others, using a set of impersonal rules, through which they objectively and critically sort given information filtering out any subjectivity. These people exclude feelings and personal beliefs to develop their knowledge objectively. Their learning attempts to separate the knowledge from the source of knowledge so as to evaluate the knowledge itself.

Connected Knowers develop their understanding from the perspective of the knower. These people prefer to connect with the knower and try to understand her/him and their subjectivism and the way they formed their knowledge as well as their knowledge. These individuals trust knowledge that comes from personal experience rather than being handed down from authority. They value learning and knowledge that is woven into their personal relationships, surroundings and environment. These people do not try to view knowledge as cold and impersonal but try to include the emotion and personality of the knower and see this as adding to the knowledge rather than detracting from it. Their knowledge of truth develops through care for others and their relationships with others (for a more complete explanation see Belenky et al., 1986)

Buerk (1985) describes how these forms of knowing (Gilligan, 1992) can be used to explain the way people from each group develop their mathematical thinking and learning. Separate knowers prefer to develop their mathematical knowledge from an objective, individual and critical perspective. They prefer to learn their mathematical understanding individually from an authoritative perspective, critically analyzing the information supplied via formal, structured and explicit instruction. Ideas need to be fully developed and thought through before separate knowers are prepared to discuss them. Hypothetical and tentative talk is not a preferred method of forming and evolving their thoughts. Connected knowers value the knowledge of others that is based upon their personal and particular understanding and grounded in first hand experience. Relationships both between the learner and knower and the knower and their own knowledge are seen as important, almost essential for the development of new knowledge. Discussions are important to connected knowers and the forming of knowledge may often occur through tentative and hypothetical talk between learner and learner or learner and knower. It is the form of the knowing that is central to connected knowers rather than the content, (Buerk, 1985).

Traditional mathematics teaching is based on an authoritative figure (usually the teacher) giving out information in a non-contextual way without relevance to the life of most of the students. Learning is based on remembering and correctly applying often complex and unconnected algorithms. The examples, exercises and problems used are usually contrived and bear no relevance to or reflect few of the issues relevant to young people. Mathematics classrooms are predominantly arranged in ways that encourage students to work individually. The opportunity to discuss and talk through issues to form knowledge is often rare. Students often get the feeling that mathematics classrooms are different to their experiences in other subjects. In mathematics classrooms the answers are always known and this offers students little opportunity for creativity and discovery. Such a view of mathematics classrooms suits the preferred learning style of separate knowers but is in contrast to the preferences of connected knowers. Teaching in a connected way includes:

- giving students the opportunity to find their own voice;
- have first hand experiences with mathematics;
- have the opportunity to pose their own problems;
- experience doubt in their understanding that requires questioning of themselves and reaffirming their belief in themselves;
- have support through challenging phases without being given answers; and,
- being given the opportunity to explore within the curriculum.

(see Becker, (1995), and Buerk, (1985)).

THE CONTINUING INVESTIGATION

Not long after the single-sex classes started in 1996, the researcher was in discussion with the teachers about the classes when it was revealed that there were differences in the attitudes and behaviours of the students. At this stage the researcher was not aware of the notion of connected and separate knowing. Generally, the boys in the SSB class preferred to work individually, they disliked group work and were not inclined to share ideas. They preferred learning from the board with the teacher as a transmitter, giving out information for the students to receive. The students in this class clearly preferred to learn as separate knowers. In the SSG class the students formed small groups at every opportunity, shared ideas and discussed their learning. They preferred group work to direct instruction and enjoyed and appreciated the opportunity to investigate problems beyond the more trivial applications associated with "normal" bookwork. The following discussion highlights this point:

> The assistant-principal (AP) took an extra of the SSG class in 1997, and came away surprised by what he saw. ... The AP commented that the girls had talked a lot in class but the talk was about the mathematics and sharing their ideas and solutions to problems. He was surprised he didn't have to continually return them to task. There was a pleasant working atmosphere in the class, they seemed to work through their difficulties by seeking help from each other and they seemed to be enjoying their mathematics. He said it was different from other mathematics classes he had sat in on.

(Keast, 1998a, pp. 154-155).

In the following year 1997, teachers changed their teaching approach to accommodate the connected knowing preference of the girls' class. The emphasis was on group work and sharing knowledge; as this appeared to be preferred by most girls.

The Students' Perspective

The girls who in the past had not experienced a lot of success in their mathematics class were more at ease in the single-sex class, they didn't feel as "lost" as before and were more willing to verbalize their mathematics in class, ask questions and share their ideas with other classmates. They appeared to improve substantially in their understanding and willingness to participate in class discussions. However, there was a small group of very able girls who objected to the methods being adopted in the class. Most of all they felt their superiority over the other girls was being undermined.

One able girl in 1997 said:

> Some girls who are really dumb at mathematics, now ask questions in class I don't know the answer to.

This had not happened before, the more traditional mathematics class, taught in a separate knowing way from the authority perspective had suited her well. She had a clear preference for separate knowing and did not like the idea of needing to change. Some girls appeared to prefer to learn in the separate knowing style rather than the connected knowing style that most of the girls exhibited.

One particular girl mentioned at Parent-Teacher interview that she felt the class had not covered as much work as the year before, and that therefore they had not learnt as much. The reasons she felt this way were based on the amount of group work covered and the lack of exercise books she had filled with practice problems! While skills practice remains an integral part of the course, many more classes are devoted to group development, sharing ideas and problem solving; this girl in particular did not view these as learning. Working in groups, discussing and developing ideas, was not as valued as copying copious quantities of notes from the board and learning algorithms to apply to set exercises. The main issue was the value placed on this style of learning by the higher achieving girls. This girl did concede that the SSG class had covered the same topics as the other class in approximately the same time. Also, she had achieved very good results in all her assessment pieces including tests. However, while she could agree that she had understood the work and the classes were enjoyable, she remained concerned as to her perceived level of difficulty of the mathematics being covered in the class. But her perceived level of difficulty was related to how difficult other students found it to understand.

> Mathematics should be hard, she said. We aren't doing enough work for a hard subject.

Her concerns about mathematics related not only to the amount of work she had completed, but to how hard it should be for *other* students to understand!

Towards the end of the first semester most girls favored the connected mode of learning compared to the separate mode, though the terms "separate" and "connected" were never mentioned directly in class. A small group of high achieving girls were not convinced that this was a better way to learn mathematics. What this highlighted was the fact that all the boys were content to be taught in the separate way without complaint, but not all girls were content to be taught in the connected way. Some of these girls operated as separate knowers, though none of the boys operated in the connected mode.

Teacher Perspective

From observation of the teachers' classes and from conversations with them the teachers found they changed their teaching methods to reflect the style preferred by the students.

The single-sex boys' class was taught in a very traditional and separate knowing way with the boys learning in a very individualized and independent way. In this class studentsresponded well to games and competition in their tasks, where the completion of the task required boys to work on their own. In the single-sex girls' class, it was observed that the girls formed small learning groups based on the tables where they sat. Their learning was a sharing process with lots of discussion and developing of ideas in a connected way. Whenever they were given the opportunity the girls would form small groups, discuss their learning and share their ideas. They always sought connection with each other. It was found to be too ineffective to teach the girls in the more traditional way. Boys in the single-sex class disliked group work and it was found to be very difficult to get boys involved in discussions of their understanding of mathematics.

School Dilemmas

It appears students prefer to learn via one of two learning styles. It is not clear if students alter their learning preference depending on the subject, time of day, or if this changes over time. Anecdotally some girls appeared to change from separate knowing to connected knowing over the course of the year. So, what are the implications for teachers in the classroom? If teaching is directed towards one method of learning (eg separate knowing) what is the impact on those students who prefer to learn via the other method (eg a more connected approach)? Teachers may need to view problem solving (and problem posing) activities, group work and discussion as important aspects of teaching for connected knowers rather than as required curriculum initiatives. Teaching mathematics has traditionally been taught in a way to encourage separate knowing and support separate knowers. This has alienated a large number of connected knowers (generally girls) who prefer to learn in a more connected way.

The unanswered questions the school staff and this researcher now grapple with are:

- Is it possible to identify which students operate in which learning mode and then group them accordingly, with their classes structured for their preferred learning style?
- What percentage of girls and boys prefer to learn as separate or connected knowers?
- Do students learn all aspects of mathematics according to their preferred learning style or can they swap between styles for different aspects of work and different subjects?
- Do students change learning styles for different classes, topics or over time
- Should those girls who prefer separate knowing in mathematics be taught with the boys?

These are similar to those issues raised by Morrow (1996).

CONCLUSION

There are two learning styles (separate and connected knowing) and two associated teaching styles (separate and connected teaching). There is anecdotal evidence that traditional mathematics teaching is mostly separate teaching. Separate knowers relate well to this style of teaching as it is representative of their preferred learning style. However, this style of teaching alienates connected knowers, who find it difficult to make connections with mathematics in these classes. Connected teaching with an emphasis on linking mathematics in the classroom to the students and their experiences encourages connected knowing students. But what are the implications of connected teaching on separate knowers? In this study it was difficult to encourage separate knowers (including some very high achieving girls) to participate in activities that encouraged connected knowing. However, over the course of the year some girls who began by preferring separate teaching had changed to preferring connected teaching by the end of the year.

From this study and the work of Morrow (1996) and Becker (1996) a different picture of mathematics teaching is starting emerge, where teachers encourage students to make connection with their own experiences and the experiences of others. Morrow (1996) contends that many connected knowers (particularly girls) in mathematics spend much of their time listening to the ideas of others. She suggests that if connected knowers are to gain a sense of their own voice in mathematics then teachers need to give them opportunities to verbalize their mathematics knowledge. It is through such discussion that students form, modify and develop their thoughts into ideas. For students who prefer to learn as connected knowers, the support of small groups and the role of hypothetical and tentative talk is important, almost essential, in the development of their ideas and understanding.

Such work is grounded in the ideas proposed by Belenky et al., (1986). To encourage more connected knowing in mathematics by both connected and separate knowers, topics need to begin from a context which provides relevance through real world application and needs. In order to promote appropriate student talk, this should be done in small groups to provide opportunities for students to interact and verbalize their understandings as they develop. The emphasis needs to be on problem solving and problem posing, with the teacher modeling the problem solving and posing process. Mathematics needs to be modeled as not always being completely known, but that there are alternative paths to the same solution, also that not all problems have neat and clinical solutions. In the real world many problems have no solutions, and approximations are required. Teachers need to show students that they make mistakes, take wrong paths, back track, create assumptions, analyze and evaluate their work, check assumptions, alter them and continue.

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Endnote

[1] (For a brief discussion of the outcomes see Keast,1997a; Keast, 1997b; Keast, 1998b. For a more comprehensive discussion of the outcomes see Keast,1998a).

GRADE 4, 6 AND 8 STUDENTS' STRATEGIES IN AREA MEASUREMENT

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This paper investigates Grades 4, 6 and 8 students' confusion between area and perimeter. Students were given area judgment tasks involving rectangles and their responses analysed in terms of the operations used and the strategies exhibited. Students were found to use both additive (Perimeter) and multiplicative (Area) judgment rules, and 7 different strategies. Those using an additive judgment rule tended to rely on rulers or fingers for measuring length and to align the rectangles vertically, whilst those using a multiplicative judgment rule tended to use overlay and partitioning strategies.

It has been well documented in the literature (for example, Hart, 1981, Hirstein, 1981 and Kidman, 1997) that students often become confused between area and perimeter. This paper reports on a study investigating this confusion through the use of Information Integration Theory (IIT) (Anderson & Cuneo, 1978) which identifies groups of students who employ additive and multiplicative judgment rules when making comparisons between rectangular shapes.

Initial research into children's understanding of area and perimeter was conducted by Piaget and his associates (Piaget, Inhelder, & Szeminska, 1960). Since then, Piaget's description of the development of area concepts has been the subject of a wide variety of studies attempting to replicate, disprove, extend or explain his conclusions. These studies (for example, Hart, 1981, and Hirstein, 1981) covered the development of the major concepts of conservation and transitivity and dealt with the following four basic concerns:

(1) difficulties with validating the reality of individual cognitive operations and describing the stages of their development - this issue led to studies which replicated (with slight variation) some of Piaget's procedures and materials;

(2) difficulties in investigating the relationships between different cognitive skills -this second issue was addressed by researchers administering a series of different tasks to the same sample of children to establish a developmental hierarchy;

(3) misconceptions children have relating to area - information on this issue was obtained by analysing incorrect answers provided by children and adults; and

(4) difficulties with identifying the nature of the progression between stages of development - this was dealt with through training studies.

The research questions gradually changed over time. Piaget and his associates published their findings in 1960. During the next 15 years, most area research centred around replicating Piaget's work. After 1975, replicating studies were rarely seen, with the exception of Lu's work (1981, 1991) from China. Area research focused on investigating relationships between different cognitive skills, and on large scale assessments of the performance of children on a variety of area tasks (Bell, Hughes, & Rogers, 1975; Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Foxman, Ruddock, Badger,& Martini, 1982; Hart, 1981). Studies concerned with student misconceptions emerged from this availability of data related to children's understanding.

Using the data from the mathematics section of the second National Assessment of Educational Performance, Hirstein (1981) studied the causes of errors and found that the difficulties shown by the students seemed to result from misconceptions about area rather than from computational weaknesses. Students were often unable to select the appropriate procedure when the link between the question and the computation was not explicitly made. This finding was supported by Foxman, Ruddock, Badger and Martini (1982) who found that students were unclear in the choice of correct strategy in a practical test. However, the particular problem students have with area is confusion with perimeter (Foxman *et al.*, 1982; Gholam, 1994; Hirstein, 1981; Hirstein, Lamb, & Osborne 1978; Kidman, 1997; Nitabach & Lehrer, 1996). Students found the process of obtaining a shape's measurements difficult, especially with regard to which dimensions to consider and how to count the units along the selected dimensions. A number of studies have also found confusion with reading and writing the Standard International area measurement units (Baturo & Nason, 1996; Foxman *et al.*, 1981).

This paper reports on a study investigating the confusion between area and perimeter. This study determined Grades 4, 6 and 8 students' area judgment rules by using Information Integration Theory (IIT) (Anderson and Cuneo, 1978). The aims of the study were: (1) to identify groups of students who have additive judgment rules, and groups of students who have multiplicative judgment rules; (2) to identify the strategies for judging area used by these students; and (3) to relate strategy usage to judgment rule.

METHOD

The study used a multi-method design where the quantitative methodology of IIT was combined with the qualitative methodology of a semi-structured clinical interview. Thirty six students, from the same private college, were interviewed, twelve students from each of the three grade levels, 4, 6 and 8. The students represented a range of mathematical abilities, one third each of below average, average, and above average. (A comprehensive outline of the methodology of the study, including how IIT determines area judgement rules, is provided in Kidman, 1997, and Kidman & Cooper, 1996.)

In the interview, students were provided with 16 rectangular wooden pieces painted to represent chocolate bars. Four widths (3, 6, 9 and 12 cm) factorially combined with the same heights made the 16 stimuli pieces used in the experiment. As the pieces were presented to them, the students were asked to rate the area of the wooden pieces by stating how happy or sad they would be to receive that amount of chocolate in relation to two end anchor pieces. To obtain a rating of the their judgments, the student was provided with a 19 point scale with two end points. The student responses for these 16 stimuli were plotted (using the methodology of Anderson and Cuneo (1978)) against the length of the rectangles.

If the resulting plot was parallel (a collection of parallel lines or curves), this reflected a perception of area where doubling the lengths of both the sides of the rectangle was seen as doubling the area. Therefore, students with parallel plots were considered to have additive area judgments, that is, the students perceived the area of a rectangle in terms of the *sum* of its dimensions. This perception was considered a confusion between area and perimeter for rectangles (Kidman, 1997), and the area integration rule it represented was denoted as *Area = Height + Width*.

If the plot was fan shaped (expanding lines or curves), this reflected a perception where doubling the sides more than doubles the area of the rectangle. Therefore, students with fan-shaped plots were considered to have multiplicative area judgements, that is, the students tended to see the area of a rectangle in terms of the *product* of its dimensions. The area judgement rule represented by this perception was denoted as *Area = Height x Width*.

The students' actions as they determined their rating for each wooden piece were documented, and the students were also asked to describe their thinking when they were determining their rating.

RESULTS AND DISCUSSION

The actions and statements of the students revealed that all students understood what they had to do with the ratings and saw the rating as determining area (see Kidman, 1997; Kidman & Cooper, 1996.) For all but one of the interviews, the resulting plots were obviously additive or multiplicative, with clear parallel or fan-shaped plots. The groups of students who applied additive and multiplicative area judgment rules, the seven strategies used for judging area, and the strategy usage in relation to area judgment rules, are now provided.

Judgment rules

Both the additive and multiplicative judgment rules were evident in the students' responses. 19 of the 36 students employed an additive judgment rule, while 16 employed the multiplicative judgment rule. The occurrence of the additive rule was consistent across the grades indicating a strong misconception for the area of a rectangle being related to the sum of the rectangles' dimensions.

It was not possible to determine a judgment rule for one Grade 4 student (Ben). This particular plot had four intersecting locations and no obvious parallel curves or diverging lines. Plots with lines crossing indicate a very poor conception of area. This student judged a rectangle with smaller dimensions as having a larger area than a rectangle with larger dimensions.

Strategies

There were seven strategies used by the students. Most of the students displaying an additive judgment rule used more than one strategy, whereas most of the students displaying a miltiplicative judgment rule employed only one strategy.

1. Ruler Request strategy (RR). When asked if there were any unclear aspects of the experiment that were of concern, most students indicated that they fully understood the requirements and were ready to begin the actual experiment. However, the facial expressions of 8 students, 5 from Grade 4 and 3 from Grade 6, indicated that they were not happy to proceed to begin. These students all requested the use of a ruler when asked what the problem was: "Can I use a ruler, please?" (Jemma, Tom and Sam); "I need a ruler to do this" (Martin, Ben and Jack); "This can't be done properly without a ruler" (Jodie and Angie). Jack (Grade 4) explained that he needed to "measure the chocolate pieces to see if one was bigger or not" . He wanted to measure the longest sides of each piece. Jodie (Grade 6) insisted that "we always use rulers to measure. You see, without one you can't measure something". Ben (Grade 4) needed "a small ruler to do this". Ben explained that "a big ruler is the one what the teacher used on the board, but we use small ones about this long ... [Ben indicated a length of approximately 25 cm]. They change the size [of the stick] to fit our books".

With the exception of Peter (Grade 6) and Ben (Grade 4), all students requesting the use of a ruler employed an additive judgment rule. Ben's plot was not able to be classified as either additive or multiplicative, while Peter exhibited a multiplicative judgment rule.

2. *Index Finger strategy (IF).* Eight students, four students from Grade 4 and two each from Grades 6 and 8, used their index finger to repeatedly measure one side of the rectangular test piece. Initially the student placed an index finger adjacent and parallel to one side of the rectangular piece. This is then repeated by moving the finger along to the next adjacent parallel position. The student repeated this procedure along the edge of the rectangular piece to the opposite side. A value is then assigned to the piece and a rating made. It was a strategy used only by students employing the additive judgment rule. It seems that, in the absence of a ruler, some of the children who perceive the area of a rectangle will

double if you double the length of both sides will improvise by using the index finger. The length of the finger is overlaid on the test piece and the number of times this occurs is directly related to the rating response.

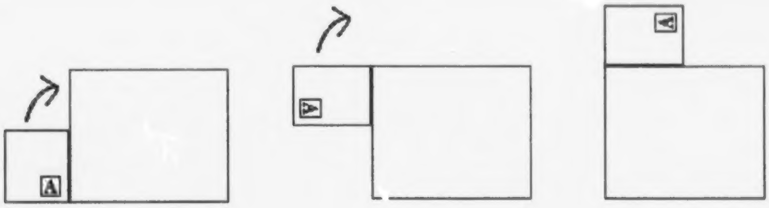
3. *Vertical Alignment strategy (VA).* Each of the wooden pieces were presented to the students in a uniform manner, but some pieces were presented in a horizontal alignment (lying flat ▭), while others had a vertical alignment (standing tall ▯). Six students from Grade 4, another six from Grade 6, and four from Grade 8, rotated the rectangular test pieces when presented with a horizontal alignment. Students using this strategy mostly employed an additive judgment rule (80% of the additive students favoured a vertical alignment of the rectangular test pieces). The strategy was particularly prevalent for the larger pieces.

Rhea (Grade 6) explained that the alignment of the test pieces was very important: "You gave me it [the test piece] like this ... [horizontal alignment] ... kind of lying down, but chocolate bars are stacked like this [she realigns the test piece to a vertical orientation] on shop shelves, so you need them put here in the same way ... like the way the wrapper would go". However, most students did not know why they repeatedly realigned the test pieces, for example, Sam (Grade 6) simply stated "It just feels better", while Steven (Grade 8) claimed "It makes things easier".

4. *Test piece Rotation strategy (TR).* One student from Grade 4 (additive judgment rule), and two students from Grade 8 (one additive and one multiplicative judgment rule) employed a strategy where they measured the outside edge of the rectangular piece (rotating the piece so each edge can be compared) against the end anchors (see Figure 1).

Sue explained: "You get the piece and match up the side with the ... um ... this one [selected the large end anchor] and turn the piece all around it[illegible]dge. That gives you the count of its size. If the count is big, you rate it small". Sue [illegible] some understanding of inverse proportion, but confused perimeter (the distance aroun[illegible]ch shape) and area (the internal space of the shape).

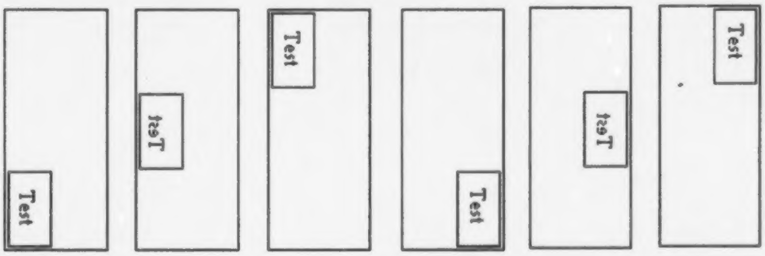
Figure 1
Test Piece Rotated Around the End Anchor



5. *Test piece overlay on end Anchor strategy (TA).* Three Grade 4, two Grade 4 and two Grade 6 students used a strategy of overlaying the test pieces onto the large end anchor (see Figure 2). The test piece was moved over the surface of the end anchor in a series of flip and slide transformations. The students counted the number of times the test pieces were moved to cover the end anchor. Only students employing a multiplicative judgment rule displayed this strategy.

While this strategy shows an understanding of area as being coverage of space, it also reveals a possible misconception for the students. It had never occurred to both Matt and Mick (Grade 8 students) that the shape of the overlay piece was important except that as Matt pointed out "circles are no good, they leave gaps", while Mick knew that "you need shapes with points that fit together". John (Grade 4) thought that rectangles could only be used to tile rectangles, squares to tile squares, and so on.

Figure 2
Test Piece Overlay Strategy



The students using the test piece overlay strategy were unable to articulate the process by which they converted an overlay quantity into a rating response. As Peter (Grade 4) stated, "... just pointed to where it seemed right".

6. *End anchor overlaid onto Test piece strategy (AT).* The small end anchor was a square. Two students from Grade 6 and one from Grade 8 (all multiplicative judgment rule users) employed a strategy where they overlaid the small end anchor onto the test pieces. Jenny (Grade 6) stated "... counted how many squares, see these little squares [indicating the small end anchor], I want to see how many of them cover this bit [holding up the test piece]". Jay (Grade 8) explained "it has to be the small square that you use because it doesn't change each time like the bigger ones do. You have to have the same measuring piece all the time, and this [indicating the small end anchor] is good because it is a square. Well I think it looks like one ... um ... the sides look the same".

The students counted how many of the small end anchors fitted onto a test piece, demonstrated this using a series of randomised flip and slide transformations and then decided on the rating. Again, the students using this strategy were unable to articulate the process they used in converting an overlay quantity into a rating response. Anne (Grade 6) said she "just remembered from the piece before, and then made it bigger or smaller".

7. *Partitioning strategy(P).* One student from Grade 4, two from Grade 6 and two from Grade 8 (all using a multiplicative judgment rule) displayed partitioning strategies. These students were familiar with the method of partitioning both the length and width of a rectangular shape, and integrating these values using multiplicative reasoning.

One of the students, Kim (Grade 4) was unable to visualise such partitions. Instead she made imaginary marks along the two salient dimensions with her finger (see Figure 3). Initially Kim partitioned the width of the test piece into what appeared to be 1 cm lengths, maintaining a mental count of the partitions. Upon completion of the width, her attention focused on the length of the test piece and she repeated the partitioning process, again with 1 cm partitions, maintaining a mental count as she progressed. Kim explained she "... got the width number and times it by the length number. That gives the number of squares fitting onto the shape to tell how big it is". Kim had helped her grandfather lay paving tiles, and he had explained this to her.

As the four students from Grades 6 and 8 all explained, they imagined the chocolate pieces already divided into squares "like proper chocolate is supposed to be" (Wade, Grade 6), counted the squares on two dimensions, and multiplied "... them together, like you do for area sums" (Phillip, Grade 8).

Figure 2
Kim's Imaginary Partitioning Marks

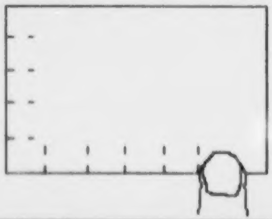


Table 1
Area Judgement Rules and Strategies

Student	Rule	RR	IF	VA	TR	TA	AT	P
Jemma 4	A	▨	▨	▨				
Cassie 4	A		▨	▨				
Stacey 4	A		▨					
Martin 4	A	▨		▨	▨			
Tom 4	A	▨		▨				
Tony 4	A		▨	▨				
Jack 4	A	▨		▨				
Dianne 6	A			▨				
Rhea 6	A		▨	▨				
Jodie 6	A	▨		▨				
Angie 6	A	▨		▨				
Sean 6	A		▨	▨				
Sam 6	A	▨		▨				
Carol 8	A		▨					
Brooke 8	A		▨					
Elle 8	A			▨				
Kelly 8	A			▨				
Sue 8	A			▨	▨			
Steven 8	A			▨				
Ben 4	?	▨						
Total	***19A 1?***	***8***	***8***	***16***	***2***	***0***	***0***	***0***
Alice 4	M					▨		
Cathy 4	M					▨		
Kim 4	M							▨
John 4	M					▨		
Jenny 6	M						▨	
Anne 6	M						▨	
Josè 6	M					▨		
Wade 6	M							▨
Peter 6	M	▨				▨		
Mark 6	M							▨
Helen 8	M				▨			
Phillip 8	M							▨
Mick 8	M					▨		
Matt 8	M					▨		
Jay 8	M						▨	
Kevin 8	M							▨
Total	***16***	***1***	***0***	***0***	***1***	***7***	***3***	***5***

KEY Rule:
A = Additive judgment rule
M = Multiplicative judgment rule
? = Unclear - intersecting lines

Strategy:
RR = Requested Ruler
IF = Index Finger
VA = Vertical Alignment
TR = Test piece Rotated
TA = Test piece overlay on end Anchor
AT = end Anchor overlay on Test
P = Partitioning

Link between strategy usage and judgment rules

Table 1 shows the seven strategies exhibited by the students in the study, for each of the two area judgment rules. It is clear that there is a strong link between judgment rule and the strategies employed. Students with a misconception of area relating to the additive properties of perimeter tended to require a ruler or use an index finger for measurement, and prefered a vertical alignment of the rectangle. Students using a multiplicative judgment rule tended to use overlaying strategies, involving both flips and slides of either the test piece or the small end anchor, and the more sophisticated partitioning strategy. The strategy of test rotation did not appear to be linked with either judgment rule.

CONCLUSION

The differences between the grades were not as apparent as expected (as can be seen in Table 1). The misconception of area of rectangles being dependent on the sum of the rectangles' dimensions is fairly constant across the grades. As a group, the Grade 8 students do not seem to have progressed beyond the Grade 6 level and not much beyond the Grade 4 level.

This experiment has, therefore, supported the claims of Anderson and Cuneo (1978) that students use both additive and multiplicative judgment rules. The findings of Hirstein (1981) and Hirstein *et al.* (1978) have also been supported in that there is confusion between area and perimeter. Around 50% of students from all grades exhibited judgments that showed they were using the perimeter rule to determine area. Those students using an additive judgment rule tend to want something with which to measure (ruler or index finger) and prefer a vertical alignment. Students using a multiplicative judgment rule tended to use overlay strategies as well as partitioning.

The findings of the study suggests that these students, particularly those in Grades 6 and 8, may not have had sufficient opportunity to explore practically the spatial foundations of area and perimeter and the relationships between them. The author suggests structured classroom activities of a practical nature (for example, the use of geoboards and dot paper) be more widely used to develop the notions of area and perimeter, and a delay in the presentation of area and perimeter formulae.

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Endnote

1. The assistance of Assoc. Prof. T.J. Cooper is appreciated in the preparation of this paper.

AGENCY/NUMERACY: A POSTSTRUCTURALIST ANALYSIS OF THE RELATIONSHIP BETWEEN CLASSROOM MATHEMATICS AND NUMERACY IN NEW TIMES

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The key issue canvassed in this paper is whether or not current teaching practices in mathematics are likely to foster or inhibit numerate behaviours in students. I understand numerate behaviours to comprise both intellectual (mathematical) and self (social) knowledges (Willis, 1998) which are interwoven and interdependent. Numerate behaviours are fostered where student initiation and construction of mathematical ideas is genuinely valued such that they are authorised or enabled to speak and write their developing constructions with respect. A poststructuralist analysis of classroom practice in three (3) primary mathematics classrooms shows that although students are actively engaged in the construction of knowledge, ***what*** *they learn of mathematics and of themselves as numerate individuals may not be conducive to the construction of a sense of agency so necessary for lifelong learners of the new millenium.*

INTRODUCTION

The field of mathematics education is currently saturated with debate and research on the possibilities and limitations of students' active involvement in learning mathematics (Boaler, 1998; Hiebert et al., 1996; Yackel & Cobb, 1996). In Australia, *A National Statement on Mathematics for Australian Schools* (1990) states "It is now widely accepted that learning is best thought of as an active and productive process on the part of the learner" (p. 16) where "teachers assume a considerable responsibility for creating the best possible conditions for learning for all students" (p. 18). However, it is my belief that much more work needs to be done to understand better the "active and productive" learning process and the teacher's role in establishing and maintaining a learning environment that fosters not only the construction of powerful mathematical knowledges but a sense of empowerment or agency in learners as well.

With recent definitions of numeracy (Willis, 1998) stressing both the intellectual and social aspects of learning and applying mathematics, it is imperative that educators and researchers establish clear and unambiguous understandings of what mathematical and social knowledges will best equip students for the new millenium. Although many educators, including the three (3) whose classrooms I videoed for this research, have willingly embraced "constructivist" and "problem solving" approaches to teaching mathematics, it is not clear that these approaches necessarily foster numeracy as "the *competence* and *disposition* to use mathematics to meet the general demands of life at home, in paid work, and for participation in community and civic life" (Willis, 1998, p. 71, my emphasis). In this paper I argue that although active and collaborative aspects of learning are encouraged, in many ways the students' experiences of mathematics are little different from those encountered in "transmission" classrooms. I use the poststructuralist concepts of power/knowledge and subjectivity to show how classroom activities and practices continue to deprive students of choice and the possibility of taking themselves up as numerate agents.

AGENCY

Poststructuralist concepts can be used to show that agency for learners is not an unproblematic issue. "Constructivist" and "problem-solving" approaches are premised on humanistic understandings of the individual agent which is contrary to poststructuralist notions of subjectivity (where the individual is subjected to, and in turn influences, power/knowledge relations in discourses). The humanist individual is seen to have a unified, rational and coherent identity; s/he interacts with an external social world and internalises norms and values. S/he takes rational control; those not making appropriate rational choices are seen to be in some way deficient. Extrapolated to the mathematics classroom, the competent and responsible humanist agent learns the mathematics that s/he is later able to apply in the external world of work and leisure.

Poststructuralist understandings of subjectivity and power/knowledge relationships in discourses would have it that agency is not so easily won. Indeed, agency can in no way be considered an attribute of an individual, a disposition, but is rather a matter of constituted subjectivity, and how one has been, and is being, positioned within discourse(s). The concept of subjectivity allows no external/internal divide between the individual and society (or the classroom context) nor the active/passive binary regarding learning; to live is to learn and power/knowledge relations abide in all interactions constituting subjectivities either to liberatory and empowering ends or to lack of engagement and indifference. Teaching mathematics can be regarded as a discourse, where classroom activities and practices (such as completing worksheets, streaming) are implicated in constituting subjectivities for agency, or not. Using a poststructuralist framing, (developing) agency in a mathematics classroom would appear to be premised on:

(a) The recognition and affirmation of the learner's prior (and present) constituted(ing) and intellectual knowledges; and

(b) The construction of the learner as one with the right to speak and be heard; and as one who has the right and ability to go beyond given meanings and forge something new.

I will elaborate on each of these in the methodology section below as these comprise the analytic framework of my research; the concepts of subjectivity, as in (a) above, and power/knowledge, as in (b).

METHODOLOGY

At the end of 1997 I videoed the mathematics lessons of several primary and early childhood teachers who were recognised as competent and innovative in this field. All teachers videoed were committed to having students actively engaged in learning, as is reflected in the video extracts. I have chosen three (3) lessons for analysis: one from a Year 1 class playing number games; a Year 3 lesson on shapes and a Year 5 cooking lesson. I intend to show that although these teaching interactions engaged all children in the active construction of knowledge, that is they were *engaging* as in (a) above, they were not necessarily *enabling*, as in (b).

It is not easy to measure engagement; indeed one can merely infer. Engagement is a qualitative entity, a process, and it has to do with constituted subjectivity, "who" a person is and how well classroom activities jell with or reproduce previously constituted knowledges and understandings. Within the discourse of mathematics, engagement is influenced by how the learner sees that the content (mathematics) that is to be learned and the processes of learning "fit" with preconstituted needs and desires. For example, it could be that many students will engage with whatever the teacher presents because they have been constituted to "know" mathematics as a powerful ticket to future employment opportunities; others will engage only minimally because they may perceive that what is to be learned is irrelevant to their lives or that the processes of learning are alienating (for example some cultures/individuals may find the competitiveness of many activities off-putting).

Enablement has to do with agency or the extent to which a particular individual is authorised to establish him/herself as a competent, "numerate" being within the power/knowledge nexus in any context. Using poststructuralist concepts, agency is not an unproblematic issue. It does not necessarily follow from merely knowing a lot of mathematics nor is it a personal attribute or disposition. Rather agency is constituted through engagement in enabling interactions. In the classroom, enabling interactions are those where the learner can speak his/her developing understandings and conjectures and be heard with respect as learner and initiate into a community of inquiry. Beyond the classroom, agency as numerate subject is again constituted (or not) according to one's ability to speak and use the mathematical knowledges considered powerful in a particular context. For example, for life and work in the twenty-first century, enablement/agency beyond the classroom would seem to be premised on the construction of, and the ability to use, higher level thinking skills such as conjecturing, patterning and making connections between mathematical ideas rather than the reproduction of facts, skills and procedures more relevant to a past era (National Council of Teachers of Mathematics, 1991).

THE DATA

In the Year 1 classroom, the children were in groups playing mathematical games. One group was throwing small bean bags at a chart headed with tens and ones, each child endeavouring to make the largest number. After each turn, the children would write the number they had made on a chart. A second group was involved in an activity throwing a die and putting beads onto a "ten" frame. When there were ten on the frame, the idea was to put these ten onto a pipe cleaner, and start again. The overall aim was to see who could get the most pipe cleaners filled with beads. Another group was playing a concentration game; turning over cards two at a time to try to make a pair. The cards had pictures on them, not mathematical signs or symbols. The last group I videoed in this room was playing an "odds and evens" game; adding the faces of the dice and, if the answer were "odd" advancing in one direction, if "even" the other. The idea was to get to the edge of the board first.

The Year 3 children were also sitting in groups when I entered the room. The teacher began the lesson by discussing the difference between 3-D and 2-D shapes, getting one child to draw a shape on the blackboard. The children all appeared to be listening intently and watching what was happening. The first group encountered was making 3-D shapes with constructo-straws; copying from pictures of shapes on a wall chart. After they were finished, they were to write down the number of faces, edges and vertices for each shape. A second group was (rather confusedly) looking for pictures of 3-D shapes in a magazine. Completing a worksheet was the task set for another small group where the children had to colour various shapes certain colours; for example, they were asked to colour the three-sided polygon blue. Some children were constructing shapes from nets using scissors to cut them out and sticky tape to stick them together.

The cooking lesson in Year 5 went on for an hour and a half. Before beginning the recipe, the teacher went through the rules for group work in detail and assigned roles for managers, collectors and recorders. The children sat in groups, and one person went to the front of the room to collect an ingredient only after the teacher had indicated that this was now an appropriate thing to do. The groups were rather large, with children standing around desks put together to form a rectangle, so vision and physical involvement were sometimes difficult. However, I noted occasions where the children were taking care to ensure that all group members were given an opportunity to measure ingredients and stir the shortbread.

ANALYSIS OF THE DATA

In each of the classrooms I visited it is clear that the students are actively involved in learning. Indeed, as previously mentioned a poststructuralist reading of practice does not allow the active/passive binary; to live is to be constituted through discourses of one sort or another, all discourses positioning learners in ways that enable and empower or limit the ability to act in powerful ways. My reading of what I observed in these classrooms is that although all the activities were *engaging* in that the students appeared to find the activities relevant and enjoyable, ultimately they may not have been *enabling* in that students were not authorised to take themselves up as agentic numerate individuals in/for new times.

The classrooms I looked at in this research were all from the one school which draws students from a relatively high socio-economic area. There were some Japanese students in the school who were learning English as a second language; the remaining students were largely Anglo-Australian. There were no Aboriginal or Islander students as would commonly be the case in most schools in this geographical area of North Queensland. Using the poststructuralist concept of subjectivity as an analytic tool, it is perhaps not surprising to find that the great majority of children in these classes were actively engaged in set activities. The classroom mores and uses of language would largely reflect those experienced at home, making the classroom community a welcoming, or at least comfortable, place for most of the students to be. Also, through involvement in home-based discourses, these students would most likely have been constituted to know mathematics as an important and powerful subject deserving their full effort and attention. However, if we take seriously new definitions of numeracy, it may be that engagement in mathematical activities as depicted in these classrooms is not sufficient to ensure either the intellectual or social competencies required for numerate participation in society and work in the future. To better substantiate this suggestion I will use the poststructuralist concept of power/knowledge to show how students continue to experience mathematics as disconnected facts and procedures and themselves as reproducers of the teacher's superior knowledge. Thus *what* they are learning, the intellectual or mathematical knowledge, and *how* they learn it, merely reproduce traditional views of the nature of mathematics and teacher/student roles.

From the data it is clear that in all the classes videoed the children are engaged in activities that might readily be recognised as "mathematical". In the Year 1 classroom the children throwing bean bags at a "tens and ones" chart are learning how to write and read numbers to 100; another group is practising making groups of ten; yet another is consolidating visualisation skills in a memory game and a final group is playing a game that relies on the children's knowledge of prime and composite numbers. The year 3 class is learning to make and name 3-D shapes and describing the various attributes. One group is completing a worksheet which, in this case, is more a consolidation or assessment, rather than a teaching/ learning, activity. The Year 5 class was learning to recognise and use metric measures in making shortbread.

While it is impossible to list accurately all the intellectual knowledge developed, reinforced or assessed in these lessons, it is possible to highlight important knowledges that are conspicuous by their absence. In the classrooms I visited there was no space nor time for exploration of patterns and relationships in mathematics and no fostering of conjecture and communication of student initiated mathematical ideas. Although the teachers believed themselves to be fostering the students' construction of mathematical knowledge, the mathematics I saw in my visits comprises what I see as "service" mathematics where children learn facts and skills: what 350g looks like, how to write numbers, names of shapes. Although important, an emphasis on such low-level mathematical knowledge, to the exclusion of the previously mentioned content, reproduces a view of mathematics as comprising disparate facts and skills and right/wrong answers. Just as importantly it excludes students from a knowledge of the richness and regularity of mathematics, of mathematics as an activity in which one can engage on one's own terms to powerful ends.

While teachers hold on tightly to a view of mathematics as isolated skills, procedures and formulae to be "given" to students, they will not be encouraged to attempt more investigatory or inquiry approaches to teaching which may be more empowering or enabling for students. In a poststructuralist sense, enablement or agency for students is constituted in discourse. Thus mathematics, as a discourse, could be taught in more investigative ways that construct the student as a valued member of a community of inquiry with the right to initiate and explore new ideas and meanings. This would require a fracturing of the power/knowledge nexus as it currently operates in classrooms to explicitly recognise both the intellectual knowledges and subjectivities that students bring with them, and which are then (re)constituted in mathematics' classrooms. In the classrooms I visited, although the children were engaged in and enjoying the mathematical activities, there was no evidence of students involving themselves in numerate behaviour in the sense of having the agency to initiate and communicate mathematical ideas. All the activities they undertook were planned by the teacher, and had clear parameters set as to "correct" performance. For example, the "games" the Year 1 students were playing were initiated and tightly controlled by the teacher, such that any variation was seen as an aberration. One student who was throwing a die and putting beads on the "ten" frame merely disposed of extra beads when there were more than ten. Group members remonstrated and showed the correct procedure for playing the "game". In the Year 3 class I did not observe (although if may have been done later) students and teacher coming together to talk about what they had learned about 3-D shapes. Although they had established some facts about these shapes, the learning environment was not one that fostered conjecture and exploration of the similarities and differences in properties of the various shapes.

CONCLUSION

Humanistic discourses stress the importance of confidence and a positive disposition if students are to be able to establish themselves as numerate individuals in the world of work and leisure. It is as if these are personal attributes that one could easily nurture if only one applies oneself assiduously enough to the mathematics. A poststructuralist understanding of the individual constituted through discourse places the onus squarely on the uses of language and practices of school mathematics to construct students who know themselves as competent and capable numerate persons.

However, there are no methods of teaching nor theories of learning that hold all the answers - in the end there are just interactions between teachers and students that are more or less abling for individual learners. Numeracy is not a gift but a social practice always in process; it is contextual and always constituted by, and constitutive of, learners. Thus as educators we can never be sure that our students who have constructed mathematical knowledge in school will consider themselves, and be considered by others in authority in a particular context, as numerate. I am thinking here of women and Aboriginal and Torres Strait Islanders, for example, who, although they do well at the mathematics, may not even consider careers requiring high levels of mathematics as they see themselves as not really belonging in this field. On the other hand, it may be that a constituted positive subjectivity or sense of agency in school mathematics could be highly correlated to future success in disparate fields even though one may not have fully grasped all the set mathematical content?

It has to be accepted, of course, that in mathematics, unlike other discourses, a curious situation exists whereby one can find oneself in a relatively powerful position with regard to employment and further education prospects simply by having amassed lots of mathematical knowledge and passed the exams. As Paechter (1998, p. 65) states:

> Mathematics provides a fantasy of power and control which, although at least on one level chimerical, is part of a discourse in which those seen as having 'mastery' of it are given a real (in the sense of exercisable) power, arising from its possession, rather than from the knowledge directly.

However, this power comes at a cost. In order to compete successfully in the discourse of school mathematics one must abandon or place on hold one's already constituted and embodied personhood which "while pleasurable because of the illusion of mastery that it brings, is also painful, a denial of the self" (Walkerdine, in Paechter, 1998, p. 66). As can be seen from this research, even so-called "constructivist" and "problem-solving" methods of teaching mathematics can rob learners of a sense of agency as they are unable to meaningfully access relationships of power/knowledge in the classroom.

In this paper I have used the poststructuralist concepts of subjectivity and power/knowledge as analytic tools to look beyond the surface of active involvement in teaching-mathematics-as-usual. I have suggested that if educators truly value numeracy as a social practice, much needs to be done to turn around the crippling effects of teacher and textual authority on student learning, especially, and perhaps most damagingly, where the rhetoric of personal sense-making and "problem solving" endures. In assisting teachers to think more carefully about what they do, and the effects of their uses of language and classroom practices on students, a poststructuralist analysis aims to cause an easing of the iron grip of tradition; it hopes to facilitate an opening up of classroom interactions and activities to student initiated voices and meanings in communal processes of inquiry and exploration.

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ASSESSMENT OF STUDENTS' UNDERSTANDING IN GEOMETRY: THE DIFFICULTIES IN WRITING GOOD QUESTIONS

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This paper focuses on the difficulties encountered in writing good questions for the assessment of students' understanding in geometry. A set of questions considered suitable for use in the classroom as a written test, were trialed on 106 Year 10 students. Following assessment of the students' responses, the questions were analysed with regard to their ability to elicit statements demonstrating students' understanding in geometry.

INTRODUCTION

There is need to be able to assess concurrently students' understanding in geometry. The availability of a reliable assessment procedure provides a teacher with information on the effectiveness of the education as well as providing a tool for teachers to provide evidence and feedback on what students know, understand and can do. Lawrie (1998, p.54) observed that clinical interviews, while giving an in-depth knowledge about students' understanding, are very time consuming, and, hence, are not particularly suitable for assessing large numbers of students. English (1998) emphasised that it is important that accuracy is not equated with understanding. Her study demonstrated the need for a range of questions designed to reveal a student's understanding. As interest in the application of mathematical knowledge, strategic knowledge, higher-order thinking, and problem solving become more prevalent, there is a concurrent need for assessment instruments that provide measures of these processes. The nature of mathematics and pedagogical approaches for teaching mathematics warrant consideration of specific assessment techniques in the area of mathematics. For example, deductive proof is prominent in mathematics in establishing truth (Webb, 1992, p.671). If Webb's (p.667) approach is adopted, i.e., that the method of assessment should depend largely on the purpose of assessment and the information that is to be gleaned from it, then there is need for a set of questions suitable for use in a classroom situation as a written test, the responses to which provide the teacher with a measurement of the degree of understanding of the topic by the students.

A set of items (known as the UGAT test (Understanding in Geometry Assessment Test)), suitable for testing for a student's van Hiele level of understanding of the square has been produced at the University of New England by Lawrie and Pegg (in press). This paper reports on the difficulties encountered in writing the UGAT items.

BACKGROUND

The Nature of Test Items

Item content and structure is not all that important asserted Biggs and Collis (1982, p.52), provided that the task set, or question asked, allows higher-level responses to be considered relevant. For example, a question that begins 'List the various factors that' is setting a van Hiele Level 2 criterion. On the other hand, 'Compare and contrast' invites a van Hiele Level 3 response.

Smith (1989) made a comparison between two geometry tests, a modified version of Usiskin's (1982) multiple-choice test and the written test developed by De Villiers and Njisane (1987), in which the nature of the questions ranged from simple yes/no items to those requiring construction of a formal proof. While both tests were found to be useful for the allocation of students' van Hiele levels, the open-ended type questions gave more consistent results than did the multiple-choice questions.

Ellerton and Clements (1997, p.157) reported that in their research, designed to test the effectiveness of multiple-choice and short-answer questions,

> over one third of responses ... were such that (a) correct answers were given by students who did not have a sound understanding of the correct mathematical knowledge, skills, concepts and relationships which the questions were intended to cover; or (b) incorrect answers were given by students who had partial or full understanding.

Hence, questions suitable for the UGAT test would be open questions which invited higher-level responses and which encouraged students to display their understanding in geometry. Responses to the questions would be assessed for the van Hiele level of understanding in geometry displayed therein.

Van Hiele Level Descriptors

A brief description of the first four van Hiele (1986) levels, the ones commonly displayed by secondary students, and, hence, the most relevant to this study, is given:

Level 1 A figure is seen as a total entity and as a specific shape. Properties play no explicit part in the recognition of the shape.

Level 2 The figure is identified by its geometric properties rather than by its overall shape. However, the properties are seen in isolation.

Level 3 The significance of the properties is seen. Properties are ordered logically and relationships between the properties are recognised.

Level 4 Logical reasoning is developed. Geometric proofs are constructed with meaning. Necessary and sufficient conditions are used with understanding.

DESIGN

The study was designed in four stages:

1. the selection of several questions, each investigating understanding of the concept square;
2. trial of the initial set of questions on Year 10 students;
3. discussion of the ability of each test item to ascertain a student's van Hiele level of understanding based on the analysis of the students' responses to that item; and
4. production of a final and smaller set of test items to be known as the UGAT test for the concept square.

In selecting appropriate questions, several factors were considered. Each question was phrased so as to probe for a statement indicating the depth of the student's reasoning in geometry. While responses at van Hiele Levels 1 and 2 were be short answers identifying figures or listing knowledge about properties, responses at Levels 3 and 4 would include a justification of the knowledge displayed. Hence, questions probing for expression of Levels 3 and 4 reasoning should include phrases such as 'How do you know?', 'What can you say about ... ?', 'Why?', 'Give your reasons.', 'Compare and contrast ...'. Such questions were considered suitable for inclusion in the first UGAT test assessing understanding of the square.

Since the focus of the NSW Years 9 and 10 syllabus is no higher than early Level 4, the questions were to test for understanding of the first four van Hiele levels. They consisted of items designed by, or derived from those used by Burger and Shaughnessy (1986) and Mayberry (1981), together with questions designed by Lawrie and Pegg. All questions except one were to have the potential to be answered at more than one van Hiele level. The exception was a question designed solely to determine whether a student could select the squares from a group of figures (Level 1). The final trial selection was a set of fifteen items able to be completed within a forty-minute lesson.

The trial UGAT test was given to 106 Year 10 students from three secondary schools in Armidale, one government school, one systemic Catholic school and one private school. In Years 9 and 10 in NSW, mathematics is taught at three levels, Advanced, Intermediate and Standard. The 106 students came from six classes which represented all levels, three Advanced classes, two Intermediate classes, and one Standard class. Each student response was assessed for the van Hiele level displayed therein.

The responses to each question were analysed to evaluate whether that question had the potential to elicit statements expressing understanding in geometry. This involved an evaluation of the ease of coding the students' responses to a question, together with the consistency of the responses in relation to the students' other answers. A final set of questions known as the UGAT test (square) was then formulated.

DISCUSSION

Although all questions selected for the trial test paper were considered to have the potential to elicit responses which could be assessed for understanding, not all questions provided the expected information. In particular, it was found that the responses to some questions did not, by themselves, give the required information. Following analysis of the students' responses, the questions were categorised into five groups. These are described in Table 1. Questions in Groups 1 and 2, and the amended version of those in Group 3 were considered suitable for the UGAT test. Questions in Groups 4 and 5 were found to be unsuitable questions.

Table 1
Classification of Question Types

	Description of questions	Questions
Group 1	questions which elicited responses exhibiting the expected van Hiele levels	2, 10 and 14
Group 2	questions, the answers to which failed to represent all the expected van Hiele levels but otherwise produced assessable responses	3, 6 and 15
Group 3	questions, the responses to which indicated need for some modification in either the phrasing or the design	1, 5, 8, 11 and 13
Group 4	questions whose responses could be categorised only with consideration of a student's responses overall	4 and 12
Group 5	questions which elicited mainly rote, recalled or prompted responses, and, hence, were considered not assessable	7 and 9

Discussion of Types of Questions

Each question type is discussed through an analysis of the responses to relevant questions.

Group 1

Q.14 was designed to test whether students have understanding of necessary and sufficient conditions in a definition (Level 4), appreciation of a definition (Level 3), or awareness of properties (Level 2). Responses to the last two parts of the question, 'Which are the least clues necessary to identify the figure?' and 'Why?' are the main indicators of the depth of a student's reasoning.

Analysis of responses to Q.14 showed it to be a question which encouraged students to demonstrate the depth of their understanding. All except five students attempted the question, and all responses were able to be coded for a van Hiele level of understanding. A weakness in the question lies in the prompting of earlier questions that the figure is a square. However, this weakness is countermanded by the following explanations. Students gave responses reflecting all three van Hiele Levels. Examples of typical responses to the final part 'Why?' are given for the three levels.

Question 14

a. I am a quadrilateral.
b. My diagonals are perpendicular bisectors of each other.
c. One angle is a right angle.
d. All angles are right angles.
e. Opposite sides are parallel.
f. Opposite sides are equal.
g. Adjacent sides are equal.

What am I?
Which are the least clues necessary to identify the figure?
Why?

N04 *To show that it has 4 sides (a), and if the diagonals are perpendicular bisectors then all 4 sides must be equal (b). (c) to show that it is not a rhombus.* (Level 4, applying necessary and sufficient conditions.)

N02 *because quadrilateral says it has four sides, all angles at right angles means it must be some type of rectangle and adjacent sides equal means that all sides must in this case be equal.* (Level 3, showing understanding of implications of properties, but including too many conditions.)

O07 *If all angles are right angles, there can only be four sides and if all sides are equal, it must be a square* (Level 2, listing of properties.)

Two interesting patterns were perceived in the answers to this question. First, in response to 'Which are the least clues necessary to identify the figure?', the low achievers tended to write in full all conditions whereas the high achievers generally listed the code letters only. Second, many of the students giving Level 2 answers left a blank for the last question part, finding the final question part as superfluous.

Group 2

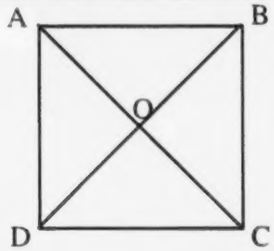
Responses to Q.3 should determine whether a student can identify a named angle and give its size (Level 2), and then explain how they know (Level 2 and (maybe) Level 3).

Question 3

ABCD is a square.

∠ CBO = ______°
How do you know?

∠ AOD = ______°
How do you know?



The question was expected to elicit responses representing van Hiele Levels 2 and 3. However, on assessing the responses, none were found to reflect Level 3 reasoning. It is considered that the structure of the question fails to encourage students to look for relationships between properties. Hence, the expectation of Level 3 responses is considered unrealistic. To assess students as not having Level 3 understanding for this question is not reasonable, consequently, this question is considered suitable for inclusion solely as an assessment of Level 2 knowledge.

All except five students gave assessable responses. Two responses were considered to demonstrate early sequencing, but not of sufficient complexity to warrant a Level 3 grading.

N06 *∠CBO = 45°. The axis-of-sym[m]etry cuts the 90° angle exactly in half. ∴ it is 45°*

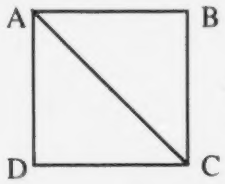
O86 *∠BOC is a right angle. 180° - 90° = 90°. The two angles are equal so ∠CBO = 45°*

Many low achievers interpreted ∠CBO as meaning 'the angle sum of ΔCBO'. This elicited many responses of "180°, because the angles of a triangle add to 180°."

Group 3

Q.13 contains a proof using congruency, in which the reasons for each step of the proof were to be completed together with a final statement of what had been proved. The responses to the question were expected to identify to what degree students could sequence their reasoning (Level 3). Level 2 responses were not expected to include acceptable statements for the last two parts.

Question 13
Complete the missing steps for this proof.



ABCD is a square

In triangles ADC, ABC,
AD = AB (equal sides of a square)
CD = CB (____________)
∠ADC = ∠ABC (____________)
Therefore the triangles are congruent (____________)
Therefore ∠DAC = ∠BAC,
and ∠DCA = ∠BCA (____________)

What has been proved?

Results were encouraging, with ninety-three students giving assessable responses. It was found to be easy to distinguish between Level 2 and Level 3 responses, and to grade the depth of understanding displayed in each answer within the level. However, several students giving Level 2 responses found the given reason in the first line of the proof '(equal sides of a square)' a useful prompt, and so completed the next two lines of proof with "(equal sides of a square)" and "(equal angles of a square)". They seldom made any further statements. Hence, it appears that such a prompt may, in fact, inhibit the thinking of some students. It was decided that if the prompt in the first line of the proof were omitted, Q.13 would be a better question.

This question has now been amended twice. Originally, the question was designed to end with a concluding statement 'Therefore the diagonal bisects the angles of the square.' instead of the question, 'What has been proved?' However, it was recognised that in such a form, most of the question could be completed with recalled facts, and, therefore, is of little value.

More responses to this question (those of seventeen students), were considered to reflect Level 3 understanding than for any other of the trial questions. This is considered to reflect the amount of attention given in the classroom to congruency proofs.

Group 4

The intention of Q.4 is to differentiate between students reasoning at van Hiele Levels 4, 3 and 2. A Level 4 response is (b) and (e), necessary and sufficient conditions for a square, while a Level 3 response would be likely to include too many details, showing necessary but not sufficient conditions. A student responding at Level 2 circles as many properties as is known, often excluding both (a) and (b).

Question 4
Circle the smallest combination of the following which guarantees a figure to be a square.
a. It is a parallelogram.
b. It is a rectangle.
c. It has right angles.
d. Opposite sides are parallel.
e. Adjacent sides are equal in length.
f. Opposite sides are equal in length.

It was found that very few responses to the question could be categorised without examining the student's responses to all other questions. For example, a response of (b) and (e), as well as being a Level 4 response, was found to come from a Level 3 student attempting to minimise conditions without understanding the implications. The same response was given also by a student whose responses to all other questions reflected van Hiele Level 2 understanding. Another response, selecting (b), (d), (e) and (f), was given by a large number of students. Some of these students had demonstrated Level 3 reasoning in several questions, while others had responded to all questions with Level 2 answers. It was not possible to code this answer without recourse to a student's overall performance. Compounding this difficulty in assessing the level displayed in responses were two extra factors, namely, the attempts by some students to minimise their response, and the prompting factor resulting from the listing of many properties.

As a consequence of the above analysis, it is considered that Q.4 is not a good question for ascertaining a student's understanding in geometry.

Group 5

Two questions, Q.7 and Q.9, which were expected to give information about students' understanding in geometry, were found to elicit responses that appeared to be mainly recalled or prompted statements. Hence, the questions were considered not to generate sufficient assessable information.

Question 7
Will two squares always be similar? YES NO
Give reasons for your answers.

In Q.7, the reasons for a yes/no answer were expected to reveal whether a student understood the implications of the angle and/or side properties of a square (Level 3). Responses were of the kind "all angles are equal", "they will always be the same shape (different size)" and "the sides are in proportion". While only five students failed to give a reason and nine students commented on the proportionality of the sides, the remainder of the students (ninety-two) focused almost equally either on the equality of the angles or on the shape being the same but the sizes different. It was considered the responses did not indicate sufficiently the amount of understanding of the students.

Question 9
ABCD is a parallelogram.
AB and BC are equal.
$\angle$ ABC is a right angle.

What have we proved?

Responses to Q.9 were expected to show whether a student understood the essence of a proof. It was considered that students reasoning at van Hiele Level 3 would show appreciation of sequencing but not an overview of the proof, while students reasoning at Level 2 would select a single statement from the proof.

While no responses were considered to indicate Level 4 understanding, a few students demonstrated Level 3 reasoning in their responses. Many students attempted to draw a diagram of the figure, and very many stated that the figure was a square, probably prompted by the other questions in the paper. The responses were grouped into the following types of answers.

Responses considered assessable :

a) a diagram together with some (incomplete) explanation of reasoning

b) a statement showing some reasoning; this was often a one-step development, combining a property of a parallelogram together with a piece of the given information, e.g."opposite angles are equal, therefore ∠ADC is also a right angle"

c) a diagram correctly showing given information, with or without a bare statement "This is a square, rectangle or rhombus"

Responses that were considered to be non-assessable:

a) no answer and nonsense statements

b) incorrect diagrams (such as a triangle) and diagrams of a standard parallelogram with no markings and no statement

c) statements solely repeating information given in the question

d) responses: "ABCD is a square"

The majority of responses (two-thirds) were found to be not assessable.

The weakness in this question lies in its design. There were very few steps (three) in the proof, and there is no request for justification of a student's statement. The most interesting responses were those containing students' attempts to draw the figure. Many of these students started with a standard parallelogram, then modified or drew over it to arrive at the figure of a square. Often, they appeared not to know how to interpret what they had drawn. Most of these students would be reasoning at van Hiele Level 2. While Q.9 could be amended so as to elicit a greater number of assessable responses, it is considered that Q.13 discussed above provided the same information in a more suitable format.

CONCLUSION

Whereas fifteen questions were designed to specifications described in this paper, analysis of the students' responses indicated that not all the questions were able to elicit the expected information about students' understanding in geometry. Six questions were found to provide responses which indicated a student's van Hiele level of understanding, although responses to three of these were not across as many levels as expected. Responses to a further five of the questions indicated that, with amendment, these questions should elicit responses giving the desired insight into student understanding. The remaining four questions were considered not suitable in that responses to them did not provide the researcher with sufficient information with which to assess students' van Hiele levels of understanding satisfactorily. In summary, good questions for the purpose of ascertaining students' van Hiele levels of understanding in geometry by means of a written paper do not result automatically from the application of a design formula.

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EXTENDED COMMON ASSESSMENT TASKS IN VCE MATHEMATICS:VALIDITY, RELIABILITY AND OTHER ISSUES

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Since its inception in 1990, the Victorian Certificate of Education (VCE) Mathematics study has incorporated extended school based and assessed common assessment tasks (CATs) in its assessment regime. These tasks have been externally set by panel, graded by teachers and reviewed by various formal mechanisms. Key issues associated with the use of such tasks include validity, reliability, authentication and equity. We discuss several of these issues in this paper.

INTRODUCTION

The assessment regime constructed for the Victorian Certificate of Education (VCE) Mathematics study responded to the body of research and strong direction in mathematics pedagogy that supported the development of a broader assessment repertoire than one based solely or substantially on examinations. This regime was implemented to provide a robust picture of student learning and achievement. A unique feature of the approach taken in the VCE Mathematics study was to construct extended Common Assessment Tasks (CATs) of 2-4 weeks duration over a prescribed period, which would be set externally by panel (consisting of two or three distinct starting points based on a broad theme or context) and required the use of problem solving, modeling, or investigative approaches (see Leigh-Lancaster, 1995). These tasks would then be graded by teachers using set criteria and task-related advice provided on an annual basis. Teachers and students would be required to attest to the authenticity of student work (student ownership of unacknowledged work) and work from each school is externally verified. The first VCE Mathematics study allocated an equal weighting to each of the two school assessed extended CATs (CAT 1 - Investigative Project, CAT 2 - Challenging Problem) and the two examination CATs. This assessment structure had a significant effect on approaches taken to teaching and learning mathematics.

REVIEW

The VCE has 'lived in interesting times', a scenario which in various forms has been played out in the review of senior secondary curricula throughout Australia. The 1990's have been a decade of intense competition for access to positions in post secondary education or employment opportunities, with mathematics occupying a particularly significant area of the curriculum in these 'high stakes' contexts.

In the early years the VCE was reviewed on an almost annual basis (see Eyers, 1990; Northfield, 1991; Brown, Hill & Masters, 1993). Whereas the work of Brown and others found a high level of reliability for school assessed mathematics CATs, several issues were identified in association with the use of this type of school assessed extended CAT, namely, validity, reliability, authenticity, workload and equity (for the latter, see Leder et al., 1995). From 1994 VCE Mathematics assessment consisted of one school assessed CAT and two examination CATs, with each CAT weighted equally. Student (and teacher) workload and authentication had emerged as major issues for the school-based task, and this re-configuration addressed both of these concerns. For the remaining Mathematical Methods and Specialist Mathematics school assessed tasks, both now conducted over a prescribed period of 2 weeks, there was also an associated test, externally set, but teacher-marked according to a marking scheme provided by the Board of Studies.

The VCE was again reviewed in 1997, with a recommendation for assessment to move to a school assessed coursework and examination regime with essentially the same weighting of components. School assessed coursework will not include CATs or other tasks involving significant amounts of work over an extended period or work which is not directly supervised. Rather, it will consist of a collection of smaller, locally devised tasks from a specified selection of task types. The detail of school assessed coursework for the re-accredited VCE Mathematics study 2000 - 3 is currently being elaborated for a statewide information program in the second half of 1999. It is envisaged that this approach will provide for greater local control of teacher and student workload in this component of the assessment regime, and ease the systemic load of authentication requirements, while retaining significant aspects of the earlier CAT based approach within a more flexible framework. The Board is currently working with Cambridge University Press to produce a CD ROM resource, based on these CATs, to support teachers in devising suitable tasks for school assessed coursework.

VALIDITY, RELIABILITY AND OTHER ISSUES

Validity

The validity of extended mathematics CATs has, in general, been strongly endorsed by both mathematics educators and teachers. The CATs are widely regarded as valuable learning activities for students, with an acknowledged close relation between the relevant mathematics study design content, task design and context specification. This is due to several factors including the design brief for the task, the expertise of the setting panels and vetters, extensive vetting process key learning area manager, sitter vetters, study specialist vetters and NESB vetters. Moreover, there is a robust annual discourse on the extended CATs within the mathematics education and mathematics teaching community.

Authentication

Teachers and the Board have been keenly aware of the issue of authentication since the inception of the VCE. This has been dealt with by strict progress consultations, draft, logbook and journal requirements. Both student and teacher are required to attest to the authenticity of any unacknowledged work, with the teachers being required to not consider unauthenticated parts of the CAT for assessment purposes. Schools can require students to attend interviews to establish authenticity of sections of work as appropriate. The requirement for an interview occurs automatically if the student result on the test component compared to the project component is outside a specified discrepancy range. Student results for the CAT may be adjusted as an outcome of these processes. The general feeling amongst teachers, however, is that while these processes work effectively, they require a lot of time and effort, and can produce tension in the student teacher relationship.

Equity

Another important issue, which has received increased attention, is that of equity. Whereas the extended mathematics CATs appear to have provided female students with increased opportunity to demonstrate their mathematical capabilities (see Cox, 1996), there has been a critique of extended tasks on equity grounds based on likely access, or non-access, to resources across socio-economic groups. A notable aspect of this critique is access to supporting technology such as graphics calculators or other 'mathematically able software' such as computer algebra systems. Although the extent to which access to particular technologies may *enhance* the quality of student response within the context of this type of task is not clear (since this would depend on, amongst other things, the effectiveness of such use), differences in access to available technologies are likely to vary in schools both within and across sectors and also with demographic factors. The use of different technologies within all types of mathematics assessment is a pertinent and current issue, with a keen awareness of equity considerations in Board directions on this issue.

Reliability

Reliability analysis is essentially concerned with estimating how well an entity and its constituents have been measured. The assessment tasks of VCE studies provide a rich source of data for this type of statistical analysis. The following material is part of a comprehensive study (Rowe, 1998, 1999c) which places VCE Mathematics assessment within the context of reliability analysis for all VCE studies. Typically, an index of reliability (ρ_{rel}) is expressed in the form of a coefficient that is defined as the ratio of observed variance, σ_o^2 (less the measurement error variance, σ_e^2) to observed score variance, σ_o^2; or more simply, the proportion of observed variance that is true score variance. This coefficient is expressed as:

$$\rho_{rel} = \frac{\sigma_o^2 - \sigma_e^2}{\sigma_o^2}$$

where values of ρ_{rel} range from zero, indicating total unreliability, to unity, indicating perfect reliability, or zero measurement error variance.

In the case of assessment measures related to the VCE since 1995 student achievement scores for each study have consisted of three measures that have been either two examination scores and a score on one school-based common assessment task, or scores on two school-based assessed CATs and one examination. As part of the VCE assessment procedures, a fourth measure of students' 'ability' has been obtained from their scores on the General Achievement Test (GAT). While student GAT scores are not included in the final achieved study scores, they are used as an administrative, quality control device to identify those school-based assessed CAT scores that appear to be either over-scored or under-scored, in relation to what could reasonably be predicted from GAT scores (see Hill, Brown, Rowe & Turner, 1997).

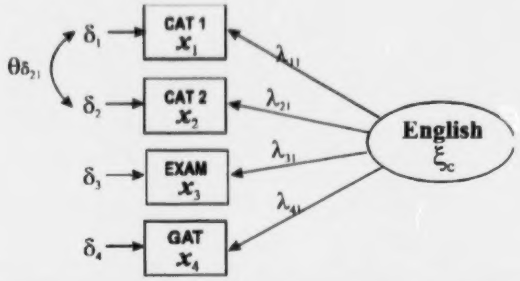
Estimating the Reliability of Assessments for VCE Studies

To estimate a reliability coefficient for each VCE study, confirmatory factor analysis was employed using LISREL (Jöreskog & Sörbom, 1998a) under a weighted least squares method of estimation and a listwise method for deleting missing data. In particular, the reliability coefficient was obtained from fitting a one-factor, congeneric measurement model to the standardised data for the four constituent measures (indicators) relevant to each study, based on a scaled variance-covariance matrix (and its asymptotic estimates) using PRELIS (Jöreskog & Sörbom, 1998b). Unlike traditional methods for computing reliability coefficients, this method makes use of factor score regression weights that minimise measurement error in the indicators contributing to each study and hence, maximise the accuracy of the reliability estimates. For explanatory research applications, the use of maximally reliable composite scores is crucial in fitting both single-level and multi-level regression models (Bryk & Raudenbush, 1992; Goldstein, 1995; Rowe, 1999a), as well as in fitting structural equation models (Bollen & Scott Long, 1993; Holmes-Smith & Rowe, 1994; McDonald, 1996; Rowe, 1999b).

This sort of model is illustrated diagrammatically for the VCE English composite (x_c), as shown in Figure 1. In this case, the assessment of VCE English since 1995 has entailed scores on two school-based assessed CATs (CAT 1 and CAT 2), one examination CAT, and a score on the GAT – each measured with error (δ_1, δ_2, δ_3, δ_4).

Figure 1

One Factor, Congeneric Measurement Model for VCE English



In matrix format, Figure 1 shows the regression of x_i (i = 1, …, 4) on ξ_c where the elements λ_{xi} are the partial regression coefficients of ξ_c in the regression of x_i on ξ_c, namely:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \lambda_{11} \\ \lambda_{21} \\ \lambda_{31} \\ \lambda_{41} \end{bmatrix} \begin{bmatrix} \xi_c \end{bmatrix} + \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix}$$

$$\text{or} \quad x_i = \lambda_{xi}\xi_c + \delta_i \tag{1.1}$$

The model assumes that the vector of observed indicators (ξ_i) are measured independently, and implies that the covariance matrix of the ξ_i's is of the form:

$$\Sigma = \lambda_{xi}\lambda_{xi}' + \Theta_\delta \tag{1.2}$$

where Θ_δ is a diagonal matrix with elements $\theta_{\delta i}$ indicating the variances of the measurement errors δ_i (i = 1, 2, 3, 4). It should be noted from Figure 1 that a covariance has been specified between the unique measurement errors for CAT 1 (δ_1) and CAT 2 (δ_2), namely $\theta_{\delta 21}$. In such instances where the assessment procedures for VCE studies entail two school-based assessed CATs, the assumption of independence of the measures is not tenable. Hence, to adjust for what is commonly known as *method effect* (i.e, non-independent measures from the same source, or repeated measures from the same source), it is essential that the errors of measurement among such indicators are specified to co-vary. For those VCE studies assessed by two examination CATs and one school-based CAT, as is the case with VCE Mathematics studies, such covariances among the measurement errors are not warranted since all constituent indicators are measured independently. From the parameters of equation [1.2] the reliability (ρ_{rel}) of a composite (ξ_c) is given as

$$\rho_{rel} = \frac{w_c'\left(\hat{\Sigma} - \hat{\Theta}_\delta\right)w_c}{w_c'\hat{\Sigma}w_c} \tag{1.3}$$

where $\hat{\Sigma}$ is the estimated variance-covariance among the factor 'loadings' for the vector of congeneric indicators (ξ's), Θ_δ is a vector of unique measurement error variances among those indicators, and w_c is a vector of factor score regression weights that maximise the reliability of the composite. For specific details of these well-established procedures, see Brown (1989), Fleishman and Benson (1987). The rationale for this approach to computing composite variables and their reliabilities has more recently been detailed in Hill et al. (1993, 1996a) and Rowe and Hill (1997, 1998).

Results of Reliability Analysis

An excerpt from the results of the reliability analysis for 53 VCE studies undertaken between 1995 and 1998 are given in Table 1, which includes the assessment type in order of occurrence during the year (E for examination CAT and S for school-assessed CAT) as well as a mean reliability estimate over four years. The data has been sorted on this last category with those studies with higher reliability (and smallest measurement error variances) placed towards the top of the table.

Table 1
Excerpts of Reliability Coefficients from 53 VCE Studies: 1995-1998

Study	Assess. Type	1995	1996	1997	1998	Mean
1. Mathematical Methods	SEE	0.945	0.948	0.950	0.953	**0.949**
2. Accounting	ESE	0.947	0.950	0.947	0.949	**0.948**
3. Specialist Mathematics	SEE	0.945	0.949	0.940	0.952	**0.947**
4. Psychology	SEE	0.936	0.937	0.941	0.941	**0.939**
5. Biology	ESE	0.934	0.938	0.942	0.936	**0.938**
6. Chemistry	ESE	0.937	0.939	0.933	0.940	**0.937**
7. Physics	ESE	0.928	0.939	0.940	0.942	**0.937**
8. Physical Education	SSE	0.941	0.936	0.901	0.912	**0.923**
9. Economics	SSE	0.920	0.922	0.908	0.907	**0.914**
10.Further Mathematics	SEE	0.907	0.920	0.896	0.917	**0.910**
13.English	SSE	0.877	0.890	0.890	0.893	**0.888**
16.Info-Tech-Info-Systems	SSE	0.858	0.837	0.876	0.898	**0.867**
43.Art	SSE	0.788	0.807	0.834	0.836	**0.816**
Means (all 53 studies)		**0.843**	**0.850**	**0.849**	**0.855**	**0.849**

Comments on the Reliability Analysis

Of the 53 VCE studies examined, 20 were assessed on the basis of two examinations and one school-based assessed CAT, while 33 studies were assessed via one examination and two school-based assessed CATs. Twenty six studies (49%) had mean reliability coefficients in excess of 0.85, ranging from a high $\rho_{rel} = 0.949$ for Mathematical Methods, to $\rho_{rel} = 0.852$ for Materials and Technology, of which 11 studies were assessed via two examinations and one school-based CAT, and 15 studies were assessed via one examination and two school-based CATs.

In general, reliability coefficients $\rho_{rel} > 0.85$ are indicative of those VCE studies in which the assessment criteria for each of the constituent elements are both clearly specified and uniformly applied by assessors and examiners. Such studies are characterised by assessment components that are more likely to be similar in terms the nature of the assessed tasks, yielding high inter-correlations among their component scores. To illustrate this phenomenon, Table 2 gives an excerpt of corresponding data from the 53 studies in 1998. For example, the squared multiple correlation values, R^2, for the two examination components (CAT 2 Facts, skills and applications task - multiple choice and short answer format, and CAT 3 Analysis task - extended response format) of Mathematical Methods in 1998 were 0.864 and 0.922, respectively, and for the school-based assessed CAT 1 Investigative project, $R^2 = 0.626$. While this CAT required students to demonstrate knowledge of core content related to the Mathematical Methods course, there is greater assessment emphasis on student written communication and presentation skills of related mathematical working compared to the examination CATs.

VCE studies with reliability coefficients $\rho_{rel} < 0.85$ are more likely to have assessment criteria that are less clearly specified and not as consistently applied by assessors and examiners. Such studies are often characterised by assessment components that are more likely to be less similar in terms the nature of the assessed tasks, often yielding low inter-correlations among their scores, such as those found in the latter part of Table 2.

Table 2
Excerpts of Reliability Data from 53 VCE Studies: 1998

Study	Assess. Type	CAT 1 R^2	CAT 2 R^2	CAT 3 R^2	GAT R^2	ρ_{rel}
1. Mathematical Methods	SEE	0.626	0.864	0.922	0.338	**0.953**
2. Specialist Mathematics	SEE	0.575	0.882	0.912	0.288	**0.952**
3. Accounting	ESE	0.883	0.560	0.897	0.492	**0.949**
4. Physics	ESE	0.891	0.486	0.858	0.505	**0.942**
5. Psychology	SEE	0.568	0.864	0.868	0.627	**0.941**
6. Chemistry	ESE	0.888	0.484	0.856	0.457	**0.940**
7. Biology	ESE	0.859	0.531	0.852	0.635	**0.936**
8. Further Mathematics	SEE	0.401	0.827	0.824	0.473	**0.917**
9. Physical Education	SSE	0.444	0.446	0.890	0.559	**0.912**
10. Economics	SSE	0.485	0.508	0.881	0.524	**0.907**
13. English	SSE	0.640	0.660	0.786	0.683	**0.893**
16. Legal studies	SSE	0.584	0.577	0.814	0.558	**0.882**
37. Art	SSE	0.609	0.437	0.697	0.492	**0.836**
43. Music-History & Styles	SSE	0.624	0.662	0.490	0.356	**0.823**
Mean (all 53 studies)						**0.855**

The results which have in part been presented in this paper, indicate that the VCE assessment procedures have performed effectively in terms of reliability. For the 53 studies with the largest student enrolments between 1995 and 1998, the mean reliability coefficient was 0.849, indicating an improvement from 0.843 in 1995 to 0.855 in 1998. This is a strong result given the diversity across study types and content, in particular in the case of the VCE Mathematics studies. However, the magnitude of a reliability coefficient is not necessarily commensurate with validity – both content validity and criterion-related validity (see Allen & Yen, 1979). While it is possible to have a highly reliable assessment that lacks validity, a valid assessment that has low reliability is of limited value. The content validity of assessment tasks – including face validity and logical validity – may only be established via a rational analysis of their content based on professional judgment, albeit by consensus, which, in the case of VCE studies, is more properly determined by the respective accreditation and assessment panels.

SUMMARY

The preceding presentation indicates that extended common assessment tasks can be incorporated within a study to provide valid assessment as part of a highly reliable assessment regime. Quality tasks which are relevant, challenging and stimulating for both students and teachers alike can be devised by strongly constituted setting panels on a sustainable basis, with supporting appropriate and effective authentication and verification procedures in place. However there is a substantial commitment of resources and effort required by all concerned - the system, teachers and students. When this type of assessment takes place in a 'high stakes' environment, issues of workload, authentication and equity become highly significant, both in terms of perception and practice, especially when the same type of assessment is an integral part of five or six subjects studied in the final year of secondary education. It is timely for the Board of Studies to move on in terms of the school assessed component of its VCE studies. For Mathematics, the experience with extended CATs has provided a strong resource base for school assessed coursework in the re-accredited Mathematics study 2000- 3.

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FREE PROBLEM-POSING: YEAR 3/4 STUDENTS CONSTRUCTING PROBLEMS FOR FRIENDS TO SOLVE

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This paper investigates the way in which two children constructed and designed mathematics problems for friends to solve. The two children, of different ages, designed problems for one another and for other friends over a ten week period. The way in which the children engaged in problem solving prior to, and after, formulating or posing a problem was explored. Insights into the children's mathematical abilities were identified through the problem-posing activities.

LITERATURE REVIEW

Problem posing is an important companion to problem solving and lies at the heart of mathematical activity (Kilpatrick, 1987). Stoyanova (1998) defined problem posing as "the process by which, on the basis of their mathematical experience, students construct personal interpretations of specific situations and formulate them as meaningful well-structured mathematical problems" (p. 165). Until recently, few studies have examined the mathematical processes employed by the problem poser when constructing a problem. As Silver (1993),commented:

> despite the interest, however, there is no coherent, comprehensive account of problem posing as part of the mathematics curriculum and instruction, nor has there been systematic research of mathematical problem posing. (p. 66)

More recently, some educators have recognised the importance of promoting problem-posing opportunities in the curriculum (English & Halford, 1995; Silver, 1995). These studies suggested that when proposing a problem to solve, the originator of the problem will usually investigate the type of processes required to solve the problem. Importantly, the solution to the problem is likely to also be considered.

Silver (1995) identified four types of problem-posing experiences that provide opportunities for children to engage in mathematical activity. He argued that problem posing could occur *prior* to problem solving when problems were being generated from a particular situation, *during* problem solving when the individual intentionally changes the problem's goals or conditions, or *after* solving a problem when experiences from the problem-solving context are modified or applied to new situations. The way in which children engage in problem solving prior to, and after, constructing or posing a problem will be investigated in the present study.

Stoyanova (1998) identified a number of categories that could be used by teachers and researchers to identify different problem-posing situations. These categories included: a) free; b) semi-structured; and c) structured problem-posing situations. One of the situations described in the free category included *problems written for a friend*. In such cases, a student creates a problem for a friend to solve. Some researchers (Ellerton, 1986; Mamona-Downs, 1993) have found that, for motivational purposes, it is helpful to have someone in mind when designing problems. In the present study, the friend would be someone in the problem poser's class. Moreover, the friend would attempt to generate a solution to the problem designed for them. Thus, the problem poser would receive feedback on the solution to the problem. The feedback obtained from this stage of the process fosters a reflective component of the problem-solving process.

Ellerton (1986) found that encouraging students to write problems for a friend was a useful way of understanding that person's mathematical ability. In such problem-solving situations the problem poser is forced to consider the individual for whom they are designing the problem. As Stoyanova (1998) commented:

> there is a strong acceptance among researchers and educators of the notion that students' ability in posing quality problems provides a useful indication of potential mathematical talent. (p. 172)

The very fact that a student must consider the mathematical ability of another person when engaged in free problem-solving situations requires reflection and careful planning. In order to complete the task successfully, the problem poser might not only focus on the underlying structures of the problem but also the extent to which the problem solving will be able to interpret the components of the problem. Such metacognitive thinking processes encourage mathematical power (Lowrie, in press).

Problem-posing situation allow children to have some control over the curriculum content and the type of learning activities presented in the classroom. Furthermore, the tasks or the activities children construct may provide insights into their beliefs or attitudes they have toward mathematics. The way in which the problem poser represent problems, for example, may reflect the type of problem-solving experiences they have been accustomed to solving in the classroom. The present study will investigate whether feedback obtained by the problem poser—through interactions with the classroom teacher, the researcher, and other members of the class—influences the type of problems they pose over time.

METHOD

This investigation was conducted over a ten week period (one term of the school year) in a composite Year 3/4 class. Throughout the term the children were provided with opportunities to construct (pose) a range of different problems in free, semi-structured and structured situations. The present study will report on the findings associated with children engaged in free problem-solving situations.

When developing a problem, the students were asked to *write a problem for a friend* or a *particular student in the class*. It was envisaged that these two scenarios would create different types of problem-solving situations for the children. In the first instance, students would choose the person they would write the problem for whereas in the second case the teacher identified the student for whom the problem poser would construct the problem for.

The classroom teacher provided opportunities for the children to engage in problem-posing situations twice a week. Children were encouraged to discuss solutions with each other after they had exchanged and solved problems. Moreover, the children were able to modify problems posed if the task was either too easy or too difficult for the intended problem solver. This reflective stage of the process was seen as an important component of the problem-posing process. The children had not been accustomed to posing problems and as a result cooperative learning experiences and discussion sessions influenced the way in which the children constructed problems as the study progressed. The researcher visited the class on five occasions throughout the term. These visits were at regular intervals throughout the duration of the study and provided opportunities for the researcher to talk to the children and the classroom teacher. Several of the discussions with the two case-study participants were audio taped.

The Students

The two students investigated in the study were from a composite Year 3/4 class from a small school in a large rural city. John was an eight year old boy who was considered by his classroom teacher to be a capable student who was particularly good at problem solving. Bettina was a nine year old girl who was considered to be an outstanding student. The two children appeared to be "at ease" when being interviewed by the researcher and were able to express and discuss the way in which they posed the problems in great detail.

When asked to *write a problem for a friend* John enjoyed designing problems for his best friend Tom. It appeared that Tom's mathematical ability was not quite as advanced as his own. Nevertheless, they enjoyed writing problems for each other and worked well in cooperative problem-solving situations. Bettina did not choose a close friend to construct problems for. She enjoyed writing problems for Joe because "he really liked solving hard problems because it was challenging." Like Bettina, he was a very capable student. They also worked well cooperatively and seemed to thrive on each other's desire to learn.

When required to design problems for one another the two children had a reasonably good understanding of the other's mathematical ability. John commented that Bettina was "the best at maths in the whole class" and that "she could get all the hard problems right." Bettina realised that John was "good at mathematics but was only in Year 3 so he couldn't do what some of the Year 4's could do." Throughout the study the two children developed a strong rapport with one another.

The Procedure

The investigation traced the way two of the children in the class engaged in problem-posing activities throughout the term. These two children were selected for investigation for a number of reasons.

- The children were of different ages allowing comparisons across grades and different levels of mathematical ability.
- Both children designed problems for friends who were at a stage of mathematical development similar to their own.
- The children were asked to solve problems for one another.
- A detailed case study analysis (Yin, 1994) of the two children provided the researcher with an opportunity to trace the way in which the children constructed and developed problem-posing tasks over an extended period of time.

The case study format of the study ensured that a rich, detailed, analysis of the students' worksamples and reflections could be gathered over an extended period of time (a ten week period). Interviews with the children were audio taped with information collected about the way in which the students posed problems for their friends and for each other. When asked to *pose a problem for a friend* the two children designed problems for children in their grade (Year 3 or 4). On other occasions they were required to pose problems for each other (thus, one student was required to solve problems for a person younger than themself, the other problems for someone older than themself). Direct questioning (see Lowrie, 1998a) was used to stimulate the children to think metacognitively. Lowrie (1998b) found that when young children were able to interact in discussion sessions that stimulated them to consider "what-if" situations they were able to engage in increasingly sophisticated problem-solving activities. These questioning techniques were used to encourage the children to discuss their thinking processes prior, and after, constructing problems (see Silver, 1995).

Another focus of the present study was to assess whether the types of problems posed by the children changed over the ten week period. It was anticipated that interactions with classroom teacher, the researcher, and other students would influence the way they, and other members of the class, constructed and designed problems over the term.

THE RESEARCH QUESTIONS

Three research questions were formulated for the study.

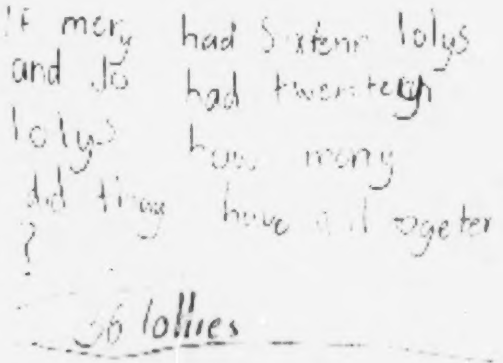
1. Were the children able to consider the needs and interests of a peer when posing a mathematical problem for them to solve?
2. Did a student's mathematical understandings impact on their ability to construct "well designed" problems?
3. To what extent did an individual's personal beliefs about mathematics influence the types of problems they formulated and did the type of problems they pose change over time?

RESULTS

Initial problem-posing situations

From the outset of the investigation it was evident that John and Bettina were attempting to consider the needs of the problem solver when designing problems for friends in their class to solve. This is in line with Mamona-Downs (1993) suggestion that the type of question asked will be influenced by the person for whom the problem is being designed. When posing a problem for Tom (someone at or near his ability level) to solve John constructed a problem that required the addition of two 2-digit numbers (see Figure 1). In contrast, the problem intended for Bettina was a two-step problem requiring addition and subtraction operations (see Figure 2). The two problems were solved successfully by the respective students.

Figure 1
John's Problem for a Friend



If mery had Sixtenn lolys
and Jo had twenteegh
lolys how many
did they have all togeter
?

36 lollies

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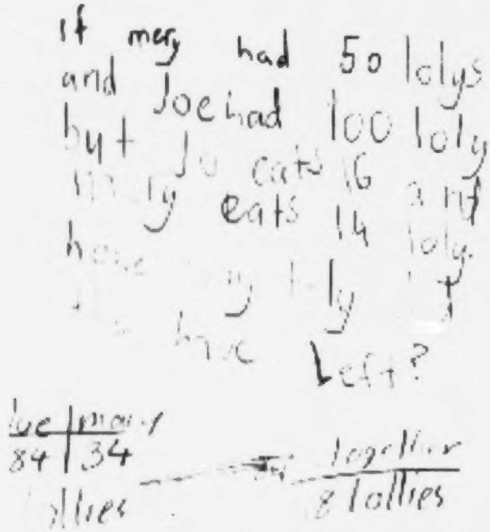
Figure 2
John's Problem for Bettina



if mery had 50 lolys
and Joe had 100 loly
but Jo eats 16 and
mary eats 14 loly.
how many lolys do
they have left?

Joe | mary
84 | 34
lollies

together
8 lollies

John showed that he had considered the solutions to the particular problems, or at least solution paths, before asking others to solve the tasks. The following section details other problems John had posed for Tom and Bettina respectively.

> Jason saw 17 birds the first day. The next day he saw 14 birds. How many did he see altogether? (problem for Tom).
>
> If Joe had 110 102 lollies and ate 467 and bought 764 144 and ate 14 and then bought 224 213 how many did he have? (problem for Bettina)

The problem designed for Tom was similar to the one proposed at an earlier time (see Figure 1). Similarly, the problem prepared for Bettina required a two-step process but in this instance the numerals were considerably larger than in the first problem (see Figure 2). An audio-taped interview conducted with John on the second visit to the site indicated the extent to which he attempted to adapt the problems to the students needs.

> Tom will have to trade to get the proper answer...Bettina said my last problem was too easy so I gave her big numbers to work out this time. She will have to do lots of trading and think about when to add or take away...I couldn't get this one right but I think she can.

Although he commented that he did not think that he could obtain a correct solution to this problem, John's comments indicate that he had thought about the processes Bettina would need to use in order to obtain a correct answer. Interestingly, he felt that 5-digit numbers substantially increased the difficulty of the problem.

John found the first problem Bettina designed for him to be quite difficult.

> You have twenty dollars ($20). You go to the newsagency and buy 2 pencils (40c), 1 notebook ($1), stationary pack ($10.50), stickers (10c), and a rubber and sharpener ($1). How many dollars did I spend? How much change did I get? (Bettina's first problem for John)

Although he was able to select the appropriate problem-solving strategies to complete the task after she had read the problem to him, he needed some assistance in order to obtain the *how much change did I get?* component of the question. It would be fair to say that the problem was not well worded. With some guidance from Bettina, however, he was able to obtain a "correct" solution.

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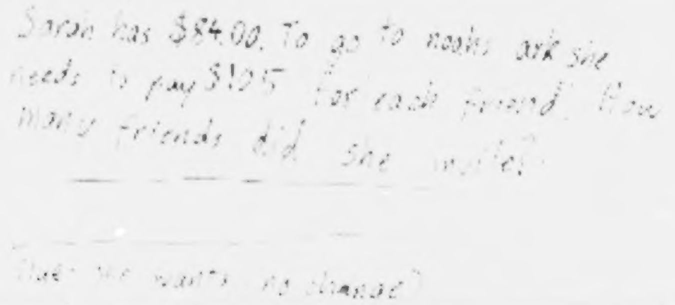
One of the most pleasing aspects of the problem-posing activities undertaken in the present study was the fact that the children were willing to work with one another to solve the problems. As Bettina commented:

> At first he didn't really understand all of the problem. He was able to add all the dollars together but he found adding the cents tricky. I helped him with that but he really did most of the work himself.

In subsequent problem-posing situations she was conscious of the mathematical language she used in problems posed for him (see Figure 3).

Figure 3
Bettina's Problem for John



Sarah has $84.00. To go to noahs ark she needs to pay $10.50 for each friend. How many friends did she invite?

Hint she wants no change

The content and structure of the initial questions were associated with traditional one- and two-step word problems. Each problem had one correct answer. This is not surprising considering the children were not accustomed to developing their own problems. It could be argued that the traditional format of these problems provided the children with a useful proforma.

Insights into Children's Mathematical Ability

As the children became more competent in designing problems, and more comfortable in working with one another, the way they structured and representated problems changed. Bettina, for example, used less language in the problems posed for John. She also provided him with clues such as "she wants no change" when representing the problem (see Figure 3).

The problems Bettina designed were becoming increasingly sophisticated. In the early weeks of the study she did not need to have a solution in order to pose the problem. After three weeks her problems contained information that could only be designed if she had first sought a solution herself. This type of processing is consistent with findings obtained by other researchers (for example, Ellerton, 1986; Silver, 1995). In the problem posed in Figure 3, for example, she needed to select amounts of money that could be evenly shared. Thus, eight friends @ $10.50 per head would cost $84.

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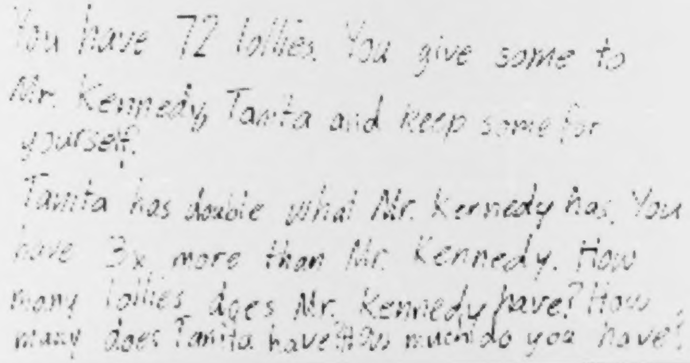
Figure 4
Bettina's Problem for a Friend



You have 72 lollies. You give some to
Mr. Kennedy, Tanta and keep some for
yourself.
Tanta has double what Mr. Kennedy has. You
have 3x more than Mr. Kennedy. How
many lollies does Mr. Kennedy have? How
many does Tanita have? How much do you have?

In the problem represented in Figure 4, which was much more difficult than the type of problems she was designing for John, also required forward planning in its construction. This is a quite difficult ratio problem for children in Year 4 and was again designed for Joe. By indicating that there was a total of 72 lollies to be shared, and that three people were to receive different proportions of the total Bettina must have calculated that the amounts would be 36, 24 and 12 respectively. As Bettina commented "I had to know what the answer would be otherwise the problem wouldn't work." It could be argued that the way in which the problem was structured provided useful insights into Bettina's mathematical ability.

CONCLUSION AND IMPLICATIONS

This study investigated the way in which primary-aged children engaged in problem-posing situations that required them to pose problems for friends. Although the students had not been exposed to such problem-solving activities in the past the problem originator was able to consider the needs and interests of a peer when posing mathematical problems. Not surprisingly, a student's mathematical knowledge had an impact on their ability to generate a variety of problems. It was more difficult for John to construct challenging problems for Bettina to solve because he did not posses the necessary knowledge or mathematical processes to extend her in a problem-solving environment. His only option was to use larger numbers in his problems. It was much easier for him to design problems for someone at or near his level of development. Although this finding may have implications for classroom practice, it needs to be recognised that John did gain a great deal from designing problems for older children. One positive consequence from these problem-posing activities was the fact that he was able to work cooperatively with a peer who challenged him to engage in mathematics at a higher level than he was accustomed to.

These problem-posing experiences also helped the children think about the problem-solving process in more sophisticated ways. In designing problems for friends to solve, the children were required to consider the type of understandings and processes needed to complete a task and also decide whether these understandings and processes were appropriate for the problem-solver to complete. These experiences also provided opportunities for the children to talk to one another about their solution methods in cooperative situations. When designing new problems for their friends to solve, the two children considered one another's comments and insights on a regular basis. From a classroom perspective, problem-posing activities help children to more fully understand the problem-solving process and help the teacher to gain insights into the children's mathematical ability.

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CHILDREN'S GENERATED WORD PROBLEMS: A CASE STUDY

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This paper investigates the performance of Grade 3 children's generation of word problems from photographs. It looks at the relationship between this performance and performance on a grading instrument; whether children tend to generate characteristic categories of word problems in relation to specific photographs and to what extent student generated word problems reflect their mathematical understandings and thinking.

Ellerton, (1986) wrote that:

> As the substance and style of the problem made up by each child uniquely reflects that child's mathematical experiences and ideas, the made up problem is a particularly useful tool for studying mathematically talented children for whom routine tasks are usually completed quickly and accurately. (p. 261)

Student generated word problems have been investigated by numerous researchers (e.g., Ellerton, 1986; Greenes, Schulman & Spungin, 1992; and Silverman, Winograd & Strohauer, 1992) with children of different ages. Ellerton (1986) had her Grade 7 and 8 students generate and solve difficult problems for their friends. She found that more capable students generated problems of greater complexity than less able students. Providing students with an opportunity to generate mathematics problems achieved a balance between a passive mathematical environment, and one in which an active communication of mathematical ideas became evident. The problems posed also gave an insight into the mathematical understandings of her students. Greenes et al. (1992) required their middle school students (Grades 6 and 7) to generate mathematics stories for the class to solve. Investigated was the relationship between children's understanding of mathematical concepts and their communication of mathematical ideas. The findings suggested that, through discussion, children were able to clarify their concepts and develop an understanding of the mathematics involved in the problems. Silverman et al. (1992), working with Grade 5 students, used student generated word problems to replace reliance upon textbook problems. They initially used word problems from texts, teachers, and students, to introduce problem construction and solution strategies to his class. Group sharing of problems enabled the communication of mathematical ideas and concepts. The complexity of problems generated was not considered in this study.

Fennema and Peterson (1986) have described this complexity. They referred to high-level mathematical tasks as those requiring reflection whilst low-level mathematical tasks required only the use of rote knowledge and the application of an algorithm.

In regard to word problems, the Victorian Curriculum and Standards Framework: Mathematics[1] (Board of Studies, [CSF], 1995) lists several 'learning outcomes[2]' for Level 2 students (Years 3 & 4) namely:

> generate word problems using specified numbers and operations (p. 54);
> use mathematical language appropriately … (p. 102);
> generate mathematical questions from … familiar contexts (p. 104).

The CSF also advances a curriculum focus on generated stories, real situations, pictures and number sentences, that involve collections of objects, money and informal units of measurement. This requires students "to devise situations which may give rise to symbolically expressed operations (e.g., tell a story)." (p. 53) Students are to use their mathematical knowledge and understandings to generate and discuss problems drawn from the mathematics of familiar situations. The question of how this curriculum focus has been applied in the classroom is unknown. The present study seeks to address this lack of knowledge.

STUDY

In this exploratory qualitative research study, the following questions were considered:

1. Can a range of young students generate word problems from photographs?
2. Is there a relationship between a student's ability to generate a word problem and that student's performance on the grading instrument?
3. What is the nature of word problems generated in this way?
4. Do such word problems provide an insight into children's mathematical understandings and thinking?

In this study a word problem has been operationally defined as any problem signifying a mathematical relationship in words to which a solution was to be sought.

METHOD

This study was conducted with a small sample of Year 3 (8-10 year old) children towards the end of their school year. For a week before the study, I visited the class during the mathematics lessons to work as an aide. This allowed the class to get to know me and for me to meet the class and become accustomed to their practice. During this time the teacher gave her students two sheets of commercially produced word problems.

At the beginning of the study, I administered a performance instrument (described in the next section) to the class of 28 children. Each child's performance score was ranked. From the 28 children who completed the instrument, six were selected for the in-depth study (two students from each performance level – above average, average, and below average – see Table 1).

PERFORMANCE INSTRUMENT

This instrument was developed to select children for the interview. It determined their relative performance in solving number problems. The instrument was based upon the arithmetic operations stipulated in the CSF mathematics statement (Board of Studies, 1995). It consisted of 23 number problems. There were 11 high-level and 12 low-level problems. It involved the four arithmetic operations and contained problems that could be derived from the photographs presented to children during the interview. The format of this instrument was based upon that devised by Cai (1995). This format was familiar to students. Previous experience had indicated that:

i. it was typical of class exercises presented by teachers,

ii. the low-level number problems presented in the instrument were typical of those presented to students at this year level,

iii. high-level number problems were not often presented to students at this level.

PHOTOGRAPHS AND TASK

A number of objects (oranges, cutlery, and cupboards) were photographed, scanned and the black and white images electronically enlarged and printed. The prints were laminated. (See Table 3 for the objects on each of the 14 photographs.)

In an individual task setting, the children were told that they would be shown some photographs and asked to "Make up a difficult mathematics word problem or story for your friend to solve". The children were then shown the photographs and asked to select one. They were also told that they need not write down their word problem or story, as I would do that for them. If the child's word problem seemed ambiguous, the child was questioned about its meaning. Once a student had generated a 'story', s/he was invited to provide a solution equation for the problem. The child was then asked to select another photograph. In all, each child selected four photographs, generated word problems and provided solution equations.

RESULTS

Although the six children generated 24 word problems, not all problems were adequate; Two would not be able to be solved by 'friends'. Initially all children had trouble understanding what was required of them: to use the photograph to 'make up a difficult mathematics word problem for your friend to solve'. Once this was understood, all students attempted to generate a word problem. All children showed a lack of command of mathematical language in that they did not have knowledge of mathematical terms other than for the four arithmetic operations (addition, subtraction, multiplication, and division).

Note: As the child ranked 27 was absent from school, it was decided to use the child with the next higher ranking in her place.

Table 1
Students, Ranking, Gender, and Performance Level on the Grading Instrument of the Interview Students.

Student[1]	**Student ranking**	**Gender**	**Performance level**		
			Task Complexity		
			High n=11	**Low n=12**	**Total Score**
Mary	1	Female	10	12	22
Noel	2	Male	11	10	21
Olive	18	Female	8	10	18
David	19	Male	8	10	18
Irene	26	Female	5	9	14
Henry	28	Male	5	8	13

Table 1 (above) shows that there were clear differences in the students' performance ranging from near perfect scores to satisfactory but lower scores. The selection of children compared favourably with their teacher's estimate of their achievements. For instance she reported that Irene was not achieving very good results in class, and that Henry had had difficulty meeting the requirements of the Year 3 syllabus. It was fortuitous that the selection of children for the interview consisted of three girls and three boys. With the exception of Noel, the children successfully solved fewer high-level tasks than low-level tasks.

The more capable students (Mary, and Noel) were able to overcome their language difficulties and generate solvable word problems. The average students (Olive, and David) differed in their ability to use language to describe their word problems. Olive showed evident difficulty in verbalising her problems whilst David, apart from the first problem, was able to state his word problems relatively clearly. The two below average students (Irene, and Henry) also differed in their ability to verbalise their word problems. Irene began well but quickly faltered. Henry really found the task difficult and was unable to generate four solvable word problems. All the word problems generated by these six students were representative of low level mathematics activities. They tended to develop word problems in which the answer was the unknown quantity, as opposed to making the unknown either start or change unknown. The word problems and the solution equations generated by the children are presented in Table 2 below.

Although the class teacher had commented that her class "hadn't done a lot of word problems," it was impossible to determine whether this had been the students' only practice with generating word problems. If it were, it would certainly explain their lack of familiarity with the genre and would go some way to explaining the difficulties they experienced with generating a word problem from a photograph.

Table 2
Students, Word Problems, and the Solution Equations Generated.

Student	Word problem
Mary	1. I had one banana and I went to the shop and I bought six more bananas. How many bananas have I got altogether? 1 + 6 = ☐ 2. I had three people coming over to my house for lunch. I set out a knife, a fork and a spoon for each person. How much cutlery did they get? 3 + 3 + 3 = ☐ 3. If I had twelve eggs and I broke two, how many eggs would I have left? 12 - 2 = ☐ 4. If I had four oranges and I ate three oranges and I gave one to my friend, how many would I have left? 4 - 1 = ☐
Noel	1. What's wrong with one the cupboards in this picture? (No equation required. Draw a picture.) 2. I had six oranges and I found another six on the road. How many oranges do I have altogether? 6 + 6 = ☐ 3. I had a box of Tri-ominoes. What numbers do you think the numbers go up to in the box? (No equation produced) 4. How much money can you see in the picture below? 20¢ +20¢ + 50¢ + 50¢ + 50¢ + 10¢ + 10¢ + $1-00 = ☐.
Olive	1. There were four bottles and some one poured some water into five glasses and that bottle was empty. How many bottles were left full of water? 4 - 1 = ☐ 2. There was a traffic jam because a car smashed into another car and four people were in one car and three people were in another. There were two in the front of each car and one in the back of one car and two in the back of the other and the ambulance came and took the people who were hurt away. How many people do you think got hurt? 7 - 4 = ☐ 3. Jessy had $12.50 and she bought some Skinny Milk for a dollar. How much change would she have? $12.50 - $1.00 = ☐ 4. Sam had a banana, apple, orange and sandwich for lunch. How much food altogether? 3 + 1 = ☐
David	1. There were seven bananas and one was left by itself. How many of the bananas that were left were in the two groups? 7 ÷ 2 = ☐ 2. There were three spoons, three forks, and three knives. How much cutlery is there altogether? (No equation provided.) 3. There were eight oranges and my friend gave me four more oranges. How many altogether? 8 + 4 = ☐ 4. I had a sandwich and an orange and my mum gave me another banana and an apple. How much fruit and sandwiches altogether? 2 + 2 = ☐
Irene	1. .I had twelve oranges and twelve of my friends came and all of my friends liked oranges so they asked if they could have one. And I said, "Yes!" I gave them twelve oranges. How many oranges were left? 12 - 12 = ☐ 2. It was my birthday party and my mum was baking a cake. On the recipe it said you need to put ten eggs in. My mum put twelve eggs in. How many eggs were left? 12 - 12 = ☐ 3. Five of my friends came over and it was a really hot day so my mum poured five glasses of water out. One bottle was finished. How many bottles were left? 5 - 5 = ☐ 4. I went to school and my mum put in my lunch box a sandwich, an apple, an orange and a banana. I ate my sandwich and my orange and my banana but I didn't eat my apple. How many things were left in my lunch box? 4 - 3 = ☐
Henry	1. I had fourteen oranges and fourteen friends and I squeezed four oranges. Ten oranges were left. How could we make it even for every friend I have? (No equation offered.) 2. It was my birthday party and six of my friends came. How many bananas were left? (No equation offered.) 3. There were twelve juicy oranges and six of them were eaten. How many were left? 12 - 6 = ☐ 4. There were nine knives, forks and spoons. Six of them needed to be used for dinner. How many were left? 9 - 6 = ☐

DISCUSSIONS

Generating word problems: Most of the students were able to generate some kind of word problem - albeit with continued probing. Two main areas of difficulty were observed:

1. The children found difficulty in expressing their word problems although it seemed evident from what they were saying that they had an idea of what word problems they wanted to

pose. This could be explained perhaps by a lack of familiarity with the activity. They seemed unable to talk about mathematics.

2. The children generated word problems which were representative of only low-level mathematics activities. If the work-sheets issued to the class by the class teacher a week prior to the project's inception are indicative of the children's exposure to word problems, then there was a reinforcement of both a dependence upon low-level mathematics activities and misconceptions about the four arithmetic operations (Greer, 1992) so typical in word problems often given to students.

Use of photographs: Initially students found the task difficult though once they understood the requirements of the activity they were able to generate a number of word problems. The degree of complexity of the student generated word problems differed. This is evident in the transcripts. Given the seeming lack of experience the students had with word problems in general and the requirement to generate word problems, the task of using a photograph as the basis for generating word problems was not beyond their mathematics capability.

STUDY QUESTIONS

This study, addressed the three research questions. These will now be considered in the light of the study's findings.

1. The success children had with the activity indicated that they could use photographs as the basis for generating a word problem. This success however was limited in that the 'quality' of those word problems generated was not rich in novelty of concepts. Because of a general lack of familiarity with the task, students initially found the task forbidding. However, once an appreciation of the task requirements had been gained and some level of success had been achieved, most students, no matter what their achievement level, were able to generate a word problem representative of the photograph.

2. There seems to be a relationship between the participating children's level of achievement and their generated word problems. All participants experienced some form of mathematics language difficulty but the lower the achievement, the greater the mathematics language difficulty shown. Although they all generated representative low-level word problems – the lowest achiever generated two unsolvable word problems. The solvable problems generated tended to be similar to those provided by the teacher in the work sheets so it is possible that this model of word problems may have inhibited the children's creativity. Most of the word problems were either separating or joining with a few non-routine and unsolvable problems and one of each equal groups and part-part-whole problems.

3. Each of the six students who participated in the interview was able to generate word problems. With respect to Mary and Noel, the above average pair, Mary initially had difficulty understanding what the task required but once she understood Mary was able to generate word problems. Mary generated two joining[4] and two separating word problems. Noel's word problems differed from the other participants' problems in that he generated not only a joining word problem but also two non-routine problems and a part-part-whole word problem. These three required photographs to be seen by his friend; one of these required his friend to discover the difference between the drawers, another required searching through the triangular blocks to discover the highest number, and the third required adding the coins shown in the photograph. The average pair, Olive and David, Olive had difficulty expressing her conception of her word problems but generated three separating and one joining word problems. David, on the other hand seemed to be able to express himself mathematically quite well and generated three joining and one equal groups word problem. In the case of the below average pair, Irene and Henry, Irene, after initial success, found her inadequate mathematics language skills inhibited her verbalisation of the problem concepts she wished to express. The word problems she did generate tended to be inconsistent

with the photographs from which they arose. Henry not only had great difficulty in generating word problems (he generated two unsolvable word problems) but also experienced serious mathematics language difficulty. Of those problems Henry did generate, both were separating problems.

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Table 3 below shows the frequency with which the six children selected each photograph. Also listed in the table are the categories of word problems that were generated from each photography. The number in parentheses indicates the number of word problems of each category generated from each photograph.

Table 3
The Frequency of Selection of Each Photograph and the Category of Word Problem Generated from the Photograph

Photograph	Frequency of selection (n = 24)	Category of word problem
Juice extractor and oranges	1	Unsolvable (1)
Bottles of mineral water	2	Separating (2)
A sandwich and two pieces of fruit	3	Joining (2) Separating (1)
A carton of milk, coins and a docket	2	Part-Part-Whole (1) Separating (1)
Tessellated floor tiles	0	Unrepresented
Kitchen drawers	1	Graphic display (1)
Four grapefruit, one halved.	1	Separating (1)
Seven bananas	3	Joining (1) Partition division (1) Unsolvable (1)
Three knives, three forks and three spoons	3	Joining (2) Separating (1)
Two cartons of eggs	2	Separating (2)
A train track and carriages	1	Separating (1)
A 3 x 4 array of oranges	4	Joining (2) Separating (2)

4. The children's generated word problems tended to mirror those presented immediately before the study commencing. Of the 24 word problems generated, 11 were separating, seven were joining, one of each was part-part-whole, non-routine, graphic, or partition division. Two problems were unsolvable. Not one of the solvable problems involved high-level mathematical tasks.

The unfamiliarity of the task appeared to hamper an adequate reflection of the children's understandings and thinking to be revealed. As stated above all solvable problems were low-level mathematical activities showing no evidence of metacognitive thinking. One child (Noel)

generated two problems requiring drawings to be viewed to answer his questions. This question raises some implications for mathematics education within schools.

Posing word problems appeared to be a novel experience for these children. Whilst all their problems were low-level word problems, the children who experienced the greatest difficulty

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completing the task were those of below average achievement. There was, however, no evidence to suggest that the word problems posed by the above average children were of any greater complexity than those posed by the average or below average pairs. This differed from Ellerton's (1986) findings. This could be explained by the novelty of the situation and that these children did not have an opportunity to communicate their problems to their cohort and so gain valuable feedback. Because of this lack of communication, so valuable to Greenes et al. (1992) and Silverman et al. (1992) the children were unable to clarify their understanding of the mathematics involved.

FURTHER RESEARCH

Although this study involved only a small sample of children from one class, teaching experience suggests that the findings would be similar across a number of classes and schools. Further research to determine whether this is the case is obviously needed. It would also be useful to investigate, for example,

i. whether two children working collaboratively could generate higher level problems;

ii. if several experiences of problem-posing would improve children's problem posing;

iii. whether exposure to stimuli other than photographs would make a difference;

iv. the kind and degree of support teachers might need to incorporate problem-posing into their mathematics' syllabus.

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Endnotes

[1] The CSF provides schools with formal guidelines for curriculum development in eight Key Learning Areas.

[2] This term is used in the CSF to indicate what students are expected to have achieved at the completion of each level.

[3] Pseudonyms have been used.

[4] For a full description of word problem categories see Carpenter & Moser (1982) and Ellerton (1986).

THE LEARNING NETWORKS OF ISOLATED TEACHERS: THE USE OF AUDIOGRAPHICS COMMUNICATION TECHNOLOGY TO BUILD A VIRTUAL MATHEMATICS DEPARTMENT

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Four teachers of secondary school mathematics in small rural schools, an adviser and a participant researcher are using Farsite audiographic equipment to develop a learning network. A description of the patterns of interaction within the project leads to a discussion of issues for future planning within this project and more general principles for developing later networks. We identify some skills for networking in the cyber-age, and discuss some advantages of audio-graphic networks over face-to-face contact.

Teachers who work in schools where there is only one teacher of senior mathematics are isolated from discussions about the teaching and learning of mathematics. This project is designed to offer these teachers the opportunity to share ideas with other, similar teachers on a regular basis through after school meetings using communication technologies that already exist in their schools.

The schools involved all have audiographic equipment that allows a teacher in one school to teach groups of students in other locations. This equipment has previously not been used after school hours for teachers' meetings, nor do teachers of mathematics normally use the equipment for teaching between schools. We are investigating the potential of this equipment as a tool for teacher communications and professional development related to teaching secondary school mathematics.

Liz Burge uses the metaphor of a weaving loom (in preference to an information highway which has connotations of isolated bits, metallic vehicles, streaming along concrete pathways) in her discussion of the use of conferencing technologies for learning (Burge, 1995). This more gentle, less hurried image suits the story we are about to tell. We are weaving together a network of teachers, we are weaving theory and practice, and we are tying together our ideas with those of our colleagues. For these reasons we are weaving the metaphor through this paper. Firstly we summarise briefly those things that were set in place before the weaving began: the loom is taken to be the institutional structures in which we are working, the warp threads refer to our methodology and pedagogy, and the tools link us to a discussion of the parameters under which we are working[1]. Secondly, we describe the sessions, the weaving that is occurring and the patterns that are emerging. Finally, we consider some trends and issues for the design of future patterns.

THE LOOM: BACKGROUND CONSIDERATIONS

The institutions This project links five institutions, four schools and a college of education, each of which is affected by massive changes in education. Teachers are aware that their work is substantially different from that of their city counterparts and are interested in the work of their colleagues in other isolated schools. At the College, advisers and lecturers are aware of pressures to offer courses in ways that do not disadvantage isolated learners. Teachers and policy makers in all our institutions are watching this project with interest. Each institution is supporting the develop-ment. We are all keen that it should work. Does this feature mean that it is doomed to succeed?

The people The four teachers involved in the project all responded to an invitation from Kevin, the Mathematics Adviser, to become part of a network. This network was not designed with a specific content planned for the curriculum, and there is no qualification to be gained by being part of this network. Instead it is designed to enable teachers to

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discuss those issues that interest them, in the company of an adviser who could be expected to provide some leadership and support. Teachers indicated their areas of interest before the first session began. We aim to achieve good principles of adult and distance education such as those described by Boyd (1993):

> "Good distance education is ... particularly concerned with linking the teaching-learning experiences of people with complementary educative interests and capabilities who are geographically separated. ...
>
> "... the heart of the matter ... is that responsible creative free-willed selves are struggling to reach one another, in order to help, despite constraints ..." (P237)

All the participants knew Kevin and Elaine prior to enrolling and two of them were part of an earlier project that was run when Elaine was a mathematics adviser (Mayo, 1996). The teachers knew in advance that we were aiming to develop a democratic, non-hierarchical network. The teachers have a vested interest in making the project work, but they indicated that they would continue to be involved only if they felt it was helpful to their teaching and to their use the technology.

The research In the current study Elaine takes the role of participant researcher using an emancipatory methodology[2]: she is not trying to be an objective observer. Above all this is a study about making changes that are people friendly. It supports the use of what Liz Burge describes as synomorphic communication technologies (devices that "give to" rather than "take from" the humanity of the communication) (Burge, 1995, p.154). Our work is informed by the Senge's concept of a learning organisation (Senge, 1992) and by the invitations such as that of Tony Bates (Bates, 1995) to work toward a vision for how things might be in open and distance learning.

The technologies Farsite audiographic software allows us all to link using an audio conference call, and at the same time to share a communal computer screen. We can all hear each other and we are all able to see and write on a virtual, shared whiteboard. We are unable to see each other. We are not yet using the Internet, except to refer to for resources. Between the sessions we use a variety of other technologies. Telephone, fax and post have been used to share information and materials. An e-mail link was established after the third session. In addition Kevin or Elaine visited each teacher in the first two months of the project, and Kevin has met all the teachers at other regionally organised events. A conference call without the shared whiteboard was used at the end of 1998 to evaluate progress. Despite these other connections the audiographic link remains the dominant tool of the network.

The meetings Prior to the sessions Kevin and Elaine tried to overcome potential technical problems by learning about Farsite and the bridging technology. In planning the sessions we considered protocols for communication. Essentially Kevin was to coordinate the sessions with Elaine offering contributions when they seemed necessary or appropriate. All participants are aware of the high costs of networking, both in terms of money and time. We planned to meet initially for three sessions, each of 45 minutes, spread out over seven weeks, two further sessions were later added to the 1998 package. In 1999 sessions are longer: 90 minutes each and we aim for two per term.

The network The project was promoted as being network for professional development using communications technologies. We also made it clear to teachers that we were all learners together and did not have a great deal of experience in the use of the medium. Teachers have agreed to be part of a research study. We are not a network of equals because our interests and responsibilities are different. In what ways, we wonder, are we like a virtual mathematics department?

THE WEAVING

The fabric grows as the weavers weave their threads. Our threads are the experiences and activities we each bring to each session; our tools are the audiographic system and the other communications technologies we use; the cloth that emerges is our network.

The Weaving - Kevin's Story of the Three Sessions

Getting started "During 1997 I spoke with several teachers who were enthusiastic about the possibility of running audiographic sessions for the purpose of professional development in mathematics. Elaine set up a network during 1996 and was supportive of another attempt at it.. I met with Elaine on several occasions, and spent hours familiarising myself with the software. I also sat in on a couple of sessions observing a group of year 11 students being taught agriculture using the equipment."

The format for sessions "Schools were invited to become part of a network. No experience with the equipment was expected. A suggested format for the sessions was proposed:

- short introductory activity while establishing link with participants
- main activity on previously agreed theme
- open forum for issues of the day
- something to use in tomorrow's teaching
- closure and preview of the next meeting.

Four teachers responded positively and topics were set for the three sessions. All that remained was to organise the bookings. We ended up with three 45 minute sessions during August and September. These worked out well enough for us to book a further two sessions in October and November."

Session One - Introductions. "Three of the four schools participated in the first session. For much of the session only an audio-link was operating. During this time we discussed the format of the next session and clarified the meaning of "novel approaches to class work". Once the graphic link was established there was a feeling of relief (at least on my part!) and the interactive nature of the activity on the screen appealed to the participating teachers. Somewhat ambitiously we set a task for each teacher to make a slide of a simple type-able puzzle to share in the next session."

Session Two - Novel Approaches to Class Work "All four schools participated in the audio link and three of them were also able to participate in the graphic link. The introduction was the same activity as had been used briefly in the previous session. Having something interactive on the screen while everyone joined the conference worked well. A framework for the main discussion had been faxed to members before the session: this acted as a good catalyst. We set ourselves the task of talking through any approaches we have used or seen that: make the teaching more student centred; gave the students more control and responsibility for their own learning; assisted in maintenance and retention in an interesting way; and addressed the issue of motivating students. One teacher revealed as system of portfolios he was using. This aroused interest and it was agreed to make it the focus of the next session. Teachers also wanted to share the slides they had prepared, but time had run out: the agenda of the next session was determined by the progress of this one."

Session Three - Slides and Portfolios "Everybody got linked up - for the first time! Additionally, Elaine was on an audio link from another province. The planned agenda was for everyone to: present their slide; share some information about themselves and their school; discuss portfolios; begin a problem solving activity. We didn't get beyond the sharing of slides. Two of the four teachers succeeded in sharing workbooks thereby managing a part of the session themselves. They were ecstatic, all four of them, even those who had not succeeded themselves. It was agreed that we would continue the project through the fourth term. Things had just started to happen."

Session Four - Absences "I didn't make it to this session so my knowledge is second hand. The intention was that I would out at one of the remote sites in South Westland while Elaine would be in North Canterbury with another participant. Unfortunately I got stranded by floods and didn't make it to the session. I talked to the teacher from South Westland the next day and was told that all had gone very well without me. The agenda was a continuation of the previous session and managed to get to the point of this teachers sharing a portfolio of work on measurement. Elaine had a copy of the screens with her in North Canterbury, and I understand she simply "did a Kevin" in facilitating the session."

Sessions Five - A Number Portfolio "The focus for this session was the sharing of resources suitable for a portfolio on number. Prior to the session participants had used a variety of technologies to send material to the rest of the group. They used conventional mail, faxes and e-mails with attachments. They then talked about the activities during the session, relying on the fact that everyone had a hard copy. None of the teachers chose to display their tasks on screen during the session. I provided a series of slides on screen to encourage discussion of issue such as: what is a portfolio, why use a portfolio, contents of a portfolio, and using portfolios to encourage mathematical understanding. By the end of the session, which was the last for the year, there was general consensus that there was more than enough material for teachers to use as a unit of work at the start of the following year. We finished with an expectation that the sessions would continue next year."

Pluses and Minuses "I wasn't particularly looking forward to the second session. Given the technical glitches that had happened in the first session it was worse than going to the dentist. I was at the mercy of the technical equipment. If things fell over, I didn't know when or where it would happen, and I wasn't sure if I could react in such a way as to make the conference be seen as a worth while experience. It was a little like teaching for the first time all over again. Strategies had not developed to redeem the irredeemable. A lot of this was in the head rather than apparent. Teaching is like that too.

"The technical hitches that have occurred in the sessions have provided a good learning opportunity for the whole group and have to an extent worked to our advantage. It has become clear to all the teachers that there is no single technical expert in the group and all can help. This was emphasised by my absence in the fourth session. That message is filtering in a subtle way into the exchanges on mathematics teaching. An atmosphere has been developed in attempting to master the technology that has a spin off for the culture of the group that is forming. This is becoming a group of teachers helping themselves, not just technically but mathematically as well. I may be the hub of the wheel but I am not the source of all knowledge.

"A shift of control occurs as the sessions move on. Initially I set a structure and proposed an agenda, and this is always the backstop position for the sessions, but the rate and direction of development is increasingly determined by the participants. By the third session control of the medium had significantly shifted to the teachers as they took us through their slides as they gained confidence and competence with the technology so they woke up to the possibilities of the technology. More and more they are creating their own unique micro-world of professional development."

The Weaving - Elaine's Comments about the Network

Background "As a participant researcher I am clear that "supporting" includes helping in any way, including running for help in times of technological crisis, and discussing ideas from literature and experience,. At the same time I am trying to record the process and document the progress of the network. Rarely do I talk to Kevin or any of the group without a tape recorder running. Our discussions form part of my data."

Planning sessions "The planning sessions with Kevin are valuable for teasing out ideas about how to use this project for the benefit of all participants, with the concerns of the teachers as the key issue, the development of a successful network as second, and my research as third in priority.

"We plan the audiographic sessions and the other necessary detail of the project, but the meetings also provide a venue to discuss common issues: discussions about teaching strategies and mathematical activities for the classroom are commonly raised by Kevin; I commonly introduce issues related to networking and maintaining the cohesion of the group. We philosophise too."

The audiographic sessions "There are four dominant strands of conversation running through the sessions. Initially most of the talk time was related to technology and team building, with less on mathematics and teaching. We have spent a lot of time supporting each other in using the technology and trying to help each other solve the individual problems we are having about getting it to work. The strategy has been to maintain a cheerful, philosophical approach to problems where several of us offer suggestions. Because of the supportive tone of these conversations they contribute to second strand: teambuilding. The third session included more spontaneous laughter than earlier sessions, and that laughter was linked to our glee at successes in solving technical problems. We all managed to get hooked on to the system at one time! The third strand of conversation relates to the mathematical activities that the teachers have developed to share. Except for the problems suitable for classroom that Kevin has used, mainly as discussion starters in each session, there has been little focus on mathematics content. Any mathematical discussion quickly moves on to talk about teaching strategies, ways of dealing with the very mixed classes that arise in small schools, ways of organising students to work independently, and issues related to assessment and the recording of assessment judgements."

Discussions "The discussions on assessment are leading toward a different form of dialogue. As the teachers clarify what they are trying to do and why there are differences arising. Some teachers are seeking to record very detailed information about student progress, others tend to be much more holistic and to focus on processes rather than content. Questions arise about our assumptions, and the dialogue becomes more reflective. We are all sharing resources that relate to our current discussions, commonly classroom materials are shared by teachers and articles or references to curriculum guides and documentation from Kevin and Elaine. In this way the discussion is informed by both theory and practice."

The Weaving: Elaine and Kevin Report on the Responses of One Teacher

Our assumptions about the value of the project have been confirmed by the comments made by teachers. For example one teacher emphasised the value of the networking and hinted at the satisfaction that comes with getting the system to work. She liked "Talking to other teachers with similar sorts of class structure and problems. Getting some new resources. Getting the system to work!" She also commented that "It's a great way of getting in touch with other people. [It d]oesn't feel so lonely - even though I have been to courses in Nelson and I know quite a few of the maths teachers in the area in some ways I feel a bit left out as I have some different sorts of problems than they do." These sentiments were echoed at the end of the year when we evaluated progress in an audio-conference. Teachers had suggestions about the ways of making the learning about the technology easier, and the whole tenor of the conversation was very positive.

OUR NEXT WEAVING: 1999 DEVELOPMENTS

The fabric that is evolving from the process is our network has four patterns emerging: the links within the network, the development of dynamics within the group, the evolving protocols, and the links between theory and experience. We will continue to weave these patterns.

Growth in the Links Within the Network

Despite the fact that group members have not yet met face to face there are strong connections emerging among the members. Two of them spent time together at a Heads of Department meeting late in 1998, the other two are meeting early in 1999 at an Area School's sports meeting. Kevin will also attend that meeting, and other teachers from schools and other curriculum areas plan to look in on the audio-graphic conference that coincides with the meeting. During 1998 we included some sharing of information about ourselves and our schools in the conferences, but this was less that we had hoped.

The teachers are keen to continue the project into a second year. "We are just getting into it" said one teacher. Another said "We couldn't not continue - we're developing all these great skills".

An attempt to use e-mail as an additional tool for networking has generated very little activity. For the two teachers who do not have easy access this is yet another technological and organisational hurdle to overcome. We used it sparingly during 1998 and will continue to monitor the possibilities for easy access by the teachers.

We used a digital camera to photograph teachers in their schools and tried to transmit these pictures during a session. The files proved to be too big to be transmitted efficiently within the audio-graphic system, and the graphic capability of the system did not allow detail to emerge. Experiments like this (experiments that are technically failures) have allowed us to share some information about each other and have strengthens our bonds within the group. A development for 1999 could be to establish a web page where we can post graphic items.

Development of Dynamics Within the Group

Strands that contribute to cohesiveness of the group in the first few sessions include:

- we are all learners together
- we can all contribute our suggestions
- supportiveness of those with a problem - people come first
- recognition that our preparation for new activities is now different from earlier times-no longer do we become expert before we become involved. We learn by making mistakes. This could be a fundamental skill of the cyber-age.
- laughter that results from sharing the glee of a colleague who has beaten a technical problem
- individual contact by Kevin and/or Elaine with the teachers - listening to their interests and opinions and adjusting the sessions accordingly
- reassurances that we are all facing problems
- trust that Kevin will move the discussion along, and that Elaine will watch for the things that may have been missed. Should we, and how can we, move these responsibilities on to the group? How will the dynamics change as the group evolves?

Growth in the Protocols

Kevin talks about the shift of control to the teachers. They are now leading some sections of the sessions. We wonder whether and how this shift of control can evolve. This is a culture that uses visual and auditory cues, but has no body language. There are times when silences seem unnatural, we are learning to use our voices to show we are concentrating. At other times the silences seem natural. Where Kevin uses the audiographic screen to note significant observations, or points not to be missed in later conversations there is a silence that goes unnoticed, until Elaine transcribes the tape. Our next step is to see how we can use the screen more interactively within our discussions.

Will the teachers begin to take even more control? Under what conditions? As the project moves forward the more long term protocols will emerge: projects such as this do not normally last beyond a school year.

Links between Theory and Experience.

Theory, reflection, experience, and our developing understanding all interact to allow the group to use this medium more and more effectively. Mayo and Hannah (1998) distinguish between literature that offers suggestions for distance educators and other more systemic literature that provides more theoretical models. During 1999, now that we have a collegial network in place and we are not so concerned about the nature of the technology, we are conscious of the ways in which the discussions allow us to compare and contrast our different views on the teaching learning process.

The metaphor of a building a virtual mathematics department provides us with a tool for discussing our roles and responsibilities. It could allow us to find novel ways to share resources, planning, tasks, and ideas. It implies stability, for mathematics departments normally have similar staffing from year to year. It might allow us to share ideas or theories, both formal (published) and intuitive (experiential) theories about leadership, and what constitutes good professional development, and about what good teaching is.

PATTERNS FOR THE FUTURE

We have come to believe that an element of risk taking is a pre-requisite to taking part. All of us are prepared to be seen to make mistakes, and by acknowledging this we may make it easier for others to gamble. A sense of colleagialship reduces the sense of risk taking, and we believe the fact that the teachers each knew our style before they enrolled is not a coincidence. This relationship is designed to operate for more than one year so that we can capitalise on the benefits of our past experiences. We see continuity as having a part to play in enabling teachers to become more deeply involved in discussions of issues surrounding their work. This has implications for the planning of teacher professional development opportunities. While knowledge and technical expertise are important, the human factors are far more important. Without these a network would not be established.

The use of metaphor is helping our process. The weaving, networking metaphor can be used to foster a collaborative community who work together to adapt the technology to support their needs. Adapt the technology to the people, not visa versa. Enjoy the tool, laugh at its frustrations, weave contingency plans into the fabric. The metaphor of network as a virtual mathematics department will inform our ongoing development, it will allow us to explore our future roles and to analyse the responsibilities of group members. Strategic thinking, decisive action and clear leadership at critical points has enabled the group to function effectively. We have questions about how this control can be more equitably shared, and whether it should.

There are some advantages in not having the distractions of having to comply with conventions of body language. Alternate ways of using voice and visual cues can be developed. . The interplay audiographics changes the focus of attention. Participants report that it is possible to write and think more easily when not distracted by the need to the speaker by nodding or maintaining eye contact

In future projects we would retain the distance networking (it is relevant to teachers); the multiple communications aspects (each medium has its uses); the audiographics (it is fun and there is potential for things to happen differently); the team approach to development (sharing of ideas is important for professional growth); and above all, an extended time frame so that we can capitalise on our past shared experiences.

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Endnotes

[1] Refer to an earlier version of this paper (Mayo and Hannah, 1998) for a more detailed discussion of the structures under which we were operating.

[2] Patti Lather (Lather, 1992) distinguishes between research paradigms based on the purposes of the researcher. Research can be used in four ways: to predict (scientific, quantitative methods), to understand (descriptive, qualitative methods), to emanipate (participatory methods), and to deconstruct (post-structural techniques).

AN ANALYSIS OF THE TEACHER'S PROACTIVE ROLE IN SUPPORTING THE DEVELOPMENT OF SYMBOLIZATIONS

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The analysis reported in this paper documents the teacher's role in supporting the emergence of notational schemes from the students' problem-solving activity in one first-grade (age six) classroom. Initially symbolizations were offered by the teacher as a means of clarifying and communicating students' thinking. Later, the teacher worked to achieve her pedagogical agenda by using notational schemes to highlight certain solution processes. As a result, the introduction of notational schemes served to support shifts in the students' mathematical development.

Mathematical activity is communicated via a wide range of notations, schemas, and models that can be called *symbolizations*. These symbolizations are critical to the development of mathematical power in that they provide a vehicle for communication, representation, reflection, and argumentation. However, as Lesh, Post, and Behr (1987) observe, many students have "deficient understandings about the models and languages used to represent and manipulate mathematical ideas" (p. 37). This indicates the importance of supporting students' development of ways of symbolizing to communicate their mathematical reasoning. Further, if students are to develop grounded understandings of the meaning and use of these ways of modeling and symbolizing, it is essential that they view the symbols they use as descriptors of *their* mathematical activity. Students might then accept symbols and notations as ways of recording and communicating their thinking that they can use as the need arises. Ways of symbolizing should therefore emerge from students' informal mathematical activity and be consistent with their developing concepts and strategies (Cognition and Technology Group, 1990; Lesh & Akerstrom, 1982; Thompson, 1992). In addition, they should serve as a resource that students can use to describe, communicate, and reflect on their mathematical activity (Confrey, 1990; Kaput, 1987).

The purpose of this paper is to analyze the teacher's role in the emergence of ways of symbolizing and notating in one first-grade (age six) classroom. In doing so, I will document how the ways of symbolizing served as protocols of action as the students explained and justified their mathematical reasoning (Dörfler, 1989). In particular, the paper will focus on how the teacher's efforts in redescribing and notating students' explanations and solutions supported the emergence of ways of symbolizing from the students' activity. In addition to documenting the development of the ways of symbolizing and notating, the reported analysis will also relate the use of symbolizations both to shifts in discourse and to the students' development of their own ways of notating their reasoning. Although the particular emphasis is on the teacher's role, the analysis will necessarily document the interactive constitution of ways of symbolizing and notating as they emerged from classroom mathematical activity. In this way, the teacher's role can be characterized as that of supporting the emergence of both individual and collective ways of symbolizing.

The episodes reported in this paper are taken from a classroom in which I participated in a year-long teaching experiment in close collaboration with the teacher, Ms. Smith.[1] She, in fact, viewed the research team as peers with whom she reflected daily about numerous aspects of the mathematics class. One of the goals of the teaching experiment was to develop instructional sequences designed to address quantitative concepts typically introduced in first grade. In particular, the Patterning and Partitioning instructional sequence was designed to address early number concepts by providing students with opportunities to conceptually construct patterns and to partition collections of up to ten items (cf. McClain & Cobb, in press). The Structuring Numbers instructional sequence was intended to support students' ability to flexibly structure numbers in situations where they added and subtracted with sums and differences up to twenty (cf. Cobb, Gravemeijer, et al, 1997). A critical aspect of these sequences was developing ways

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to symbolize students' thinking that could then support shifts in their ability to reason in problem situations. As a result, Ms. Smith took a proactive role in initiating the use of symbols and notations to communicate students' ways of reasoning. In addition to providing a detailed account of the teacher's proactive role, the episodes will also clarify how the development of notational schemes became realized in the classroom by developing empirically grounded analyses of the teaching-learning process as it was interactively constituted in the classroom (Cobb, Wood, Yackel, & McNeal, 1992).

METHODOLOGY

Data were collected during the 1993-94 academic year and consist of daily videotape recordings of 103 mathematics lessons, copies of all the students' written work, daily field notes that summarize classroom events, and notes from daily debriefing sessions held with the project teacher. In addition, videotaped individual clinical interviews were conducted with each student in September, December, January, and May. A method described by Cobb and Whitenack (1996) for conducting longitudinal analyses of videotape sessions guided the analyses of the data. This method is consistent with Glaser and Strauss' (1967) constant comparative method for conducting ethnographic studies. It involves constantly comparing data as they are analyzed against conjectures and speculations generated thus far in the data analysis. As issues arose while viewing classroom videorecordings, they were documented and clarified through a process of conjecture and refutation.

The interpretive framework that guided the analysis is called the emergent perspective (cf. Cobb & Yackel, 1996). This framework emerged out of attempts to coordinate individual students' mathematical development with social processes as students' learning is accounted for in the social context of the classroom. It therefore places the students' and teacher's activity in social context by explicitly coordinating sociological and psychological perspectives. The psychological perspective is constructivist and treats mathematical development as a process of self-organization in which the learner reorganizes his or her activity in an attempt to achieve purposes or goals. The sociological perspective is interactionist and views communication as a process of mutual adaptation wherein individuals negotiate mathematical meanings. From this perspective, learning is characterized as the personal reconstruction of societal means and models through negotiation in interaction. Together, the two perspectives treat mathematical learning as both a process of active individual construction and a process of enculturation into the mathematical practices of wider society.

CLASSROOM ANALYSIS

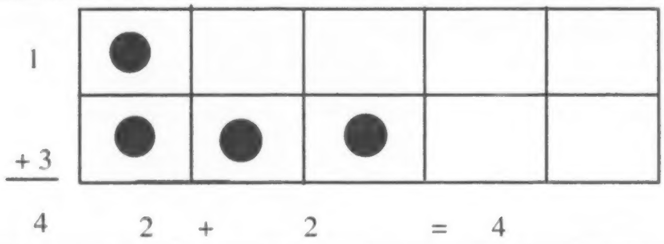
The ways of symbolizing that emerged in Ms. Smith's classroom appeared to evolve from the need to clarify and communicate students' thinking. At the beginning of the school year, students were not asked to make written records of their thinking but instead shared their solutions and explanations verbally in whole-class discussions. Ms. Smith judged that her students were unable to read well enough to be able to understand problems posed to them in text format. As a result of her desire to ensure that all students understood the task situation, she typically used the overhead projector or white board to pose tasks in a whole-class setting. This format was used throughout the Patterning and Partitioning instructional sequence and the first part of the Structuring Numbers instructional sequence. Students explained and justified their solutions verbally, often while relying on graphics from the overhead projector. During these activities, Ms. Smith would often introduce ways of symbolizing in an attempt to clarify a student's thinking either for herself or for other students. In contrast, during the second part of the Structuring Numbers sequence students completed individual activity sheets more frequently and were asked to make a record of their thinking so that others might understand their reasoning. These activities were subsequently discussed in the whole-class setting during which time Ms. Smith typically redescribed and notated students' explanations of their activity. As students participated in tasks from the instructional sequences, the use of notational schemes came to serve as thinking devices as students began to use the written records on the board as a means of reflecting on and comparing their own and other students' mathematical activity (Wertsch & Toma, 1995).

The Development of Number Sentences

In analyzing the development of number sentences in Ms. Smith's classroom, it is important to note that the use of *grouping* solutions emerged early in the school year as an accepted way to solve tasks (cf. McClain & Cobb, 1997). Grouping came to be interpreted as a solution process that involved grouping numbers or parts of numbers to quantify a collection as opposed to having to count the entire collection by ones. For instance, a student might identify the number of tiles in a domino pattern for six as *two groups of three* or as *a group of four and a group of two.* As Ms. Smith worked to support shifts in the students' ways of reasoning away from counting toward grouping, she would ask for *different* grouping ways to solve tasks. As a result, students learned to distinguish features of certain grouping solutions that caused them to be mathematically different. Although Ms. Smith still accepted counting solutions from those students whom she judged were currently unable to offer grouping solutions, grouping quantities to solve mathematical tasks became an accepted and valued way of solving tasks.

The emergence of number sentences as a means of recording the student's activity first occurred two weeks after the introduction of activities from the Patterning and Partitioning instructional sequence. Ms. Smith had developed the scenario of a pumpkin seller named Earl who sold his pumpkins in crates of ten. Using a single ten frame to represent Earl's crate, Ms. Smith posed problems by placing counters in the cells of the ten frame and asking students to determine how many pumpkins were in the crate (see Figure 1).

Figure 1
Ten Frame with Notation



Ms. Smith flashed (showed for three or four seconds on the overhead projector) a horizontal, single ten frame containing a certain number of chips and the students would tell how many chips/pumpkins they saw and how they saw them. Many responses built on the visual arrangement of the chips and were facilitated by the students' realization that each row in the crate held five pumpkins. However, it should be noted that some students attempted to count what they had seen by ones.

During one task, Ms. Smith flashed a horizontal, single ten frame containing four chips. In describing what she saw, Amy commented, "I saw three plus one." Ms. Smith then redescribed Amy's solution as "a group of three and a group of one" and notated it in vertical format to correspond with the placement of chips (see Figure 1). After the redescription and notation of Amy's explanation, Ms. Smith asked if anyone saw it a different way. Teri then responded that she saw "two going up and two going across." In response to Teri's explanation, Ms. Smith redescribed Teri's solution as "two groups of two" and notated as shown in Figure 1, careful to place the numerals so that, again, their placement corresponded with the placement of the chips. Through her notation, Ms. Smith implicitly indicated that she particularly valued both Amy's and Teri's solution method. The recasting of both explanations in terms of grouping further highlighted this fact. In addition, the number sentences provided a record for other students to use in reflecting on how Amy and Teri had each solved the task.

Throughout the Patterning and Partitioning instructional sequence, Ms. Smith regularly used conventional number sentences as a means of clarifying students' ways of reasoning. In this way, the notation served as a written description of the students' thinking. It is not clear what specific meanings the students may have given the number sentences. However, they appeared to be able to use these records as a way to compare and contrast their solution with ones that had already been offered. For this reason the number sentences supported students' reflection on their prior activity. This is indicated by the fact that in offering different ways to solve tasks, not only did students rarely duplicate previously offered solutions, but they also developed ways to clarify similarities and differences in theirs and others' solution processes. As a result, they were able to reason about features of the solution processes that allowed them to be interpreted as different. Further, their subsequent flexible use of number sentences indicates that they began to utilize the offered sentences as a way to think about and interpret their own strategies. The absence of individual student work and lack of direct evidence of the students' actual interpretation of number sentences prevent definitive claims. However, it is reasonable to suggest that students' participation in these classroom activities did give rise to learning opportunities which supported the development of number sentences as an accepted way of communicating mathematically.

As students continued to participate in activities from the Patterning and Partitioning instructional sequence during the next three weeks, an important shift occurred in the nature of the tasks in that the solution was no longer the total number of items (e.g. chips in the ten frame). The solution now entailed a process of grouping the collection in some way to find the total. In this way, students' prior activity of solving the task then became on object of reflection as they learned to judge for themselves what constituted a *different* solution. As a result of Ms. Smith's use of number sentences to communicate students' thinking, different was often interpreted in terms of how the solution might be symbolized. In other words, different solutions would require a different number sentence. This is a significant shift in that it now places the students' prior activity of solving the task in the background, diminishing the importance of simply finding the sum.

The Development of Notational Schemes

As students began participating in activities from the Structuring Numbers instructional sequence, the use of *going through ten* and *doubles* strategies emerged as taken-as-shared ways of solving tasks. For example, students often solved an addition task such as 7 + 8 by partitioning the eight into seven and one and reasoning, seven and seven is 14, and one more is 15. Ms. Smith devised a simple method of notating this reasoning by using an inverted "V" symbol that came to signify the partitioning or decomposing of a number. Ms. Smith would typically follow the "V" notation with the number sentences that expressed the result of the partitioning (see Figure 2).

Figure 2
Notating Decomposition of Numbers

7 + 8 =
/ \
7 1
7 + 7 = 14
14 + 1 = 15

It could be argued that some of the students did not actually conceptually partition the eight, but instead reasoned 7 + 7 = 14, so one more is 15. This created a situation where students' counting-based solutions were expressed as collection-based solutions although neither is arguably more sophisticated. As a result, the "V" did not necessarily fit with the students' activity. However, Ms. Smith introduced the "V" notation and called it "splitting."

As an illustration, consider an incident that occurred on December 7. Students were asked to work individually on sheets composed of context problems supported by a graphic. Their task was to make a record of their solution processes so that others might understand their reasoning. The focus in this part of the lesson was on effectively communicating their mathematical thinking — not imitating a given notational system. During the subsequent whole-class discussion, Ms. Smith asked students to share their solution methods verbally as she redescribed and notated their activity. The first task to be discussed was: *there are eight people on the bus and six more get on.* The first offered solution involved using a doubles strategy.

> Kitty: I took one off the 8 and I put it on to the 6 to make 7 plus 7 and I know 7 plus 7 makes 14.

As she spoke, Ms. Smith redescribed and notated the solution as shown in Figure 3.

Figure 3
Teacher's Notation to Describe Kitty's Solution

```
 8 + 6 =
 / \
7   1
7 + 7 = 14
```

After questions and discussion, Ms. Smith asked for a different way. Jane then explained that she partitioned the numbers differently.

> Jane: I stayed with the 6 but I broke it up into 3 and 3 and when it had the 3 it made 11 and 3 more... it made... uhm... it made 13 and one more is 14.

Again, Ms. Smith redescribed and notated the solution (see Figure 4), attempting to clarify to the students how Jane's explanation differed from Kitty's.

Figure 4
Notating Jane's Solution

```
 8 + 6 =
     / \
     3  3
 8 + 3 = 11
11 + 3 = 14
```

Symbolizing the two solutions offered opportunities for the other students to clarify for themselves how these solutions compared to each other and to their own. In doing so, the students came to be able to judge for themselves if they had solved the task differently. This decision was, again, based in part on how they would envision their method being symbolized. As a result, the notational schemes continued to provide a means of highlighting the critical aspects of different solution strategies, even in new task situations.

In examining the written records the students made on the activity sheets for the task of *there are eight people on the bus and six more get on*, it is important to note that there was diversity in the students' notational schemes (see Figure 5).

Figure 5
Sample of Students' Notation Schemes

```
 8 + 6          8 + 6                6   8                8 + 6 = 7
 / \            / \                  / \                  / \
 4 4            1   7              8  2  4   10 + 4 = 14   7  1
 6 + 4 = 10                                               7 + 7 = 14
10 + 4 = 14    6 + 1 = 7 + 7 = 14   8 + 2 = 10
```

Only the first student's way of notating is entirely consistent with that of the teacher. Although the other three students used elements of the teacher's scheme, they adapted them in original ways. Even when students' verbal explanations were redescribed and notated by Ms. Smith in a manner consistent with her original notational scheme, the students worked to devise notational schemes that expressed their thinking. As a consequence, although students might understand ways of talking about their activity that were expressed in terms of the teacher's notation scheme, they continued to solve tasks using a range of different, personally meaningful notation schemes.

As the instructional sequence progressed, Ms. Smith, on her own initiative, began to use the partitioning symbol "V" only if she judged that students had in fact partitioned an addend to arrive at the solution. It could be argued that in this way Ms. Smith was adapting to the students' solution methods; she stepped back from her own activity and reflected on the students' activity. An example can be seen in solutions to another task posed on December 7. The task is: *There are nine people on the bus and seven more get on.*

The first solution was offered by Anne.

> Anne: I thought about if I uhm. . . if I had ten and seven that would be seventeen and if I had nine and seven that would be sixteen. (Ms. Smith notates, see Figure 6).

Figure 6
Ms. Smith's Notating Anne's Solution

$$\begin{array}{c} 10 + 7 = 17 \\ 9 + 7 = 16 \end{array}$$

In this instance Ms. Smith judged that Anne had used the known relationship $10 + 7 = 17$ and simply compensated by subtracting one. This did not imply a partitioning of the ten.

Next, Dan offered a partitioning solution which was notated with the "V" symbol as shown in Figure 7.

Figure 7
Ms. Smith's Notating Dan's Solution

$$\begin{array}{c} 9 + 7 \\ \quad / \backslash \\ \; 1 \;\; 6 \\ 10 + 6 = 16 \end{array}$$

> Dan: The way I did it was kinda like her (points to Anne's solution as notated by Ms. Smith). What I thought of is also ten. I have to take one out of the seven to make nine [sic] and I have the seven then is six so I had to make ten so. . . ten plus six is nineteen. . . sixteen.

In this episode, Ms. Smith judged Anne's solution to be different from Dan's. She discriminated between Dan's partitioning solution and Anne's use of a known relationship and subsequent counting. This is evidenced by the manner in which she notated these two solutions. Initially the students had to adapt to Ms. Smith's notational schemes. Here, Ms. Smith is adapting to the students.

It is important to note that Dan was able to use the written description of Anne's solution process to compare how his solution was not only like but also different from Anne's. This is evidenced by his opening statement as he clarified that he also tried to make ten. However, he judged his solution to be mathematically different. This was made possible by his reflection on and analysis of Anne's solution as supported by the notation. In this way, the notational schemes became thinking devices that supported reflective shifts in the students' activity.

CONCLUSION

Throughout this paper, we have attempted to document crucial aspects of Ms. Smith's proactive role in supporting her students' mathematical learning by guiding the development of ways of symbolizing and notating. For Ms. Smith, the notational schemes emerged from the students' attempts to explain and justify their thinking. They were not predetermined schemes imposed by the instructional sequence. The analysis of Ms. Smith's role in introducing these schemes indicates that while they became taken-as-shared, the teacher played a central role in initiating their development. In analyzing the classroom interactions, it appeared that Ms. Smith's use of ways of symbolizing provided a way for the students to organize and reflect on their activity — providing an opportunity for shifts in thinking to occur. These ways of symbolizing enabled reflection on and analysis of the students' prior mathematical activity. This is evidenced by the fact that on their own initiative, students often referred to the notation to explain their thinking to other children during whole-class discussions. In addition, students began to use the records as a means of comparing solutions, thereby initiating shifts in the discourse such that features of their reasoning became explicit topics of conversation. As such, the students' participation in such discourse supported their reflection on and mathematization of their prior activity.

However, it could be argued that these ways of symbolizing were imposed by the teacher to fit with her interpretation of the instructional tasks. Although the development of notational schemes used in Ms. Smith's classroom appeared to support shifts in the students' use of notation, the lack of explicit discussion about the students' notational schemes could have arguably served to delimit the extent of this shift. After students had worked individually on tasks, the subsequent whole-class discussions did not focus on the students' ways of representing their thinking. What typically transpired was that Ms. Smith copied a problem statement on the white board and asked the students to explain how they thought about it. Although numerous mathematically different interpretations were solicited from the students, each was represented by Ms. Smith using her notational scheme. In reflecting back on this aspect of classroom activities, the research team acknowledged the need for students to discuss their own notational schemes. Conversations with Ms. Smith during the classroom episodes had focused on her redescribing and notating student responses in an attempt to initiate shifts in mathematical discourse. In retrospect it appears that the students' role in developing notational schemes might have been brought to the fore more prominently. Further, it could be argued that in problematic situations that arose involving notation, asking students how they might notate the *problem* would have provided students an opportunity to play a larger role in the establishment of the notational schemes.

Nonetheless, Ms. Smith's proactive role in guiding the development of ways of notating appears to have been critical in supporting her students' mathematical development. The children increasingly notated on their own initiative as they solved problems while working both individually and in groups. These records helped them distance themselves from their ongoing activity and thus reflect on what they were doing. Consequently, the use of notation contributed to the productiveness of whole-class discussions by helping to make individual children's contributions explicit topics of conversation that could be compared and contrasted. It was as they participated in these discussions that the teacher guided her students' transition from informal, pragmatic problem solving to more sophisticated yet personally-meaningful mathematical activity.

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Endnote

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Acknowledgement

The analysis reported in this paper was supported by the National Science Foundation under grant number REC 9604982 and by the Office of Educational Research and Improvement under grant number R305A60007.

LEARNING CALCULUS WITH SUPERCALCULATORS

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A CAS-active introductory calculus course was trialled with three year 11 classes of students who each had a TI-92 calculator. Access to the CAS enabled the students to perform "year 12" differentiations with the same level of success as year 12 students, but without affecting their conceptual learning or their development of by-hand skills. The use of the CAS was influenced by teachers' cognitive "privileging". Some observations are made on the suitability of the TI-92 for such a course.

INTRODUCTION

Computer algebra systems (CAS), such as Maple, Mathematica and Derive, have been used by mathematicians for at least 20 years, but it is only in recent years that advances in technology have enabled CAS to be incorporated into graphics calculators—producing so called 'supercalculators', such as the Texas Instruments TI-92 calculator. The changes currently being incorporated into the secondary mathematics curriculum as a result of the availability of graphing calculators are insignificant compared with the potential impact of supercalculators. This is illustrated, for example, by McCrae's (1996) finding that, whereas the availability of a graphing calculator would impact on only 6% of the 1994 Victorian Certificate of Education Specialist Mathematics paper, about 79% would be affected by the availability of a CAS. Similarly, Shumway (1989) found that as much as 90% of the exercises in most USA mathematics texts could be computed directly by a CAS.

However, as those mathematicians who have most strongly influenced the development of supercalculators from a pedagogic perspective have observed:

> We have much to learn about CAS in the teaching and learning of mathematics. We need to understand how CAS and paper and pencil procedures can and should co-exist. We need to learn what paper and pencil skills are necessary to use CAS effectively. We need to determine what mental computations in algebra and calculus are important for student learning. We need to learn how to test with CAS because there is much evidence that the only thing that really influences what our students learn is 'that what is tested'. We must learn how to teach the next level above 'doing' or practising procedures—thinking about mathematics. (Waits, Demana & Kutzler, 1997, p.5)

Tynan and Asp (1998) investigated the impact of the availability of a TI-92 calculator on the performance of year 9 students on a range of symbol manipulation tasks. They found that, compared to a class that used a graphing calculator with no in-built symbol manipulation features, the TI-92 students were more inclined to persist with algebraic methods for equation solving tasks. They also found that there was no significant difference between the two classes in their ability to perform by-hand algebraic manipulations.

Just as the transition from primary mathematics to junior secondary mathematics can be characterised by the introduction of algebra, the transition to senior mathematics can be characterised by the introduction of calculus. Hillel (1993, p. 29) has observed that CAS has the potential for affecting a major reorientation of the teaching of calculus so as to "focus on concepts, problem-solving skills and general investigative activities rather than manipulative skills", but adds that the "real tough" pedagogical question is "how much of the manipulative aspect can be eliminated while still sustaining conceptual learning"?

Keller and Russell (1997) note that despite the fact that several studies concerned with using a CAS in calculus have found improved understanding of the concepts of calculus with no significant loss of by-hand manipulative skill, concern remains that decreased emphasis on skills will lead to decreased ability to solve calculus problems symbolically. A major reason for the lack of impact of the research on practice is that in all of these studies either students' access to CAS was limited to computer laboratory sessions or the relevant examinations had to be done without access to CAS (or both)—meaning that there was still a strong incentive for students to develop their by-hand skills. The advent of supercalculators enables research to be conducted without these restrictions on CAS accessibility in what we might term a CAS-active environment.

In their study of first-year engineering students, Keller and Russell found that unrestricted access to a CAS actually empowered students to solve calculus problems symbolically. This paper reports some of the findings of a study that involved the development and piloting of a CAS-active introductory calculus course for year 11 students. The main objectives of the study were to determine whether the availability of the CAS enabled the students to solve more advanced problems than those normally encountered in such an introductory course, without adversely affecting their level of conceptual learning, and to investigate the effect of different teaching approaches in a CAS-active environment. Also, the general suitability of the TI-92 (the CAS platform involved in the study) for use in such a course was documented.

METHODOLOGY

During 1998, a CAS-active introductory calculus course was developed and then piloted in three year 11 classes in Victoria, two from a girl's grammar school and one from a State high school. Each student was loaned a TI-92 calculator for use in class and at home. The students were already familiar with the TI-83 graphical calculator and used the TI-92 in a two-week unit on circular functions prior to the commencement of the calculus unit.

Students from both schools were in the lowest ability mathematics classes in their year level and the classes were similar in size: there were 16 girls in one of the grammar school classes and 24 in the other, whilst there were 19 students (13 girls and 6 boys) in the high school class. The three teachers were experienced in teaching year 11 and 12 mathematics with the TI-83 graphical calculator but inexperienced in using the TI-92 with its additional CAS capability. The three teachers had quite different teaching styles.

Twenty lessons were devoted to teaching differentiation with a much stronger emphasis on conceptual understanding than had been the case in previous years. Introductory rates of change were explored using real data with a CBR linked to the TI-92. A program was written to demonstrate dynamically the connection between the tangent at a given point and a series of moving secant lines drawn very close to the point. The derivative was described in numerical, graphical and algebraic representations. Students were encouraged to discover the rules for differentiating polynomials by induction and given the opportunity to apply their knowledge in a wide variety of contexts.

The overall scope of the course was identical to that normally covered in the same period and included some topics that officially are part of the year 12 course: deducing the graph of the gradient function from the graph of a function, derivatives of rational powers, and the chain and product rules. The extra time devoted to concept development and applications was offset by less time spent on practising differentiation techniques, with the teachers being asked not to place the usual emphasis on the development of by-hand skills. (A two-week block had been set aside after the completion of the course for student's by-hand skills to be brought up to scratch if this proved necessary.)

The students were given an informal test about half way through the seven-week teaching period. In most weeks they completed a brief questionnaire and a few short questions designed to monitor their progress and log sheets which gave them the opportunity to describe their feelings about using the TI-92 to learn calculus. Three tests were administered in week 8 and interviews were conducted with 17 selected students the following week. The teachers completed journals and a questionnaire and participated in a taped interview and regular informal discussions. A Research Assistant observed approximately half of the lessons taught and completed a journal after each lesson.

The students were able to use their TI-92 in the first two of the three final tests, but were only permitted to use a (non-graphics) scientific calculator in the third test. At the end of each question part on Tests 1 and 2, a YES or NO box had to be checked to indicate whether the TI-92 had been used to help answer the item. The tests were not typical year 11 calculus tests. They were designed to test the students' conceptual understanding and whether they could do relatively advanced differentiations. Many of the items on the tests came from past year 12 examination papers.

Test 1 consisted of 14 multiple-choice questions and several items requiring short answer responses. Test 2 consisted of items requiring extended responses and Test 3 comprised a selection of items from Tests 1 and 2. For comparison, the multiple-choice questions were also given to another year 11 class (15 students), of comparable mathematical ability, and one mainstream year 12 class (21 students) from the high school. The year 11 class had just completed the normal introductory calculus course and the year 12 class had completed the calculus part of their course; neither of these 2 classes had access to a CAS for the test. This paper looks at the comparative performances of the five classes on the 14 multiple-choice questions.

RESULTS AND DISCUSSION

Table 1 shows the mean scores of each class for 13 of the multiple-choice questions (one question has been excluded from the analysis because not all classes covered the topic). Each question was awarded 0 (incorrect) or 2 (correct), so that the maximum possible mark was 26. Classes A, B and C are the year 11 CAS classes, class D is the year 11 non-CAS class and class E is the year 12 class. The values of N are given throughout this discussion, since not all students were present for each test.

Table 1
Mean Scores for Multiple-Choice Questions by Class

	Class A	Class B	Class C	Class D	Class E
N	15	19	16	15	21
Mean	14.0	14.1	14.3	8.4	17.0

The means for the three CAS classes are much higher than the non-CAS year 11 mean and although they aren't as high as the year 12 class mean, they are quite pleasing in view of the difficulty of the questions. It is likely that the CAS students must have used their TI-92s to advantage and this will become clearer in the following analysis.

Twelve of the 13 questions can be readily classified as either being a *core* item or a *symbolic* item (see Kendal & Stacey, in press). Core items have high conceptual and low procedural demands; there is no advantage in using a CAS in such questions. One example was question 9, which gave the graph of an unknown function and required the graph of its derivative to be selected from five alternatives. Symbolic items have high demands on algebraic procedures and low conceptual demands; use of a CAS can be a distinct advantage in answering such questions. One example was question 3: "If $y = (4x^2+9)^7$, then dy/dx is equal to ..." Access to a CAS makes this question almost trivial.

The first 6 multiple-choice questions were symbolic items and these questions were also included on Test 3—the test given to the CAS students about a week after Test 1 and for which the students were not allowed to use their TI-92s. Table 2 shows the mean scores of each class for these questions, including the second attempts for classes A, B and C. Only those students who sat for both Tests 1 and 3 were included in the calculations for classes A, B and C (N=15 for class A, 16 for class B and 13 for class C).

Table 2
Mean Scores for Symbolic Items by Class

Question:	1	2	3	4	5	6	Total
Class A—with CAS	1.87	1.73	0.80	1.60	1.33	0.80	8.1
Class A—without CAS	1.87	0.40	0	0.13	0.80	0.80	4.0
Class B—with CAS	2	1.63	1.00	1.50	1.00	1.25	8.4
Class B—without CAS	1.75	0.63	0.50	0.75	1.00	0.88	5.5
Class C—with CAS	2	1.54	0.92	1.69	0.62	0.77	7.5
Class C—without CAS	1.54	0.15	0	0	0.77	0.77	3.2
—with CAS mean	*1.95*	*1.64*	*0.91*	*1.59*	*1.00*	*0.95*	*8.0*
—without CAS mean	*1.73*	*0.41*	*0.18*	*0.32*	*0.86*	*0.82*	*4.3*
Class D (year 11)	1.73	0.27	0.13	0.13	0.53	0.67	3.5
Class E (year 12)	2	0.86	1.05	0.95	1.43	1.43	7.7

The "with CAS" class mean totals are higher than the year 12 class mean total for two of the classes (A and B), and comparable for the other class (C). Notwithstanding the variations between questions, this indicates that access to a CAS enables year 11 students to solve symbolic items of "year 12 standard" with the same degree of success as year 12 students. Also, the "without CAS" class mean totals are higher than the non-CAS year 11 class mean total for two of the classes (A and B) and slightly lower for class C. This is in agreement with the results for question 1, the only one of the questions that was clearly of year 11 standard (a cubic differentiation), and indicates that the CAS students did not suffer unduly from less emphasis on by-hand skills in their course. The variations in results between CAS classes will be discussed later.

Table 3 shows the mean scores for each of the five classes on 5 questions that were clearly core items. (One "clearly core" question has been excluded from the analysis because, arguably, it has two answers.) Values of N are as given in Table 1. The scores indicate that the conceptual understanding of the CAS students was generally better than that of the non-CAS year 11 students, but not as good as the year 12 students. The results are more favourable to the CAS students if question 7, on which each CAS class performed particularly badly, is ignored. This question required students to calculate the slope of a curve at two points by calculating the gradients of the tangents at those points. Analysis of students' responses revealed that they were just as likely to calculate the slope of the secant through the two points or to choose the relevant x- or y-coordinate as the slope of the curve. This aspect of the course obviously needs re-examination before further trialling.

Table 3
Mean Scores for Core Items by Class

Question:	7	9	10	11	14	Total	Total (without q7)
Class A	0.27	0.67	0.93	1.47	0.80	4.1	3.9
Class B	0.21	1.05	1.16	1.16	0.74	4.3	4.1
Class C	0.38	1.00	1.00	1.75	1.13	5.3	4.9
A, B, C mean	*0.28*	*0.92*	*1.04*	*1.44*	*0.88*	*4.6*	*4.3*
Class D	0.67	0.53	0.40	1.73	0.67	4.0	3.3
Class E	1.62	1.05	1.52	1.91	0.86	6.9	5.3

Comparing the CAS classes to each other, class C students demonstrated the highest level of conceptual understanding. To help determine which class made best use of the CAS, Table 4 further analyses the responses to the 6 symbolic items referred to in Table 2. For each class, it shows the percentage of total student responses in which students reported using their TI-92, and the "overall", "with CAS" and "without CAS" success rates. It indicates that greater success does not necessarily follow from higher CAS use: class A students chose to use their TI-92 almost twice as often as did class B and C students, but were less successful overall than class B students. On the other hand, it is likely that class B and C students would have profited from using their CAS more often.

Table 4
Percentage of Items Correct by CAS use and Class

	Class A (N=15)	Class B (N=19)	Class C (N=16)
Total no. of items	90	114	96
% correctly answered	67.8	69.3	60.4
% for which CAS used	64.4	36.8	35.4
% correct when CAS used	79.3	95.2	82.4
% correct otherwise	46.9	54.2	48.4

Kendal and Stacey (in press) extend this analysis to the remaining Test 1 and Test 2 questions, some of which are *options* items: they can be solved graphically or algebraically. They also look at the incidence of conceptual and procedural errors and compare the characteristics of the three teachers involved. Kendal and Stacey's summary of the relative behaviours of the three CAS classes is reproduced as Table 5.

Table 5
Summary of the Behaviours of the Three CAS Classes (Kendal & Stacey, in press)

Behaviour	Class A	Class B	Class C
Use of calculator	most frequent	least frequent	frequent
Decision to use calculator	too frequent	discriminating	discriminating
Preferred approach	algebra by CAS	algebra by hand	graphical
Algebra proficiency	moderate by hand	higher by hand	lower by hand
Graphical skills	lower	moderate	higher
Procedural competence	good	good	good
Conceptual understanding	lower	moderate	higher

Berger (1998) describes Wertsch's concept of "privileging" as "the social setting and values which may elevate one form of mental functioning over another and in this way privilege a particular form of mental operation such as algebraic or graphical reasoning" (p. 19). Kendal and Stacey attribute the differences between the behaviours of the three classes to the different privileging of the teachers involved. According to them, Teacher A privileged technological and algebraic approaches, Teacher B privileged conceptual understanding and by-hand algebraic approaches and Teacher C privileged graphical approaches and conceptual understanding.

SUITABILITY OF THE TI-92

The students in the CAS classes were very enthusiastic about the ability of the TI-92 to "do both graphs and algebra". Otherwise, the feature that they most often regarded as "great" was the ability to view and recall previous work. The most frustrating feature was the TI-92's general complexity: "too many buttons and not knowing what they all do", "there are too many pull down menus and they are too complicated". It would have helped if each student could have been given a manual, but a better user interface is what is really needed.

The fact that (identical, but) different factorisations can arise depending on the mode setting often led to confusion: for example, factor($2x^2 + 3x - 2$) gives $(x+2)(2x-1)$ in auto or exact mode but $2.(x-.5)(x+2.)$ in approximate mode. However, as noted by Tynan and Asp (1998, p. 626), probably the most frustrating feature of the TI-92 is the way in which its auto-simplify feature alters the form of expressions. Any reader who doubts this is invited to try and derive the formula for the solution of the general quadratic equation in its usual form.

CONCLUSION

Before long, supercalculators will cost no more than non-CAS graphing calculators do at present. Every kid in school will have one and we must be prepared. This study indicates that, if calculus is introduced with a strong emphasis on conceptual understanding, then year 11 students can use a CAS to do "year 12" problems. It will no longer be necessary for students to spend so much time learning differentiation and antidifferentiation techniques; many common assessment items will be trivialised. We must act now to revise mathematics syllabuses and assessment regimes accordingly.

Further studies are needed to investigate the impact of CAS on the learning of calculus. Perhaps more importantly though, its impact on the learning of algebra needs thorough study. Indications are that, paradoxically, a strong algebra facility is needed to get the best out of a CAS. Each CAS has a mind of its own and weak points that need to be overcome.

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TEACHER MEDIATION OF STUDENT CONSTRUCTIONS OF ALGEBRA KNOWLEDGE

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This study suggests that classroom research into students' knowledge construction is incomplete without data about teachers' knowledge constructions and practice. Teachers' knowledge related to algebra and teaching algebra were identified and categorised using three knowledge domains identified by Shulman (1987) as knowledge of epistemology, pedagogy and pedagogical content of a algebra. Diverse and idiosyncratic examples of algebra knowledge were articulated by each of the study teachers in relation to the epistemology and pedagogy of algebra, but examples of pedagogical content knowledge were seldom easily identified.

Previous studies about teachers' knowledge and beliefs have mentioned the relationship between teacher knowledge on student learning (Southwell, 1997; Dossey, 1992), and the need for improved teacher knowledge (Borko and Putnam, 1996; Dossey, 1992; Mitchelmore, 1995; Prawat, 1992; Shulman, 1987). My research looked beyond these observations and suggests that there are important if not critical reasons for not only acknowledging but characterising teachers' knowledge constructions when investigating classroom learning. Shulman states that the teacher

> communicates, whether consciously or not, ideas about the ways in which "truth" is determined in a field and a set of attitudes and values that markedly influence student understanding. (Shulman, 1987, p. 9)

Accepting Shulman's (1987) view, and the views of the researchers mentioned above, suggests that classroom research into students' construction of knowledge cannot be divorced from investigations into teachers' knowledge constructions. Classroom learning is viewed in the present study from the perspective of teacher mediation between a content domain and student knowledge constructions, and emphasises that knowledge construction in classroom settings is the result of a partnership between the teacher and students where knowledge is developed as it is shared. This research built upon the idea of "classroom negotiation of meaning" (Clarke, 1996). Clarke's (1996) study was concerned with two foci. One was the individual students' sense making. The other foci was the "social domain within which teachers' and learners' individual contributions play a key role in social sense-making and in the utilisation of mathematics and science concepts and skills" (Clarke, 1996, p. 4). Clarke's (1996) research emphasises that research into classroom learning involves examination of the "symbiotic relationship of social and individual negotiation of meaning: symbiotic in the sense of being mutually dependent and mutually supportive" (Clarke, 1996, p. 4). This suggests that the teacher is an important element in the highly complex learning environment of the student. Whether researchers adapt constructivist or socio-constructivist views of learning, acknowledgment is made of the significance of individual knowledge constructions (Lerman, 1994). This suggests that if teachers each have their own conceptions of mathematics then the knowledge constructions of teachers must play a crucial role in the knowledge construction of students. Classroom research needs to investigate the nature of teachers' knowledge constructions and particularly teachers' interpretations of specific content domains (such as algebra) and how teachers choose to make these interpretations visible for students via different representational forms.

The research design of this study therefore required the development of instruments and methods for identifying and analysing the forms of teacher knowledge constructions. Because learning cannot be separated from the effects of cultural influences and personal values (Southwell, 1997), beliefs about and reasons for teachers' choices of content, content

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representation and style of instruction also formed important aspects of the present research. The diverse nature of the data required for the current study made it obligatory to include in the research methods: Classroom observations, questionnaires and interviews. The data collection instruments created opportunities for teachers to express their knowledge and beliefs about algebra in each of these knowledge domains. This paper provides an explanation of the ways in which this research complements classroom research into students' knowledge construction.

A Methodology for Investigating Teacher Knowledge

A useful understanding of knowledge construction in classrooms must recognise the form of teachers' knowledge constructions and the form in which teachers' express their understandings within a particular content domain. This means that research into students' construction of knowledge:

1. Must acknowledge and incorporate teacher interpretation and representation of prescribed academic content,
2. Must include document the nature of teachers' understandings,
3. Must include research into the form and patterns of teachers' instructional practices and, the identification of the criteria used by teachers to select and construct instructional materials and in making pedagogical decisions that lead to their instructional behaviours.

The methods used in this research were designed to ensure a comprehensive investigation of teacher knowledge with respect to the epistemology of algebra, the pedagogy of algebra, general pedagogy, general learning theory and the learning of algebra. Each of the research components outlined in points 1 to 3 above are described in greater detail below.

An emphasis in this research was teachers' conscious and unconscious efforts to mediate learning. A major premise was that teachers' unique constructions of their knowledge and beliefs form a significant factor influencing the creation of possibilities for instruction. The thesis presented here views the function of teacher mediation in the learning process as involving first the teachers "interpretation" of mathematics content and the "representing" of this content for students in accordance with the teachers' personal conceptions. Each of these terms "interpreting', "mediating" and "representing" are used for a specific purpose in this study to make clear different aspects of the teaching-learning process. Each of these terms are described below in relation to algebra.

Interpreting

Teachers draw on their experiences with algebra in different contexts over time to form their "interpretations" of "algebra". Part of what teachers regard to be algebra is situated as personal knowledge, knowledge they see no need to communicate, part is public knowledge, knowledge they formalise for the purpose of communicating to others, and part of their algebra knowledge they identify as suitable content for students. For the purpose of teaching, interpretation is predicated on an understanding of the structure of the subject matter and the principles of conceptual organisation of the content domain (Shulman, 1987).

Mediating

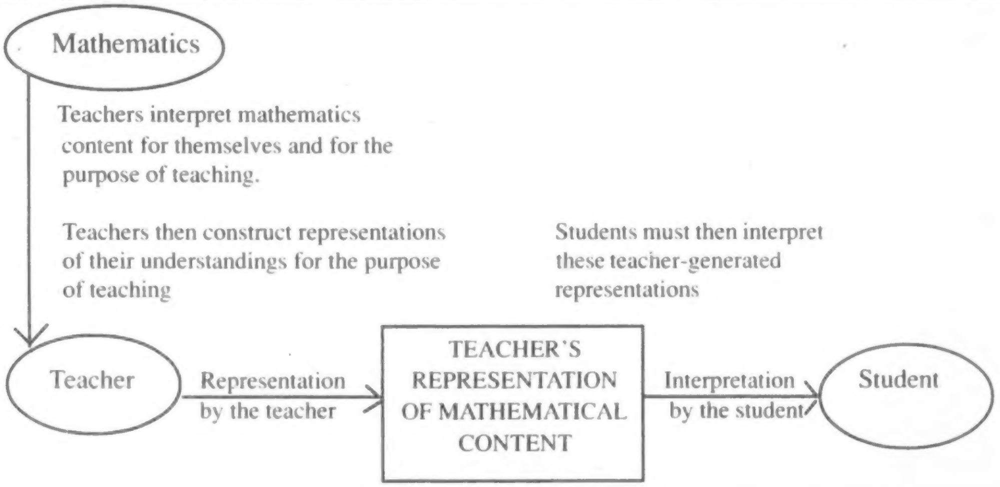
Teachers are in the position of having to "mediate" between recognised algebra content and their students' conceptions of algebra. Mediation begins with teachers' understanding of algebra, algebra curriculum, students' stages of conceptual development, how students conceive algebra knowledge and student' understanding of algebra. The mediation process involves teachers translating aspects of their personal epistemologies of algebra into representational forms they believe will be both useful and comprehensible to students. Mediation is conceived, in this study, as a dynamic process whereby teachers are continually responding to the evolution of students' knowledge constructions.

Representing

The form of teachers' representations of algebra included the algebra content they choose to include, the time allocated to different aspects of algebra, instructional methods (for example, expository styles or activity based), the representational sign systems used by the teacher (for example, words, pictures, patterns, metaphors, and graphs) and also the context in which teachers use different representations (for example, real world problems, and the extension of ideas of arithmetic).

The relationship of the acts of "interpreting" and "representing" within the overarching process of "mediation" in classroom settings is illustrated in Figure 1 below.

Figure 1:
A model of Teacher Mediation between Mathematics Content and the Student


Mathematics
Teachers interpret mathematics content for themselves and for the purpose of teaching.
Teachers then construct representations of their understandings for the purpose of teaching
Students must then interpret these teacher-generated representations
Teacher
Representation by the teacher
TEACHER'S REPRESENTATION OF MATHEMATICAL CONTENT
Interpretation by the student
Student

The model depicted in Figure 1 postulates a structure for the process whereby the teacher mediates between content and student. What is missing from Figure 1 is demonstration of the responsiveness within the mediation process, whereby the teacher acknowledges student interpretations and consequent mathematical constructions, elicits representations of the students' knowledge, and than adapts or modifies or elaborates the initial representations in ways that promote correct mathematical interpretations and constructions by the students. This "negotiative" process is discussed at greater length in Clarke (1996).

Research Principles

In the course of this research, several principles were developed that guided data collection. These principles appear to have general applicability to research into classroom teaching and learning.

Principle 1. Research into student constructions of knowledge must acknowledge and incorporate teacher interpretation and representation of prescribed academic content.

The mediating role of the teacher can be understood in terms of socio-constructivist theories of learning in which the teacher orchestrates social exchanges that give rise to knowledge construction both of the teacher and student. The practicalities of this mediating role involve interpretive acts on the part of both teacher and student, and the anticipation on the teacher's part of an audience for their representations. This suggests that teachers and students are involved in what Clarke (1996, p. 2) describes as "negotiation of meaning". It is our contention, based on this research, that comparable attention must be paid to the teachers' learnings as is commonly paid to the learning of the student.

Principle 2. Research into student construction of knowledge must include adequate study of teachers' understandings.

While recognition must be given to the sources of teachers' knowledge construction of academic content (such as might be accessed from textbooks and pre-service programs and inservice training), the resulting teachers' conceptions are not always explicit or visible. Demonstrations of teachers' understandings will include actions such as teaching practices, together with teachers' espoused beliefs and epistemologies. Any study of teacher understandings must distinguish between teacher knowledge and teacher practice. The following list builds upon the work of Shulman (1987) to identify specific indicators of teachers' understandings. These are:

(a) content knowledge;
(b) pedagogical content knowledge;
(c) general pedagogical knowledge;
(d) general instructional practices;
(e) content specific pedagogical practices;
(f) knowledge of learning processes;
(g) curriculum knowledge.

In this study, each of these indicators was studied from multiple perspectives.

Principle 3. Research into students' construction of knowledge must include research into the form and patterns of teachers' instructional practices and the identification of the criteria used by teachers to select and construct instructional materials and in making pedagogical decisions that lead to instructional behaviours.

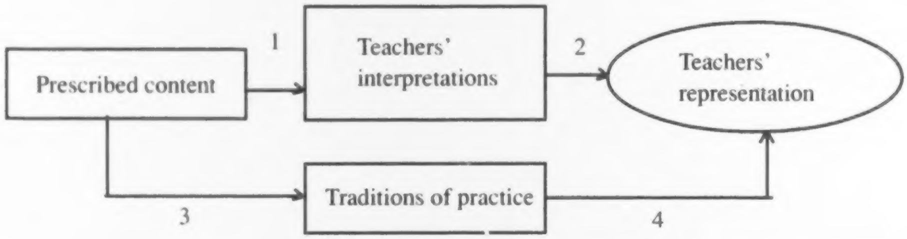
This research needs to address the manner in which teachers reflect upon their knowledge and beliefs about content and practice and the way teachers observe traditions of practice and examples of exemplary practice. Documentation of such reflections and observations should include:

- how teachers construct representations of curriculum content,
- how the representations are delivered to students,
- how teachers assess students' understanding and
- how teachers intervene in students' knowledge constructions.

Teachers may refer to their personal interpretations of prescribed content, or suggested teaching material and activities when they make choices about instructional materials and practices. Teachers' choices will depend, however, upon how they conceive the activity of teaching, learning and what constitutes important content for learning. These choices are illustrated in Figure 2.

Functioning as reflective practitioners, teachers could interpret prescribed content (1) and then develop appropriate representations of that content (2) to sustain student learning. However, it might be that the specification of content in a curriculum document cues the teacher's use of a particular instructional practice (3) (within the traditions of practice for that subject) and that the chosen practice effectively prescribes the representations to be used (4).

Figure 2
Alternate Pathways to Teacher Representations



Conceivably a teacher may devote significant lesson preparation time to lesson designs based on their knowledge of traditions of practice and innovative lessons documented and promoted by "experts". The pathway from prescribed content to teachers' representations of this content along arrows 3 and 4 conceivably bypasses any need for a teacher to first reflect on their personal interpretation of the prescribed content. Teachers' active construction of representations based on their interpretations of prescribed content and their knowledge of their students' states of knowledge is represented by arrows 1 and 2. The importance of teachers utilising both traditions of practice and their personal knowledge is illustrated by the research findings summarised later in this paper.

Principle 4 Pedagogical content knowledge is important

Throughout the research it became clear that it was difficult to identify, using primary data, examples of teachers' knowledge of pedagogical content knowledge. This raised the question: What value to researchers or educators is Shulman's (1987) concept of pedagogical content knowledge? Finding few examples of pedagogical content knowledge in the data set generated by this research suggested that either:

- knowledge cannot usefully be categorised as pedagogical content knowledge,
- the teachers possessed little pedagogical content knowledge, or
- pedagogical content knowledge exists and is important but not in a form that is easily witnessed during classroom observations, interviews or from questionnaires.

The research data suggested that teachers' pedagogical content knowledge would be most evident when the researcher discussed or observed specific incidences of student learning and teacher intervention methods with the teacher. In the case of interviews, the teachers were asked to identify and describe the concepts a particular student needed to have developed to complete a task, this student's current stage of concept development with respect to the task, the concepts the student was failing to understand and how that concept might be expressed to the student. Teachers' responses provided evidence of the teachers' pedagogical content knowledge.

Important Research Findings

Finding 1. The teachers' beliefs about content specific instructional practices were similar despite quite individual interpretations of algebra content.

A finding of this research was that while the teachers' personal epistemologies were quite diverse, their beliefs about content-specific instructional practices were quite similar.

A pattern evident in the research data related to the teachers' views about the importance of "instrumental understanding" (Skemp, 1978) of algebra and the teachers' beliefs about students' construction of knowledge. Each of the teachers studied encouraged rote learning of algebraic procedures and justified their decisions with respect to content and instructional method in terms of providing students with the instrumental understanding the teachers believed students required for their VCE studies. The teachers also explained that "the more able" students would construct their personal understanding of concepts associated with algebra skills and procedures as a consequence of routine practice and would gain no particular benefit from explorations. The teachers' adherence to teacher centred, expository, drill and practice teaching strategies was consistent with the teachers' reported beliefs about how students learn and develop concepts.

The findings above suggest that the form of algebra teaching conceived by the teachers in this study could be carried out by teachers with minimal pedagogical content knowledge and minimal teacher reflection about reasons for students' difficulties, misinterpretations or insights. This is a significant finding with respect to facilitating a better understanding of the practices of the mathematics classroom and of students' knowledge construction in classrooms.

Finding 2. There was often little association between the teachers' reported beliefs about the pedagogy of algebra and the teachers' observed instructional practices.

For the teachers investigated there was often little association between the teachers' knowledge and beliefs about broad algebra concepts they reported to be important aspects of a conception of algebra and the algebra they actually taught. These contradictions included:

1. The teachers' observed focus for instruction tended to be concerned with instrumental rather than relational understanding (Skemp, 1978) even though they reported valuing relational understanding.
2. The teachers expressed a belief that students need a context to make sense of algebra yet the teachers:
    a. usually taught algebra out of context;
    b. often taught applications as a separate topic or activity without relating the application to skills learned in other topics;
    c. made few obvious attempts to link algebra topics to each other or to other student knowledge.

The research data identified the contradictions between the teachers' algebra instruction and comments about how students learn algebra derived from:

1. the teachers' limited knowledge of the epistemology and pedagogy of algebra.
2. the teachers' limited knowledge of the concepts students need in order to develop other algebra concepts; that is, learning hierarchies as these relate to algebra.

3. the teachers' attempts to adopt content and suggested activities set out in curriculum documents without reflecting on their intended function and how the new content or activities might make algebra more accessible to students.

The consistencies and inconsistencies evident in the research data suggested an additional seven findings worthy of discussion. Each of these additional findings arose from analyses of the data from the perspective of the teacher as mediator during the student-content interactions that happen in mathematics classrooms. This perspective for the analyses made possible some extremely useful reflection on classroom learning and suggested implications for the classroom teaching and learning, and for further research. It is not possible to address all these additional findings within the scale of this paper, however, it was felt useful to list them below:

1. The teachers' curriculum choices were seldom influenced by the teachers' knowledge and beliefs of the epistemology and pedagogy of algebra but were often influenced significantly by the teachers' beliefs about "good teaching",
2. Teachers need to assist students to make the transition from procedural to structural conceptions of algebra, and recognise the importance of students' understanding of arithmetic for developing an understanding of the structural aspects of generalised arithmetic and algebra,
3. Teachers' descriptions of school algebra emphasised the manipulative aspect of algebra but they did not attend to related structural aspects,
4. The teachers' diagnosis of student difficulty with algebra did not relate to concepts identified in the literature as important for learning algebra,
5. There was a need for teachers to make connections within algebra during their teaching,
6. Teachers need to understand how students construct knowledge and
7. There was a need for the teachers to make connections between algebra content and content beyond algebra.

One general conclusion could be drawn from the combination of findings arising from this study: Teacher reflection is important.

An important finding of this research was that the teachers' often lacked the detailed knowledge needed to both identify specific student difficulties and construct situations that might assist students to overcome their difficulties. In the absence of such pedagogical content knowledge, the teachers in this study had developed beliefs that effective algebra teaching involved taking students through prescribed sequences of instruction along with the incorporation of suggested activities described in the *Curriculum and Standards Framework* (Victorian Board of Studies, 1997) documents and other teacher valued resources.

Generally the teachers believed that task selection did not require their detailed knowledge as this was the responsibility of course advisers and textbook writers. Their teaching focus was curriculum implementation as advocated by curriculum authorities.

Figure 2 above can be used to explain why these dedicated teachers were disappointed by the learning outcomes of their students. Each teacher explained that many students in their classes had conceptual difficulties with algebra but the teachers were seldom able to identify specifically the concepts the student had difficulty with. The teachers usually explained student difficulties in terms of readiness to learn abstract concepts, lack of attention or lack of practice. The more experienced teachers sought out other resources that might provide more effective teaching material. It became clear throughout the research that the teachers were attending to the process of teaching rather than learning. Paying attention to traditions of practice and new and innovative instructional materials and activities meant that the teachers were distracted from their own knowledge constructions and those of their students and hence saw no need to draw upon their personal knowledge to comprehend their students' learning. Remediation of student misconceptions or lack of understanding requires that teachers develop reflective behaviours with respect to their knowledge of content and of the learning process and their students' knowledge construction.

This suggested that the development of pedagogical content knowledge needs to occur in the context of student learning and has no meaning without it. Pedagogical content knowledge is developed as a consequence of teaching and engaging in the teaching-learning process. This interpretation of the data has important implications for teacher education programs and for research into the meaning constructions that occur in classrooms.

CONCLUSION

This research suggested reasons why it is important to recognise and incorporate teacher interpretations and representation of prescribed academic content when researching students' construction of knowledge. It was suggested that data about teachers' knowledge constructions will both add context and detail to classroom research into students' construction of knowledge. Without the inclusion of teachers' knowledge construction the interpretation of students' knowledge constructions may be flawed or incomplete. Acknowledgment has not been widely given to possibilities of students' derivations of meaning from their teachers' personal representations of their (the teachers') idiosyncratic interpretations of particular algebra content. A finding in the current study was that the teachers studied did have individual conceptions of algebra content. The fact that this small sample of teachers differed in their epistemologies of algebra suggest that classroom implications of this diversity are much wider than research currently suggests. This raises an interesting question for future research: To what extent do we want teachers to express their individual conceptions of content in their teaching, and to what extent do we want teachers to deliver "the same content"? We would argue that if teachers' have a detailed knowledge of the structure of algebra, differences in their conceptions are likely to be ones of emphasis and depth of understanding rather than of the basic nature or spirit of the concepts and skills.

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HOW TO TEACH GENERALISATIONS IN MATHEMATICS?

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This paper is in three parts. In the first part, I compare and contrast three typical methods of teaching generalisations in mathematics. In the second, I describe the theory of generalisation expounded by the Soviet psychologist Vasily Davidov. In the third part, I re-examine the three methods of teaching mathematical generalisations in the light of Davidov's theory, and make some general conclusions.

For some time, Paul White and I have been promoting the importance of the abstraction process in mathematics learning (Mitchelmore & White, 1995; White & Mitchelmore, 1996). This approach is a reaction to decontextualised mathematics teaching. It is based on the assumption that abstract mathematical generalisations are reached by searching for common elements in a range of *several* concrete situations. On the other hand, we also believe in the problem-solving approach, where a student's struggle with *one* problem may be worth twenty done as exercises following a method given by a teacher. How are these various methods of learning and teaching related?

THREE TEACHING METHODS COMPARED

The ABC Method

What I call the ABC method—teaching Abstract Before Concrete—is widely used in mathematics teaching. The theory is that "knowledge acquired in 'context-free' circumstances is supposed to be available for general application in all contexts" (Lave, 1988, p. 9). Steinbring (1989) shows that many school mathematics teachers believe that procedures should be learnt before applications, and textbooks and even curriculum guides often reflect the same order (Mitchelmore & White, 1995).

The ABC method has been widely criticised for leading (at the best of times) to *abstract-apart* knowledge—superficial knowledge that cannot be applied to problem situations and is quickly forgotten once examinations are over (Schoenfeld, 1988; White & Mitchelmore, 1996). Speaking of students who have been taught by the ABC method, Dreyfus (1991, pp. 28) writes: "They have been taught the products of the activity of scores of mathematicians in their final form, but they have not gained insight into the processes that have led mathematicians to create these products."

Empirical Methods

Diametrically opposed to the ABC method is the "concrete to abstract" approach espoused by many educators. In one manifestation of this approach, students are encouraged to model the pattern-seeking behaviour which is said to be fundamental to mathematics using concrete (manipulative or figural) materials. Two examples:

- Grade 4 students draw squares of dots: 1 by 1, 2 by 2, 3 by 3, and 4 by 4. They count the number of dots and write down the sequence obtained: 1, 4, 9, 16. They notice that these numbers go up by successive odd numbers (3, 5, 7) and infer that the next number must be 9 more than 16. Bingo! 25–it works.
- Grade 8 students draw a triangle, of whatever size and shape they like. The students then measure the three angles of their triangle. Answers vary around 180 degrees. The teacher calls this the "angle sum of a triangle theorem" and states that the angle is always 180 degrees (if you could measure it exactly).

Activities like these do show how new mathematical knowledge is often discovered, but the results are still mysterious. Some obvious questions are simply not asked: "*Why* do square numbers go up by odd numbers?" "Are you *sure* the angles in a triangle always add up to 180 degrees? Exactly?" If these questions are not asked, there is a constant danger of making false inductions. More importantly, it is answering the "Why?" questions which explains the results, and this is surely the essence of mathematics.

A Problem Solving Approach

Another reaction to the ABC method is to have students model another supposedly fundamental mathematical behaviour: problem solving. The tactic is to pose a problem and then leave students to solve it and convince others the solution is correct. An example I often use is: *Can you tessellate the plane using a scalene triangle?* (Mitchelmore, 1998, Activity P08). All sorts of geometry concerning congruence, angles, and parallels arise and become connected—in particular, angles on a straight line; the angle sum of a triangle; and corresponding, alternate, and co-interior angles formed by a line intersecting several parallel lines. In my experience, a period or two wrestling with this problem produces a far deeper understanding of these concepts and results than teaching each one separately (even using an empirical approach).

A Comparison: The Three Approaches to One Topic

The differences between the three approaches are best illustrated by considering how they would be used to teach one topic. This example was suggested by Boero and Garuti (1994), who describe a classroom investigation of indirect height measurement using shadows cast by the sun. All the Grade 8 students seemed to grasp the proportion method of calculating the inaccessible height, but most could not transfer the result to similar triangles in general. What should the teacher do next?

One method would be to present the theory of similar triangles, showing how this theory applies to height measurement and then applying it to other contexts. This is a variant of the ABC method. By presenting the theory after one application, students should at least be able to link the abstract concept of "similar triangles" to something concrete. But I predict that students would still not appreciate the generality of the concept, and would learn to manipulate problems in an abstract-apart fashion.

A second possible method is to study further contexts where similar triangles occur (e.g., scales, enlargements, gradients) and learn how to make the corresponding calculations in those contexts. Then, students can be led to notice the similarities between all such cases–between the objects studied, the relations between them, and the method of calculation. The common features can then be identified and abstracted: the objects are points, lines, and triangles; the crucial relation is similarity; and the method of calculation is proportion. This is a variant of the empirical method discussed above; the patterns that are sought in this case are much deeper, involving comparisons between relations, but the principle is the same. The concepts and relations abstracted would not be abstract-apart but *abstract-general* (i.e., general to many contexts).

A third method, a variant of the problem-solving method above, is suggested by Dörfler (1991). Take the result obtained from height measurement and ask the question: What conditions in the situation are *essential* to the result obtained? For example, the calculation involves two vertical and two horizontal lines; *must* the lines be vertical and horizontal? A short thought experiment (just rotate the page) is enough to see that the answer is clearly "no"–but at least the corresponding lines must be parallel. Must these two lines be perpendicular? This question requires some empirical investigation before coming to the same conclusion as for the first question. Must the third lines of the corresponding triangles be parallel? Again, empirical investigation is needed to reach the answer, "yes." Must all three corresponding lines be parallel, then? I can foresee much discussion and argument before coming to the conclusion that the essential condition is that the corresponding angles of the triangles be equal in size. This method of reaching a general result has much in common with the problem-solving approach outlined above. A long time is spent on one problem, but in the course of solving it many ideas are brought into relation and the *origin* of the resulting abstract concepts and the *value* of the resulting relationship is quite clear.

TYPES OF GENERALISATION

All three methods we have outlined are aimed at teaching mathematical *generalisations*–concepts and results which are generally true. The Russian psychologist Davidov has recently expounded two different types of generalisation using in teaching, including mathematics teaching (Davidov, 1972/1990). Although much of the book is full of Marxist jargon like *dialectical materialism*, there is a great deal of value in what he has to say. And since his work is not widely known in the West, I believe it is worth attempting to summarise and apply his theory. All page references below are to this book, unless otherwise stated.

Empirical Generalisation

Davidov summarises what he calls the traditional or *empirical* theory of generalisation as follows. A generalisation (also called a concept) arises from classification activity followed by "finding and singling out [properties] in a whole class of similar objects" (p. 10). The wider the range of objects or experiences included, the richer the concept. Generalisation is "inseparably linked to the operation of *abstracting*. Delineating a certain quality as a common one includes separating it from other qualities. This allows the child to convert the general quality into an independent and particular object of subsequent actions" (p. 13). The purpose of abstraction is thus to permit the application of general rules of operation in specific fields of application. Davidov argues that the processes of generalisation and abstraction are applied mainly to real objects, although they may later be applied to concepts derived from real objects. And he ascribes the pedagogical principle, "always proceed from the particular to the general," to a wide-spread belief in the psychological principle of empirical generalisation as the basis for all learning and teaching in school (not only mathematics).

The theory Davidov summarises here is essentially the same as that described by Dienes (1961), Piaget (1972), Skemp (1986), and others. While recognising that empirical generalisation has played an important role in the development of science and mathematics in the past, Davidov criticises it on several grounds:

- Classes are not formed by noticing common features. In fact, it is impossible to state the defining features of most everyday concepts (such as human). Also, classes are not formed from arbitrary collections of objects—there is always some reason for objects being grouped together (often a common function). Classifying on the basis of external characteristics does nothing towards identifying their inner connections.
- Teaching through empirical generalisation must consist of the transmission of concepts known to the teacher (who is aware of the inner connections) through examples chosen by the teacher which to the students appear to be unrelated. Also, the quality of a concept formed is determined by the objects in the class (again, chosen by the teacher) from which it has been abstracted.
- A restriction to empirical generalisation, by emphasising the link to concrete experience, discourages children from embarking on something which is qualitatively different: abstract mathematical thought.

Piaget does not escape Davidov's criticism. His theory is attacked firstly as being basically empirical and secondly as not proceeding beyond early adolescent thinking.

Implications for Teaching

Davidov cites a number of studies carried out by Soviet psychologists (mostly reported only in Russian) to support his criticism of empirical generalisation:

- Pototskii (1963) found that teachers often teach classification of problems instead of problem solving, as a result of which students cannot solve novel problems. Yaroshchuk (1957) found that nearly 90% of Grade 4 students solved those and only problems which they could identify as of a type they had solved previously.
- In one of his own studies (pp. 143-150), Davidov found that Grade 1 students who had mastered initial number concepts following the standard approach to teaching (judged by him to be based on empirical generalisation) were unable to transfer their counting skills to counting sets of sets of objects or measurement units.
- Krutetsky (1968) (also published in English in 1976) investigated 6th and 7th graders' ability to learn algebraic and other mathematical generalisations. He found that below-average students never learnt to generalise; average students learnt to generalise after many examples; and above-average students learnt to generalise after one or two examples. The below-average and the average students were apparently using empirical generalisation whereas the above-average students were using something else. Davidov interprets these results as demonstrating (a) the ineffectiveness and inefficiency of empirical generalisation, and (b) the debilitating effect of a method of teaching based on empirical generalisation.

Theoretical Generalisation

In a chapter entitled "Basic Propositions in the Dialectical Materialist Theory of Thought," much of which is taken up with references to the works of Marx, Engels, and Lenin, Davidov expounds the principles of what he calls *theoretical* or *content-based generalisation*. A few quotations should clarify the new way in which this theory uses the same terms as in empirical generalisation, plus some new ones:

> Generalization is achieved, not through simple comparison of the attributes in particular objects ... but through analysing the essence of the objects and phenomena being studied. (p. 295)
>
> To make such a generalization means to discover a principle, a necessary connection of the individual phenomena within a certain whole, the law for the formation of that whole. (p. 295)
>
> By *abstracting*, man isolates and ... mentally retains the specific nature of the real relationship of things (p. 294).
>
> Content-related abstraction and generalization underlie the formation of a scientific, *theoretical* concept. Such concept functions as a completely definite and concrete *method of connecting* the universal and the individual. (p. 296)
>
> Only when the origin of the object or a conception is clear to the student does it become possible to assert that ... the student has a concept of that object. (p. 334)
>
> The essence of a thing is none other than the basis (included in itself) for all of the changes that occur with it in interaction with other things. (p. 194)

We may note some particular aspects of these ideas:

- The emphasis on analysis, in place of comparison or similarity, as the means for identifying the concept to be abstracted. The term *content-based* emphasises that analysis can only operate on content, not on superficial appearances.
- The emphasis on connections (not only between elements but also between the general

and the particular) which exist for some reason, are necessary, the result of some law. It is analysis which brings out these necessary connections.

Terms like essence and concrete (also, elsewhere, internal, real, and true) correlate with the claim that content-based generalisation is deeply meaningful thought which "goes beyond the limits of sensory conceptions" (p. 300).

Implications for Teaching

Davidov cites several Soviet studies in support of the superiority of theoretical generalisation, including Krutetsky (1968/1976) noted above. Here are two more:

- Slavskaya (1958) showed that the detailed analysis of a basic geometrical problem led to immediate transfer to an auxiliary problem, students "singling out the essential link that connected it with the basic problem." Solving the auxiliary problem first, without any relation to the basic theorem, led only to a gradual generalisation "in the course of a detailed comparison of the features of both problems" (p. 201).
- Mashbit (1963) compared students who "ascertained initially the general structure of the solution method by analyzing particular problems which were models" with those who "solved particular problems in which the concrete conditions and the form for expressing a mathematical relationship varied." The second group made slow progress, whereas the first "travelled the path that is accessible only to the best prepared students" who used the second method (p. 329).

In proposing reforms to school mathematics teaching, Davidov (p. 320) argues that "there must be instruction ... that reproduces in compressed, abbreviated form the real historical process of the birth and development ... of knowledge. The child ... cannot independently 'acquire' what people have already attained, but he should *repeat* the discoveries of human beings in previous generations". He states the following general sequence for the introduction of new concepts using theoretical generalisation:

> 1) students' orientation in a problem situation ... whose solution requires a new concept, 2) mastery of a model for the sort of transformation of the material that discloses in it a relationship that serves as a general basis for solving any problem of the given type, 3) establishment of this relationship in an object-related or symbolic model, which permits its properties to be studied in "pure form," 4) disclosure of properties of the delineated relationship by which to deduce the conditions and methods of solving the original problem (p. 349).

He then reports briefly on an experimental primary school mathematics course which focuses on solving measurement and counting problems (in that order) and does not shy away from fractions and integers as they arise in such problems. Most remarkably, "For four years, the students solve all word problems only by setting up equations—that is, with no access to an arithmetical method. [Studies indicate that] this method, in the first place, is entirely accessible to children of 7-10 years ...; second, it substantially simplifies all of the instructional work on problems; third, it largely favors the children's development of skill in independently solving 'new' problems, one encountered *for the first time*" (p. 365).

APPLYING DAVIDOV'S THEORY

The Three Teaching Methods Classified

The three methods of teaching mathematical generalisations which we outlined in the first part of this paper can now be classified using Davidov's theory:

The "empirical method" is clearly based on the tenets of empirical generalisation. Results are recognised as properties which are common to several situations chosen by the teacher, but are not integrated into a general explanatory theory. Understanding may be superficial and the whole exercise pointless.

The "ABC method," although it is in a sense the opposite of the empirical method, is in fact also based on the theory of empirical generalisation. For the ABC method can be regarded as an attempt to short-circuit a lengthy process of empirical generalisation by presenting students with the abstract end product. Use of this end product in problem solving depends on recognising the problem as being of a type to which the abstract result may be applied–exactly as in the empirical method (but without the benefit of experience with several examples). Again, understanding is likely to be superficial and the purpose of the concepts studied obscure.

By contrast, the "problem-solving method" is clearly based on the principle of theoretical generalisation. Situations (tessellations, shadows) are analysed to identify the *structure* of the problem, the *basic* elements and the *essential connections* between them. The concepts which are formed in this way are not isolated, but take their meaning from their relations to one another.

Limitations of the Empirical Approach: One Last Example

The more I think about mathematics teaching, the more I see the limitations of empirical generalisation. Here is an example from my own textbook (Mitchelmore & Raynor, 1986), suggested by a similar case described by Peschek (1989):

In this textbook, linear relations are introduced using a double process of empirical generalisation as follows. The student is first set several arithmetical problems in familiar "linear" situations (taxi fares, electricity costs, etc.). From similarities between the various calculations within each problem, a general calculation procedure is formulated for each situation. Similarities between the calculation procedures in different situations are then identified: one quantity is always found by multiplying another quantity by a fixed number and adding another fixed number. The concept of *linear growth* is defined as such a relation. It is then found that the graph of linear growth is a straight line, and the algebraic form $y = kx + a$ is introduced.

As Peschek (1989) points out, the problem with this approach is that the essential characteristic of linear functions (that a given increment in the independent variable always leads to the same increment in the dependent variable) gets lost. It is always there implicitly, of course, but it is never made explicit. Students are never asked *why* each given situation leads to a linear function, or what types of situation are "linear." As a result, students do not learn a deep conception of linear function; instead, their concept is limited to the superficial form of the relation, essentially a procedural conception. And how is a student expected to judge whether a new situation is linear?

A Critique of Davidov's Theory

Although Davidov's theory seems to have much to recommend it, I can see some problems:

- Defining theoretical generalisation as the search for *essential* relations is unsatisfactory. How does one know what is essential? Also, what is essential in one context might not be essential in another.
- I cannot see how theoretical generalisation can take place without some empirical generalisation as its basis. For example, in generalising the shadows problem above, how does one find out that the condition that two lines be perpendicular can be relaxed to a requirement that corresponding sides be parallel, or that this condition can be relaxed to a requirement that corresponding angles be equal? Surely only by empirical investigation, not by theoretical reasoning.

- Moreover, an empirical investigation of relations can lead to an integrating structure just as a process of theoretical generalisation can. For example, the development of a language to express the communality between shadow, enlargement, and gradient situations can lead to an abstract formulation of the similar triangle properties which is at least as general as that obtained by an investigation of the essential elements in one of these situations.

CONCLUSION

Davidov's theory has much in common with several recent theories treating understanding in mathematics as "making connection between ideas, facts, or procedures" (Hiebert & Carpenter, 1992, p. 67) leading to the construction of what Ross & Hoyles (1996, p. 108) call *webbing*. I believe the theory would generously repay closer study by school and university teachers, and by teachers of teachers.

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THE CONJUNCTION FALLACY AND LONGITUDINAL DEVELOPMENT OF CHANCE EXPRESSION

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Two survey items asking for estimates of probability or frequency of everyday events (A), (B), and their conjunction (A and B), were completed by 2719 school students in grades 5 to 11. Cross-sectional and longitudinal analyses revealed chance expression improved with grade, but no change in incidence of conjunction errors. Gender differences favouring males occurred for some grades. Comparisons with responses to other probability items indicated incidence of conjunction errors is independent of development of basic chance measurement.

The conjunction fallacy arises in contexts where probabilities are considered for two events and their intersection (conjunction). According to classical probability theory, if sets A and B are defined, it is necessary that the conjunction set (A and B) is a subset of A and of B, thus P(A and B) is necessarily less than or equal to both P(A) and P(B). Previous research has found that people often violate this principle using reasoning based on a conjunction fallacy. Tversky and Kahneman (1983) and subsequent researchers used problems in social contexts, often variations of a problem that involved a character description of Linda, with respondents asked to judge the likelihood that Linda has various occupations (A), hobbies (B), or both (A and B). Tversky and Kahneman (1983) found that around 90% of university students violated the conjunction rule under certain conditions. They considered that some respondents may be averaging P(A) and P(B) to arrive at P(A and B), or thinking of the causal relations between A and B, thus rating (A and B) as more typical than (A) and (B) for the character description given. Their major finding, however, was that fewer conjunction errors (only 11%) occurred for simpler questions requesting frequency rather than probability estimates. They interpreted this as evidence that responses to probability items were often based on reasoning of typicality and intentional meaning of terms, using a more general *representative heuristic*, whereas responses to frequency items were based on extensional referents, which are countable.

Later researchers trialed variations of Linda-type tasks. Variations of response format included tasks to estimate frequencies such as "how many people out of 100..." (Fiedler, 1988), to estimate likelihood from 0 to 10 (Fisk & Pidgeon, 1997) or from 0 to 100 (Fantino, Kulik, Stolarz-Fantino & Wright, 1997), and to order alternatives (Morier & Borgida, 1984); estimating frequencies consistently yielded fewer conjunction errors. Researchers also employed variations of the components to be estimated, such as P(A), P(B), P(A and B), and P(A or B) (Morier & Borgida, 1984), as well as changing the social setting of the question to a traditional probability setting of marbles in urns (Yates & Carlson, 1986); these studies indicated that focusing attention on the components and the probability setting also reduced conjunction errors.

Only two of the research studies of the conjunction fallacy found involved school students. Fischbein and Schnarch (1997) asked students to rank P(A and B) and P(A) in a Linda type problem, and found 85% of grade 5 students committed the conjunction fallacy, whereas only 40% of grade 11 students did. Davidson (1995) also used Linda-type problems, asking young students to rate likelihoods on a 5 point scale. Conjunction errors were made by 35% of grade 2 students, rising to 57% of grade 6 students, apparently indicating increasing use of the representative heuristic with grade. No clear developmental trend emerges from these two studies, however the sharply contrasting results again draw attention to task variations of the form of expression for responses as important in accounting for incidence of conjunction errors.

Results from other studies of development of chance measurement for students of grades 3 to 11 (Watson, Collis & Moritz, 1997; Watson & Moritz, 1998) and odds for grades 6 to 11 (Moritz, Watson & Collis, 1996; Moritz, 1998) have indicated that younger students often use words or non-normative numerical expressions for measuring chance, and that sex differences favouring males are evident at selected grade levels in secondary school. Thus the expression of response to open-ended tasks and sex differences may contribute to the mixed results observed in the two studies of school students and shed light on the development of conjunction fallacy reasoning for school students.

The current study of school students' responses to conjunction problems investigated not only the incidence of conjunction errors, but also the ways that students express likelihood estimates. As the conjunction fallacy items were part of a larger survey, it was not possible to include the range of formats for questions used by all earlier researchers. Based on the experiences of other researchers it was decided to use an open-ended format asking for estimates. This format allows for a distinction to be made between responses that are appropriate, responses that explicitly express the conjunction fallacy, and responses that are undefined in terms of the context set. Rather than using in-school contexts such as marbles in urns, out-of-school contexts were used in an effort to explore student reasoning in everyday contexts and to reflect the application goals of the school curriculum. Three research questions were of interest. (1) In everyday contexts, do school students interpret two events and their conjunction in an appropriate fashion? What alternative interpretations arise and in what formats are responses offered? (2) Does student performance on these tasks improve with age, or differ between cohorts or sexes? (3) Is there an association between performance on conjunction estimation items and performance on other chance measurement items?

METHOD

Participants

Survey responses were gathered from 2719 students at 20 government primary schools, secondary schools, and matriculation colleges distributed throughout Tasmania. Responses were collected in 1993, 1995, and 1997, and totalled 3730 responses, including 785 responses from the same students surveyed again after a two-year interval, and a further 113 responses from students surveyed three times, in 1993, 1995, and 1997. The numbers of students surveyed from different schools varied across years and grades due to availability of students (see Table 1). Approximately equal numbers of males and females were surveyed in each year at each grade level. More details of the cross-sectional and longitudinal aspects of the larger study are found in Watson and Moritz (1998).

Items and Procedure

Two short answer items, shown in Figure 1, were adapted from those of Tversky and Kahneman (1983, p. 309). Both items required subjective likelihood estimates of everyday events, Item 1 in probability form and Item 2 in frequency form. For each item, it was expected that part (b) would be estimated as more likely than part (a). Item 1(c) involved the word "causing", restricting the likelihood of the event even moreso than the conjunction. These items were questions 15 and 18 of a 20-item chance and data written survey (Watson, 1994). The survey was administered to whole class groups during 45 minutes of class time. Some students who did not respond to these or later items, due to time or inclination, were excluded from this analysis.

Figure 1
Two Items Involving Conjunction Estimates

1. Please estimate:
(a) The probability that you will miss a whole week of school next year.
(b) The probability that you will get a cold next year.
(c) The probability that you will get a cold causing you to miss a whole week of school next year.
2. A health survey was conducted in a sample of 100 men in Australia of all ages and occupations. Please estimate:
(a) How many of the 100 men have had one or more heart attacks.
(b) How many of the 100 men are over 55 years old.
(c) How many of the 100 men both are over 55 years old and have had one or more heart attacks.

Coding and Analysis of Responses

Responses were entered into a spreadsheet in the form students wrote them. Responses to the two items were then coded according to (1) the numerical relation between the three numerical estimates of the probability or frequency for the three parts of each item and (2) the type of expression used. Numerical Relations included $c<min(a,b)$, $c=min(a,b)$, $min<c<max$, and $max(a,b)=c$, according to the numerical value expressed in part (c) in relation to the minimum and maximum numerical values expressed in parts (a) and (b). The distinction between $mix<c<max$ and $max(a,b)=c$ was made to further clarify the reasoning used by students, in particular to see if students might be using an averaging process (Tversky & Kahneman, 1983; Fantino, et al., 1997). The $c<min(a,b)$ and $c=min(a,b)$ categories were also coded as "correct", and $min<c<max$ and $max(a,b)=c$ as "fallacy". Word expressions were mapped to numerical values where possible: "unlikely" and "low" were assigned 0.25, "maybe", "medium" and "average" were assigned 0.50, and "likely" and "high" were assigned 0.75, with modifiers such as "very" being assigned more extreme values. An *undefined* numerical relation was entered for responses where a numerical value could not be reasonably inferred for all three parts to the item, or if word expressions to different parts could not be differentiated to determine which was intended to have a higher probability value. Categories of Expression included *frequency* (whole numbers 0-100), *percentage* (use of "%"), *fraction* (decimal, e.g., 0.25; part-whole ratio, e.g., 1/4; or part-part odds, e.g., 5/2), *yes/no* (simple use of either word), *word* expressing chance (e.g., "likely"), and *other* (e.g., "I don't know"). Expressions were also coded as Numerical (frequency, percentage, or fraction) and Non-numerical (chance word, yes/no, or other). If responses to all three parts used the same type of expression, this expression was assigned as the response expression for that item; otherwise *mixed* expression was assigned to the response.

Students' responses were analysed in three different ways. *(1) Cohort and cross-sectional analyses* using χ^2 tests involved the independent factors of cohort (1993, 1995, 1997), grade level, and sex. Responses from comparable grades collected from different cohorts were compared to investigate whether recent curriculum reform and implementation had affected incidence of the conjunction fallacy and students' expressions of chance. Responses of students from a cross-section of grades (and both sexes) were compared as one method of investigating conceptual development of students. *(2) Longitudinal analysis* was a second method for exploring conceptual development, in this case analysing differences in responses of 113 individual students gathered longitudinally over two 2-year intervals. *(3) Cross-item analyses* involved comparing students' responses to the probability (Item 1) and frequency (Item 2) forms. Responses were also compared to those of chance measurement tasks reported in previous studies (Watson, et al., 1997; Watson & Moritz, 1998) to explore understanding of other probability concepts that may impact on reasoning for conjunction tasks.

RESULTS

Cohort and Cross-Sectional Analyses

Tables 1 and 2 show the numerical relations and expressions used in responses to Items 1 and 2 respectively, for all 3730 responses grouped by grade and year of survey. The results reported include both those who were completing the items for the first time as well as those who were repeating the items in each year, as there were no significant differences in the numerical relations and expressions of non-repeating versus repeating students in comparable grades, with one exception. This exception was that repeating students in grade 7 were more likely to use numerical expressions in response to Item 1 ($\chi^2_2=17$, $p<0.001$); this may be due to the additional non-repeating students at the grade 7 level being drawn from other feeder primary schools than the repeating students.

Table 1
Percentage Responses to Item 1 Coded by Numerical Relation and Expression

Response	1993 Grade		1995 Grade					1997 Grade						
category	6	9	5	6	8	9	11	5	6	7	8	9	10	11
Numerical Relation														
c<min(a,b)*	26	39	17	19	38	31	40	19	23	26	26	33	37	49
c=min(a,b)*	23	25	21	21	23	25	28	19	19	19	27	32	30	22
min<c<max^	11	15	5	12	13	16	19	11	13	15	20	13	15	14
max(a,b)=c^	7	7	6	8	7	4	7	6	6	5	7	8	7	8
Undefined	33	14	51	40	18	24	7	46	39	36	20	15	11	8
Expression														
Frequency	1	2	4	3	4	1	1	7	5	3	2	2	4	2
Percentage	23	38	15	29	45	36	49	14	23	30	44	44	45	37
Fraction	11	24	7	9	15	21	26	14	10	10	13	8	22	39
Word	19	14	18	13	11	14	7	15	22	16	17	26	12	6
Yes/No	16	5	24	16	6	10	1	20	18	12	5	6	2	0
Mix/Other	29	17	32	30	19	18	15	29	22	29	20	14	15	16
N	307	377	435	321	361	345	215	216	215	305	186	104	292	51

* denotes correct response categories, ^ denotes conjunction fallacy categories

The results for Item 1, shown in Table 1, were consistent for comparable grades across 1993, 1995, and 1997 cohorts. The only significant difference found was that grade 9 students in later years were less likely to use numerical expressions (69% in 1993 to 55% in 1997; $\chi^2_1=9.6$, $p<0.01$). Combining the *c<min(a,b)* and *c=min(a,b)* categories in Table 1, the correct response rate improved over the grades, being fewer than half of students grade 7 or below, but more than half of students grade 8 or above. Many younger students used non-numerical expressions, resulting in the numerical relation between the three parts being undefined. Older students tended to respond with appropriate expressions of probability, such as percentage or fractional chance. The percentages of students making a conjunction error (combining *min<c<max* and *max(a,b)=c*) ranged from 11% to 27% across grade samples. The indication of increasing percentage of conjunction errors for older age groups, particularly evident in the grades surveyed in 1995, disappeared when undefined responses were removed from the analysis: the corresponding percentages of students making a conjunction error ranged from 22-34% across grade samples, with no trend across grades. More students responded in the *min<c<max* category than the *max(a,b)=c* category.

As for Item 1, the results for Item 2, shown in Table 2, were remarkably consistent for comparable grades across cohorts. The percentages of correct responses to Item 2 varied little across grades. Most students in all grades responded with expressions of frequency or of the percentage of the sample of 100 men. As a consequence, few responses had undefined numerical relations between parts. The results were also consistent across grades. The percentages of students making a conjunction error ranged from 29-44% across grade samples (32-51% when undefined responses were excluded). Of these, in contrast to Item 1, more students responded with $max(a,b)=c$ than $min<c<max$.

Table 2
Percentage Responses to Item 2 Coded by Numerical Relation and Expression

Response	1993 Grade		1995 Grade					1997 Grade						
category	6	9	5	6	8	9	11	5	6	7	8	9	10	11
Numerical Relation														
c<min(a,b)*	38	46	39	36	42	43	40	38	44	33	31	41	38	51
c=min(a,b)*	16	18	14	18	16	16	17	18	14	10	20	17	19	14
min<c<max^	12	14	13	13	16	15	17	9	12	20	12	14	13	8
max(a,b)= c^	19	16	16	17	20	15	20	21	18	24	25	19	24	20
Undefined	14	6	17	15	6	11	7	14	13	13	12	8	5	8
Expression														
Frequency	54	58	51	45	51	53	47	56	52	46	34	48	52	57
Percentage	15	21	22	27	32	20	27	14	23	30	41	30	29	25
Fraction	6	7	2	4	5	6	7	7	4	5	6	6	8	6
Mix/Other	25	13	25	24	12	20	19	22	21	19	18	16	11	12
N	307	377	435	321	361	345	215	216	215	305	186	104	292	51

* denotes correct reponse categories, ^ denotes conjunction fallacy categories

Table 3 shows the performances of female and male students at each grade level (cohorts combined) in a consolidated form (*correct, fallacy, undefined*) for the two items. For Item 1, sex differences favouring males were observed in grade 9 ($x^2_2=11.6$, $p<0.01$) and grade 11 ($x^2_2=12.8$, $p<0.01$). For Item 2, there were no significant sex differences, despite the exceptionally high incidence of conjunction errors for grade 7 females.

Table 3
Percentage Responses to Items 1 and 2 by Grade and by Sex

Response	Grade 5		Grade 6		Grade 7		Grade 8		Grade 9		Grade 10		Grade 11	
category	f	m	f	m	f	m	f	m	f	m	f	m	f	m
Item 1														
Correct	38	37	44	44	39	49	54	63	55	67	65	68	59	78
Fallacy	10	16	20	18	24	16	26	19	24	17	26	18	33	15
Undefined	52	47	36	38	37	35	20	18	21	16	9	13	8	7
Item 2														
Correct	55	54	56	55	37	48	56	55	59	63	58	58	57	60
Fallacy	29	31	31	31	52	36	36	37	31	29	35	39	36	34
Undefined	16	14	14	14	11	15	8	8	9	7	8	3	8	6
N	317	334	432	411	150	155	273	274	425	401	146	146	143	123

Longitudinal Development

The results of longitudinal analysis of responses to Items 1 and 2 by 113 students in grades 6, 8 and 10 are shown in Table 4. Responses to Item 1 involved similar distributions to those shown in Table 1: many students in grade 6 used non-numerical expressions of the probability, and thus many have an undefined numerical relation, whereas older students more often used percentage expressions. There was an increase in correct responses after grade 6 but no evidence of change in rates of conjunction errors. Responses to Item 2 involved similar distributions to those shown in Table 2: most responses were expressed as frequencies or percentages, and there was a slight improvement in correct responses after grade 6. Similar rates of conjunction errors occurred across grades. To examine within-student longitudinal change, results were consolidated by grouping numerical relations into *correct*, *fallacy*, and *undefined*. In two 2-year intervals, 1993-1995 and 1995-1997 (the latter in parentheses for the following results) for Item 1, 60 (or 63) students remained in the same grouping, 14 (or 10) students improved their response from *fallacy* to *correct*, whereas 10 (or 15) reverted from *correct* to *fallacy*. Similarly, for Item 2, 53 (or 70) students remained in the same grouping, 21 (or 16) students improved their response from *fallacy* to *correct*, and 16 (or 14) reverted from *correct* to *fallacy*. Thus overall, the longitudinal study indicates that students developed in numerically expressing chance or frequency, but percentages of conjunction errors or correct conjunction reasoning were quite stable over time, with fluctuations of improvement and reversion in similar frequencies.

Table 4
Percentage Responses to Items 1 and 2 Assessed Longitudinally (N=113)

Response category	Item 1 Grade 6	Item 1 Grade 8	Item 1 Grade 10	Item 2 Grade 6	Item 2 Grade 8	Item 2 Grade 10
Numerical Relation						
c<min(a,b)*	26	45	39	33	48	46
c=min(a,b)*	25	21	27	21	13	23
min<c<max^	12	10	16	13	17	11
max(a,b)=c^	10	7	5	19	14	17
Undefined	28	17	12	14	8	4
Expression						
Frequency	1	4	5	58	48	61
Percentage	28	50	49	15	32	25
Fraction	13	11	17	3	5	4
Word	14	11	14	-	-	-
Yes/No	13	5	2	-	-	-
Mix/Other	30	20	13	22	14	10

* denotes correct response categories, ^ denotes conjunction fallacy categories

Cross-Item Analyses

Of 3730 responses, *correct* was similar for Item 1 (1946 responses) and Item 2 (2083 responses), *fallacy* was more common for Item 2 (1235 responses) than Item 1 (732 responses), whereas *undefined* was more common for Item 1 (1052 responses) than Item 1 (412 responses). Overall 31% of responses were *correct* to both items, and 8% were a *fallacy* to both. Of those who answered Item 2 either correctly or with a fallacy, 56% or 52% respectively answered Item 1 correctly. This difference, despite producing a significant χ^2 value, reflected little actual association in responses to the two items.

To explore the association between success on conjunction estimate items and more general understanding of chance measurement, 3616 responses to Items 1 and 2 were matched to developmental levels determined by Watson and Moritz (1998), scored from 0 to 6, based on responses to three chance measurement items earlier in the survey. These levels represent increasingly complex cognitive functioning evident across responses to three items concerning (1) likelihood of numbers occurring when a 6-sided die is rolled, (2) likelihood for an outcome drawn from a bag, and (3) comparisons of likelihoods of drawing one colour of marbles from boxes with similar ratios of colours. The distribution of responses to Items 1 and 2 by chance measurement developmental level is shown in Table 5. For Item 1, higher chance measurement developmental levels were associated with more *correct* and fewer *undefined* responses; *fallacy* rates were constant across developmental levels above level 2, beyond which at least simple quantification of chance measurement is evident. For Item 2, the *fallacy* rate was again independent of developmental level. The positive association with *correct* responses was less marked than for Item 1, corresponding to the weaker of association for this item of chance measurement level with increasing numerical expressions and with decreasing *undefined* responses.

Table 5
Percentage Responses to Items 1 and 2 by Chance Measurement Level

Response	Chance Measurement Developmental Level							
category	0	1	2	2.5	3	4	5	6
Numeric Relation (Item 1)								
Correct	22	33	41	46	57	64	62	68
Fallacy	10	13	17	20	22	21	20	22
Undefined	67	54	43	34	21	15	18	10
Expression (Item 1)								
Numerical	22	33	41	46	57	67	73	78
Numerical Relation (Item 2)								
Correct	45	57	50	52	60	58	59	69
Fallacy	29	27	35	35	32	34	31	30
Undefined	27	16	15	12	8	8	10	1
Expression (Item 2)								
Numerical	71	82	84	87	91	91	89	97
N	49	177	543	889	653	914	122	269

DISCUSSION

The results of this study indicated that incidence of conjunction errors was not associated with grade, nor with chance measurement developmental level. Sex differences favouring males were observed, but there were no improvements across cohorts; both of these findings are consistent with those of Watson and Moritz (1998). The failure to observe a decrease in the fallacy with grade, as found by Fischbein and Schnarch (1997), is most likely related to the simplified form of the question which reduced the fallacy level for all grades. It may also reflect the absence of specific reference to conjunction probabilities in the Australian mathematics curriculum (Australian Education Council, 1991) and hence lack of attention in the classroom.

Students from higher grades and of higher chance measurement developmental levels were more likely, however, to express probability for Item 1 in a numerical form, and consequently to exhibit higher rates of correct responses and fewer undefined responses. This may reflect the increasing emphasis on quantitative measurement of chance in the years of schooling. Not surprisingly, expressions used in responses to the probability item (Item 1) differed from those to the frequency item (Item 2). It is surprising, however, that incidence of the conjunction fallacy was not lower for the frequency form than for the probability form, as had been found by Tversky and Kahneman (1983). *Max(a,b)=c* responses were more common than *min<c<max* responses for Item 2, but not so for Item 1. Further investigation revealed that part (c) was evaluated as the arithmetic mean of parts (a) and (b) for 65 responses to Item 1 and 103 responses to Item 2, whereas part (c) was evaluated as the sum of parts (a) and (b) for 30 responses to Item 1 and 153 responses to Item 2. These findings indicate that conjunction errors result from a variety of forms of reasoning by different students, not simply from averaging the two component parts to determine the conjunction (cf Tversky and Kahneman).

Several issues arise for the classroom from the results of this study. The generally disappointing performance indicating a relatively constant incidence of the conjunction fallacy in social contexts would indicate a need to address the topic more directly in the mathematics curriculum. That this suggestion involves the acknowledgement of making subjective probability estimates must be faced. If students are going to be making personal life decisions in risk-taking situations where various conditions and their conjunctions are involved, they need experiences discussing more than marbles in urns.

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Acknowledgement

This research was funded by the Australian Research Council, Grant Numbers A79231392 and A79800950.

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PERCEPTIONS OF MATHEMATICAL UNDERSTANDING

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Australasian mathematics educators responded to a questionnaire that sought views on what mathematical understanding is, how it can be identified, and what teachers can do to develop it. This paper presents the responses to two questions about indicators of children's understanding and differences between doing and understanding. Responses were grouped as verbal, cognitive and physical indicators but it is recognised that these are inter-related.

INTRODUCTION

There is widespread rhetoric about building on children's mathematical understanding. For instance, *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990) makes the claim that because the meanings that people construct "depend upon their existing understandings, we can only take from any situation the parts that either make sense to us or can be linked to some existing ideas we have" (p. 16). This seems logical, but the statement does not suggest how teachers can find out what children's current mathematical understandings are.

For Victorian teachers, the current *Curriculum & Standards Framework* (Board of Studies, 1995) includes as one goal of mathematical activity "depth of conceptual understanding" (p. 9). Similarly, the more detailed *Mathematics Course Advice: Schools of the Future* (Directorate of School Education, 1995) claims that it "emphasises the development of conceptual understanding" (p. 9) and that the assessment activities it includes provide an opportunity for children to crystallise "essential learning in terms of understanding the essential concepts" (p. 16). It notes that planned observation is the key to teachers forming a comprehensive picture of the mathematical understanding of all students. However, neither of these documents suggests what should be observed or what the indicators of understanding might be.

Despite this emphasis in both rhetoric and practice, when I ask teachers and teacher educators what they think mathematical understanding is, the question usually draws a blank stare or flippant response. When I distributed a questionnaire to teachers avoidance was also apparent, as the questions about the nature of understanding were the ones most frequently left blank. One elusive response that I particularly enjoyed, however, was "If I understood what it is to understand, I would tell you; then you would understand what I understood. But I don't, so you don't. Do you understand?" I did!

Madison (1982) would also have understood. He has summarised the problem as one of not being able to separate the object from the subject:

> Understanding is perhaps the most difficult thing to conceptualise, and understandably because it is that which is closest to us; in the reflective act of self-understanding, subject and object, knower and known are one and the same. Understanding is in fact that which we ourselves are ... The natural error of understanding consists in its taking the object of its believing for reality. This is natural, for it is impossible for understanding to do otherwise, for it to be anything else but belief, but is a presumption ... for understanding can never prove what it believes to be reality is indeed reality itself ... But this is not our goal. We are not attempting to overstep the bounds of experience but only to interrogate experience itself. We must attempt to draw up an inventory of what we do know, on the basis of our actual, lived experience. (pp. 153—154)

The intention of this paper is to do just what Madison suggests: to draw up an inventory of what we know on the basis of our experience. It focuses on how teacher educators think that teachers can see whether children understand aspects of their mathematics.

METHODOLOGY

The results reported in this paper are from one component of a larger study entitled "Teachers' constructions of their roles in building mathematical understanding". The project is using case studies to explore teachers' ideas about mathematical understanding and the ways that it develops, as well as what the teachers do in classrooms to advance it. A survey of teachers is being used to set the case studies in a broader context.

The data presented this paper are from a pilot of the questionnaire. Draft survey questions were trialed with mathematic teacher educators, and it is on some of the results of this trial that I wish to focus. The process of data collection was very informal: the pilot was posted on the internet site with a request that anyone with time to spare complete it and/or give me feedback on its format and content. Twenty-two mathematics educators responded. Five commented on some of the questions, and seventeen completed the full questionnaire. While the final survey will seek data from teachers, the responses provided by mathematics teacher educators were of interest to me, and they have also allowed me to trial different forms of data analysis and writing up.

The survey presented three types of questions. One set asked respondents to mark on a Likert scale how well various ideas and practices match their own. A second asked them to choose responses from a given set of possibilities that best suit their ideas, and then to add any further possibilities to the set. The third type of question was more open and followed by a blank space where respondents could write points, sentences or paragraphs. These questions were asked before the matching forced choice questions; for instance, the open question on strategies teachers can use for developing mathematical understanding appeared before a section where people mark on scales their frequency of use of particular given strategies.

This paper reports results from only two open questions that relate to evidence of understanding. The respondents had already been asked to imagine a lesson where a teacher is helping a child to understand a new mathematical concept, and had described what the teacher would do to develop the child's understanding. They were now asked, *"How would the teacher know when the child understands the new concept? In other words, what do you think constitutes evidence of understanding?"*. Because many of the respondents referred to different aspects of what children "do" in this section, I have also included in this paper responses to the question, *"Do you think that there is a difference between being able to do the necessary work in a mathematics class and understanding it? If so, what is the nature of this difference?"*

The analysis aims at description, and makes no claims about significance or generalisation. Following one tradition of interpretive analysis of qualitative data, responses were sorted into what seemed to be sensible groups and subgroups. Given the small number or respondents, it is not surprising that there is little repetition, so the results are presented in full. If a response was considered to contribute equally to two categories, the relevant words were included in each group, with the symbol ... signifying that words that have been omitted but used elsewhere.

RESULTS AND DISCUSSION

Verbal Indicators

In response to the question *"How would the teacher know when the child understands the new concept?"*, many people mentioned a child's ability to articulate ideas. This included undifferentiated talking about the concept, perhaps in response to teacher questions (see Table 1), or, more specifically, explaining it and/or teaching it to others (see Table 2).

Table 1
List of Indicators Grouped as "Ability to Talk About"

Can re-tell

By answers to questions

Monitor responses carefully

Intelligent questions and conversation

Being able to articulate understanding

Question orally, probe deeper, prompt if necessary

Be able to verbalise understanding.

Respond to probing questions with confidence

Child asks and answers relevant questions

Ask challenging questions to take a child out of the particular to the more general

I'd be looking for things said or done to provide evidence not of complete understanding as much as understanding of components first

One would need as a teacher to be aware of the web (or concept map) related to the new concept—and then look for evidence of being able to ...talk about them that fit this web

When they say the right thing (I can fiddle the programming and see how they react)

Answer the teacher's questions in a way that demonstrates a logical conceptual structure

Being able to present their logic to other members of the group

Table 2
List of Indicators Grouped as "Ability to Explain"

Able to explain to teacher or another child (2)

The ability to explain clearly or teach others

Being able to teach the concept to a peer

Can explain understanding and defend it

Explain the idea (2)

If they can explain ...in their own words (2)

Get the student to explain the concept to me in their own language

When they can teach others (2)

Explain, provide evidence

Teach other group members or respond to their explanations sensibly

Being able to teach the concept to another student

Being able to construct a model or representation

Being able to explain to a peer the underlying ideas and how these fit together to make a new and useful concept

When they can explain it to you and others in their own words, with diagrams, etc. Defend it against alternative explanations

If they can explain how they arrived at an answer—not just the process but the reasons underlying each step

Explaining in a way that presents some idea of the way they are "building" their solution

Being able to explain a concept clearly and some of the contexts in which is can be used

By their ability to explain and teach. That also serves the purpose of helping to clarify and reinforce the speaker's understanding

Being able to explain in their own language and to a child at their own level

It is often said that one really needs to understand something well to teach it, and that explaining often increases understanding. These beliefs were evident in some responses, and others stressed explanation being "at their own level", "in their own words", etc. However, one respondent pointed out that "Explaining is difficult for children, so sometimes we have to accept 'I just know it'; and in that case I usually test understanding with unfamiliar examples that require the application of the same or slightly extended knowledge and get them to 'think out loud'".

Cognitive Indicators

With a focus on the development of cognitive links, another group of responses to the same question related to children's abilities to make connections between concepts (see Table 3). Others used the idea of connected knowledge with reference to the ability of a child to apply knowledge across various contexts and situations (see Table 4). Few gave examples or suggested how such linkages could be observed.

Table 3
List of Indicators Grouped as "Ability to Link Concepts"

Having a network of foundational ideas

See how other explanations link with their own

Being able to develop a set of logical steps towards a desired end

Ability to link representations

Able to use the concept in a variety of ways (not contexts but, say, being able to think of multi- plication as lots of as well as repeated addition)

Being able to use a range of strategies, procedures and skills to solve a problem

Being able to articulate, explain, model, etc. as it applies to a general (rather than specific) *construct*; e.g. the properties of any pyramid, subtraction with internal zeros, etc.

Table 4
List of Indicators Grouped as "Ability to Apply across Contexts"

Ability to apply

Ability to translate to a new situation
Apply/transfer the idea
Be able to use it in correct contexts

Knowing how the maths in books is related to doing things

Demonstrate over a range of problems of different types

Use in an unfamiliar situation

Being able to apply that understanding across different situations or contexts

Being able to apply that understanding for various relevant situations

Being able to relationally understand a problem/situation that involves mathematical knowledge

Being able to describe or use a concept in many possible situations

Seeing the link between the real world and the mathematical model of it

The ability to use new knowledge with an unfamiliar problem

The literature on mathematical understanding frequently stresses the importance of the linking of concepts. Three examples follow. Lovell (1971) claimed that understanding suggests more than one-off observations or familiarity with single ideas: it suggests a broader perception and analysis of a concept, with an awareness of the connections between basic ideas, "a greater awareness of the information used, and ... a careful step-by-step process with concepts being formulated and defined" (p. 22). Skemp (1976) claimed that the process of developing relational understanding consists of "building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point" (p. 23). Piaget (1978) drew a distinction between "explicative" and "implicative" processes of comprehension (p. 213), the latter being connections between implicit concepts.

One respondent linked application to the growth of understanding, emphasising the value of "observing children using the concept—but in many cases understanding may not emerge until one goes on to use the concept later and can abstract it from the learning process". This brings to mind the work of Pirie and Kieren, who describe the "fractal" nature of understanding and the need to be able to "fold back" to previous stages in order to develop more refined concepts. Pirie and Kieren (1994) argue that in order that child's formal understandings not be disjoint, they must:

> grow from the child's potential mathematical structures and unfold from less formal image-based mathematical understandings. That is, formalising understanding is not an add-on to previous informal mathematical activity or understanding. The children … must recognise the patterns in their informal activities which allow them to come up with and justify methods which, to an observer, are now independent of the physical actions or images upon which they were originally based. (p. 43)

Their model of mathematical understanding involve a fractal quality that is able to be observed, and they suggest that appropriate curricula follow this model so are in themselves fractal, with teachers being (a) proactive in drawing students forward to the next level of understanding; (b) invocative, by causing the students to fold back to more foundational levels; and (c) validating of the current successful process or of existing understandings.

Sierpinska (1994) claims that generalisation and synthesis interplay with identification and discrimination—four "acts" of understanding. She uses the hermeneutic notion that when discovering that prior knowledge does not easily fit with new experience (that is, we encounter as "epistemological object") then we start understanding, i.e. knowing it in a new way. This is not a straightforward process, and articulates two perspectives that are complementary but different views of the phenomenon of coming to understand:

> Not all, perhaps, but some acts of understanding are acts of overcoming epistemological obstacles. And some acts of understanding may turn out to be acts of acquiring new epistemological obstacles. A description of the acts of understanding a mathematical concept would thus contain a list of the epistemological obstacles related to that concept, providing us with fuller information about its meaning. In many cases overcoming an epistemological obstacle and understanding are just two ways of speaking about the same thing. The first is "negative" and the other "positive". Everything depends on the point of view of the observer. Epistemological obstacles look backwards, focussing attention on what was wrong, insufficient, in our ways of knowing. Understanding looks forward to new ways of knowing. (p. 28)

Physical Indicators

A final group of responses to the question involved observation of physical activity of the children—doing (see Table 5), demonstrating understanding by representation(s) (see Table 6), or performing set tasks (see Table 7).

Table 5
List of Indicators Grouped as "Ability to Do (generally)"

When students DO the right thing (and I can slightly alter what they're doing and see if they fall in a heap)	One would need to be aware of the web (or concept map) related to the new concept—and then to look for evidence of being able to do things … that fit this web

Table 6
List of Indicators Grouped as "Ability to Represent"

Ability to represent	Being able to demonstrate or model
Being able to model	Being able to construct a model or representation
The ability to model	If they can …demonstrate (draw, give examples)
Represent the idea	I want not only [the learner being able to *explain or describe*], but evidence of being able to *do*.

This latter group echoes Mulligan's (1995) claim that the way children represent mathematical situations is closely linked with their understanding of mathematical concepts. Mulligan wrote that "we need to examine how the teaching/learning situation can be developed to promote conceptual understanding based on children's representations" and that "this firstly involves an understanding of how children use representational thinking" (p. 290).

Similarly, Carpenter (1994) said that external representations are the evidence we have of children's internal mathematical representations and that these representations should be used to develop children's understanding further, because if conceptual understanding involves the construction of connections between representations of mathematical ideas then building connections between external representations supports the construction of more coherent and useful internal representations.

Table 7
List of Indicators Grouped as "Ability to Perform Tasks"

A well-crafted written test

Correct answers

Perform a task

Complete given problems both verbally and with written working

Demonstrate understanding with an unseen example

The literature provides models of distinction between performance and understanding. For instance, Walkerdine (1989) contrasted "reproduction" (rote-learning and rule-following) with "production", which she claims is "real understanding" (p.274).

It was clear that most of the people who completed the survey distinguished between the ability to do and the ability to understand, and this showed up in two other sections of the questionnaire. First, in listing different types of mathematical understanding, many referred to Skemp's (1976) initial differentiation between instrumental and relational understanding. (He later added further categories.) Second, the survey asked the question *Do you think that there is a difference between being able to do the necessary work in a maths class and understanding it? If so, what is the nature of this difference?* Most respondents linked "doing" with rote processes, a distinction that is common in the literature.

I have grouped the responses as those that make a clear distinction between doing and understanding (Table 8), those that see possibilities for overlap (Table 9), and those that assume relative congruence (Table 10).

Table 8
Doing Versus Understanding

Able to do does not mean able to understand (e.g. you can do a subtraction algorithm without understanding why you do each step), and able to understand does not necessarily mean able to do (e.g. you can understand what subtraction is without being able to do difficult examples)

Doing is often drill and practice which may not lead to understanding

Doing the work can apply formulas, not understanding

Doing the work does not involve translating it to a new and previously unseen experience

Many students learn to do the prescribed questions mechanically, without understanding what they are doing or why

Much of school mathematics is doing—instrumental, rule and convention bound. You could train a monkey to do it. calculators can perform it

One is physical, the other is mental, and they are not necessarily related at all. I drive a car without understanding how it works, I understanding what square roots are but I do not know how to calculate them

They are quite different. You can do without understanding how it works, and you can understand how it works without doing

Table 9
Possible Overlap Between Doing and Understanding

Depends on activity—some require understanding, some do not.
The work (doing) is a blanket approach to engaging the mind. Aspects of this engagement might lead to or confirm understanding.
There's a difference in reality, but in theory school maths needs to aim at an ability to do with understanding at levels appropriate to the child's stage of development.

Table 10
Understanding Involves Doing

There's no such thing as understanding without doing. Observation of what a child does physically is the way we judge understanding, but there is other physical action going on that is not observable—and perhaps not even a conscious act of the child. Understanding *is* doing—but a doing that is hard to observe.

This latter position is not unlike that expressed by Pirie and Kieren (1989), who believe that "there is no such thing as understanding in the abstract", because understanding is "a process, grounded within a person, within a topic, within a particular environment" (p. 39). This contrasts with the stance of Sierpinska (1994) who identifies *acts of understanding* (e.g. explanations, validations) as being different from *understandings* (e.g. concepts, theories and problems), with the latter being the potential to experience an act of understanding when necessary as well as a prop for further development.

Voight (1994) describes mathematical understanding as a social relationship—a "theme" or network of taken-as-shared mathematical meanings, where "the teacher is ... dependant on the students' indications of understanding (and) the students are dependent on the teachers' understandings of their contributions" (p. 179). Voight suggests that this interdependence, as well as the unpredictable nature of meaning making, means that development of the theme is unpredictable. He notes that classroom routines (sometimes contradicting the teachers' intentions) function to minimise the risk of disorganisation.

CONCLUSION

One questionnaire respondent did not answer any of the questions, but wrote a statement about the folly of attempts to gather perceptions of understanding in the first place. The comment is more about the subject of the study than about particular indicators of understanding, but it makes a nice conclusion in that it summarises the difficulties of interpreting the data. It also takes us back to the object-subject dilemma outlined by Madison (1982) above. The respondent wrote:

> There's no such thing as mathematical understanding as an object—it is just a term we use to express the connections we make between some mathematical constructs that we have created—and people's stages or ways of coming to grips with those. Thus "good understanding" is used to describe the ability to demonstrate knowledge that matches commonly accepted knowledge and skills; "misunderstanding" is applied to knowledge that does not match (but might be perfectly viable for that person). "Partial understanding" is somewhere in between. But these are not different types of understanding (whether verb or noun) or even different levels of understanding. They are just *perceptions* of different levels of *agreement*. *Doing* is one way that learners can demonstrate that their understanding is similar to a teacher's (or more generally by what is valued in our society). When what is done illustrates a match, we say the person understands. When it doesn't, we say there is no, or less, understanding. But we are really talking about doing, not understanding. There is understanding there, but we can't see it—and can't know what it is. We can only observe what seems to be agreement.

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GRAPHICS CALCULATORS IN THE PUBLIC EXAMINATION OF CALCULUS: MISUSES AND MISCONCEPTIONS

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Graphics calculators were allowed in all tertiary entrance mathematics examinations in Western Australia for the first time in 1998. In this paper we present an analysis of students' answers to four of the examination questions in TEE Calculus and a discussion of misconceptions attributable to the technology and misuses of it.

INTRODUCTION

The National Statement on Mathematics for Australian Schools (Australian Education Council, 1991) and the Curriculum Framework for Kindergarten to Year 12 Education in Western Australia (Curriculum Council, 1998) recommend the use of technology for mathematics. Students should "make thoughtful use of technology to enhance their mathematical work" (Curriculum Council, 1998, p. 184). Allowing the use of graphics calculators in public examinations is part of the ongoing process of accommodating and encouraging adoption of technology into the whole curriculum. Candidates for the Western Australian Tertiary Entrance Examinations (TEE) were allowed graphics calculators in all mathematics subjects for the first time in 1998. In this paper the prevalence and nature of student use of the calculators in the Calculus TEE are discussed. In particular we consider calculator use that mediated against students providing answers acceptable 'in the public domain' (Noddings, 1990).

BACKGROUND

Widespread introduction of graphics calculators occurred in West Australian schools after the Secondary Education Authority (1995), now called the Curriculum Council, announced that graphics calculators would be allowed for all 1998 mathematics tertiary entrance examinations. Graphics calculators without symbolic processing and the Hewlett Packard HP38G with limited symbolic processing were approved (Secondary Education Authority, 1996; Curriculum Council, 1998) and students were advised that they would not be required to clear the memories of their calculators before the examinations. For the three hour 1998 Calculus TEE no explicit instructions were given to use a graphics calculator. The use of graphics calculators was at the students' discretion except in two part questions that specified an analytical approach. The paper was graphics calculator 'active' (Jones & McCrae, 1996) with student access to the calculators not only allowed but assumed (Secondary Education Authority, 1996) and one question could not be solved without using the technology.

DATA COLLECTION

The Calculus examination paper comprised nineteen questions of which six were selected for analysis. The selection was guided by Kemp, Kissane, and Bradley's (1996) categories "Graphics calculators are expected to be used . . . Graphics calculators are expected to be used by some students but not by others" (p. 108). For each of the six questions, examination markers were asked by the Curriculum Council to record the methods students used and the part marks awarded for answers for a sample of scripts. Systematic sampling (Cohen & Manion, 1994) resulted in 404 (21%) out of the total 1882 scripts being selected. These were the scripts of the first two candidates listed on each normal marks recording sheet. In addition, qualitative data relevant to this paper were obtained from three sources. Firstly, 172 (9%) of the 1882 examination scripts were perused to ascertain the details of students' working. These scripts were all papers in six randomly assigned bundles of scripts, bundled for marking and assigned randomly to us as examiners and markers. Secondly, three female and three male examination candidates were interviewed in the week following the examination. The students were chosen on the basis of differing abilities, based on their internal school assessments for Calculus for the year. The paper was used as a heuristic for the interviews, which had the purpose of ascertaining students' calculator use not apparent in written answers. Thirdly, two examination markers, both Heads of Mathematics' Departments and experienced in using the calculators, were interviewed after marking had finished. They were asked about their perceptions of the form and adequacy of students' solutions. The examination paper was the focus of discussion in each interview.

ANALYSIS

In the analysis, we consider errors that can be classed as misuses of graphics calculators and misconceptions about calculator outputs. We define as misuse the use of a calculator function or procedure that leads to an answer in unacceptable form or which does not relate to the question. Misuse might arise from a student not knowing the meaning or limitations of calculator functions. In accordance with the notion of publicly acceptable mathematical knowledge (Noddings, 1990) the traditional standards for the Calculus TEE determine what are acceptable or unacceptable forms. However, a rigorous approach has been adopted here that does not necessarily reflect how students' answers were penalised in deficit.

Misconceptions can be explained in terms of constructivist theory, which assumes that knowledge is personal (van Glasersfeld, 1990): students interpret or conceptualise mathematics individually. A student may correctly enter a function into a calculator and copy exactly the screen-display of the graph but we define this to amount to a misconception if the student has not taken into account the limitations of the calculator when interpreting the display. Because misuses and misconceptions overlap, with calculator limitations explaining both, they are not always differentiated in the discussion. In addition to these errors, we consider the time-efficiency of some approaches and the adequacy of written reasoning to support graphics-calculator assisted answers.

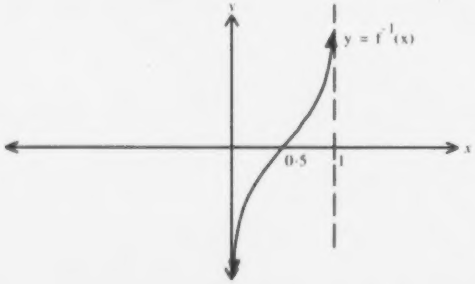
QUESTION RESPONSES

We restrict our discussion to the four questions for which errors attributable to calculator use were most widespread in the perused scripts. Each question is stated and our view of an acceptable calculator-assisted answer, informed by discussion with other markers, is given. Then, a summary of the numbers of students choosing various traditional and graphics calculator-based methods is provided, together with students' results for the various methods. The summary for each question is followed by a discussion of problematic student answers, including errors made and associated calculator usage.

For some scripts in the markers' sample, data are incomplete in that markers have identified the methods used or recorded the marks awarded for only some questions. These omissions in the recorded data are tabulated in the summaries and are accounted for by students not answering the question or using methods that were not listed as options on the data recording sheets. Calculator use associated with these methods, as suggested by the perused scripts, is discussed.

Question: Let $f(x)=\frac{1}{1+e^{-x}}$ for $-\infty < x < \infty$. Sketch the graph of the inverse of f, $f^{-1}(x)$, clearly indicating all intercepts and asymptotes.

Solution:



Results: The data summarised in Table 1 indicate whether students first graphed $f(x)$ and then reflected it over $y = x$, or first calculated $f^{-1}(x)$ and then graphed it. In view of the definitions of the given function and its inverse, students most likely used their graphics calculators for graphing in both methods.

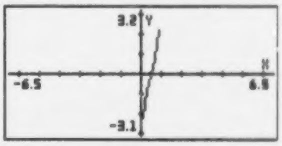
Table 1
Method Chosen by Students for Finding the Inverse Function

	Reflecting $f(x)$	Calculating $f^{-1}(x)$
No. students choosing the method	138	227
No. students whose mark was recorded	123	205
No. students with full marks	66	105
Mean mark for those with a recorded mark[1]	2.8	2.7

n = 404 (of which 39 were not recorded as choosing either of the listed methods)
[1]maximum = 4

Perusal of scripts showed that some students who chose the traditional method of calculating the inverse function $f^{-1}(x)$ sketched a graph with finite range. That is, they failed to recognise the asymptotes at $x = 1$ and $x = 1$. This misconception is an artifact of the screen resolution. The choice of an inappropriate scale (see Figure 1) results in a discretisation of the x-axis which does not enable a student to see the function values close to $x = 1$ or $x = 0$. That is, valuable information about asymptotic behaviour is not displayed and therefore lost. Another problem exhibited in students' written answers that is attributable to the calculator display (see Figure 1) was not showing the curvature of the graph, or showing it incorrectly. The mean mark of 2.7 out of a possible 4 (see Table 1) is explained by these errors, and also by students not obtaining $f^{-1}(x)$ correctly. However, 105 (51%) out of 205 students in the sample (see Table 1) who chose to calculate the inverse function scored full marks, an indication of competent calculator usage.

Figure 1
Calculator Display of $f^{-1}(x)$



Alternatives adopted by the 39 (10%) out of 404 students (see Table 1) not recorded as using the methods described above include graphing the reciprocal of $f(x)$ rather than its inverse. This is explained by students associating the graphics calculator *reciprocal* functions INVERSE and x^{-1} with the operation of finding the *inverse* of a function.

Question: Find the area of the region bounded by the curves $y = 3 + 2x - x^2$ and $y = -5$.

Solution: The area of the region is given by $\int_{-2}^{4} (3 + 2x - x^2 + 5)\,dx = 36$.

Results: The data summarised in Table 2 indicate whether students worked the integral symbolically, or gave only a numeric answer that suggests calculator evaluation.

Table 2
Method Chosen by Students for Evaluation of Integral

	Symbolic	Numeric Answer Only
No. students choosing the method	229	163
No. students whose mark was recorded	200	148
No. students with full marks	141	117
Mean mark for those with a recorded mark[1]	1.5	1.7

n = 404 (of which 12 were not recorded as choosing either of the listed methods)

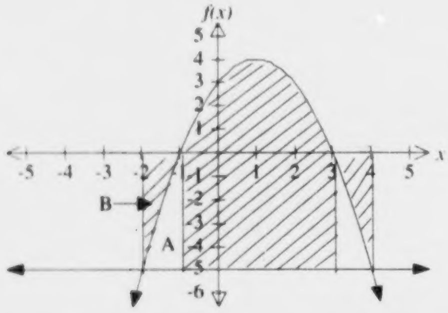
[1] maximum = 2 for the integral, after the points of intersection had been obtained.

A number of students split up the region, possibly because coordinates of graphs are readily available on graphics calculators. This approach often led to incorrect answers because of a misidentification of the integrals representing the desired part regions. For example,

$$\int_{-2}^{-1} (3 + 2x - x^2 + 5)\,dx$$

represents the area of region A (see Figure 2) but some students mistakenly evaluated area of region B, and duplicated their error for the integral between $x = 3$ and 4 (see Figure 2).

Figure 2
Incorrect Representation of the Region Enclosed by $y = 3 + 2x - x^2$ and $y = -5$

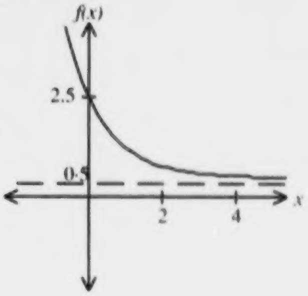


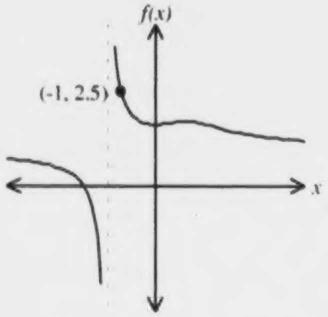
Another error was to write the answer as 35.9999, students not recognising the need to correct the inaccuracy of their calculator. The integral was relatively easy to evaluate symbolically so that the 163 (40%) out of 404 students in the sample (see Table 2) who used their calculators possibly gained no time advantage and the mean marks for the different methods were similar.

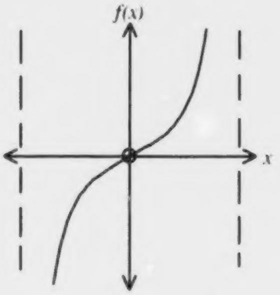
Question: Determine the following limits showing your reasoning.

(a) $\lim_{x\to\infty} \frac{e^x + 4}{2e^x}$ (b) $\lim_{x\to -1} \frac{x^3 + x^2 + 5}{x^3 + 3}$ (c) $\lim_{t\to 0} \frac{\tan^2(3t)}{t}$

Solutions:







(a) $\lim\limits_{x\to\infty} \dfrac{e^x+4}{2e^x} = 0.5$ (b) $\lim\limits_{x\to -1} \dfrac{x^3+x^2+5}{x^3+3} = 2.5$ (c) $\lim\limits_{t\to 0} \dfrac{\tan^2(3t)}{t} = 0$

Results: The data summarised in Table 3 indicate whether students used graphical or numeric methods that are both potentially graphics-calculator based, or traditional symbolic methods.

Table 3
Type of Method Chosen by Students for Evaluating Limits in Question 4(a), 4(b) and 4(c)

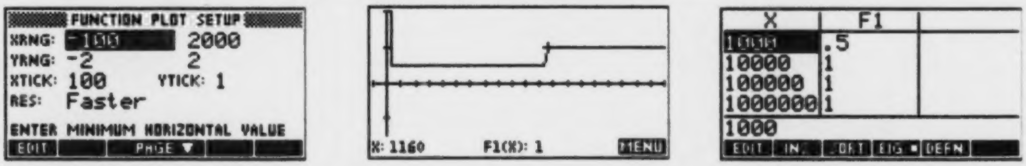
	graphical			numeric			symbolic		
	(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)
No. students choosing the method.	26	18	28	67	182	77	290	193	244
No. students whose mark was recorded	24	17	23	60	150	71	265	191	222
No. students with full marks	9	5	3	10	106	22	166	147	97
Mean mark for those with a recorded mark[1]	1.2	1.1	1.3	0.9	1.6	1.6	1.4	1.7	1.7

n=404 (of which 21 in (a), 11 in (b), 55 in (c) were not recorded as choosing any of the listed methods)
[1] maximum for (a) is 2, for (b) is 2, for (c) is 3

Only a small number of students (see Table 3) chose to provide graphical reasoning for the limits. For those who provided graphs, omissions were not including the asymptote in part (a), failing to identify the point (-1,2.5) either with its co-ordinates or with the scales on the axes in part (b) and not showing the discontinuity at $t = 0$ in part (c). The discontinuity is not easily apparent on a calculator-generated graph, leading to the misconception of portraying the function as continuous at the origin. Substitution reasoning was more commonly adopted than graphing (see Table 3) and often students' answers were displayed in tables similar to those on graphics calculators, suggesting that the technology was used rather than a scientific calculator. Insufficient reasoning was, in (a), providing only two substitutions which are not enough to show a trend; in (b), evaluating the expression at $x = -1$ without justification; and in (c) giving only one substitution either side of zero. The mean marks achieved for all limits for graphical and substitution approaches, and the proportions of students having chosen one of these approaches and receiving full marks, were lower than for traditional symbolic approaches (see Table 3). However, some of the differences were only marginal and for part (b) we question the large number of students recorded as using symbolic reasoning because perusal of scripts showed few instances of it.

Some students gave an answer of one for the limit in (a). This error is caused by the limitations to storing very large or very small numbers in graphics calculators. For example, for every $x > 1151.3$, e^x exceeds the capacity of the Hewlett Packard HP38G resulting in the graphical and calculation effects illustrated in Figure 3.

Figure 3
Graph and Table of Values for $f(x) = (e^x + 4)/(2e^x)$



FUNCTION PLOT SETUP
XRNG: -100 2000
YRNG: -2 2
XTICK: 100 YTICK: 1
RES: Faster
ENTER MINIMUM HORIZONTAL VALUE

X: 1160 F1(X): 1

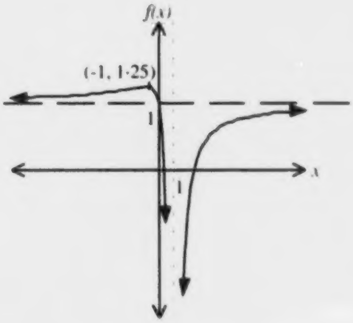
X	F1
1000	.5
10000	1
100000	1
1000000	1

1000

Question: A function is defined by the equation $f(x) = 1 - \frac{x}{(x-1)^2}$.

Sketch the graph of $f(x)$, indicating all asymptotes and the co-ordinates of any turning points.

Solution:



Results: The data summarised in Table 4 are the results for students who chose a symbolic approach to establishing the nature of the graph, and for those who appeared to use their graphics calculator by only providing the graph.

Table 4
Approach Students Chose for Graphing f(x)

	Working Shown	Graph Only
No. students choosing the method	63	321
No. students whose mark was recorded	59	289
No. students with full marks	18	66
Mean mark for those with a recorded mark[1]	3.6	3.7

n = 404 (of which 20 were not recorded as choosing either of the listed methods)
[1] maximum = 5

The 321 (79%) out of 404 students (see Table 4) who appeared to use their graphics calculators by providing the graph without working chose the time efficient option for answering this question. However, the mean mark of 3.7 and that only 66 (23%) out of 289 students providing the graph only (see Table 4) scored full marks, indicates students encountered difficulties in interpreting the calculator screen display. Errors were the inclusion of a turning point at $x = 1$, possibly found by running the cursor along to the bottom of the graph on the calculator screen; failure to identify the turning point at $x = -1$; and for the left branch of the graph to drop below the horizontal asymptote. In fact, close to half the students whose scripts were sighted failed to include the horizontal asymptote. However, judging by the mean marks (see Table 4), students who chose to provide the graph only did not score significantly differently than those who used symbolic working.

DISCUSSION

Interviews with students as to how they used their calculators were a vital part of explaining the source of students' errors, with calculator processes not being apparent from examination scripts. One student reminded us of the misuse of the INVERSE function, which perpetuates the error of plotting the reciprocal function $y = (f(x))^{-1}$ in answering a question asking for the inverse function $y = f^{-1}(x)$. Another gave a rich description of the difficulties encountered while attempting to interpret the screen-display of a graph. The analysis indicates that interpretation and transcription of graphs are major areas of difficulty for many students. In summary, errors were to interpret and copy graphical screen-displays so that (a) a function stopped on vertical asymptotes instead of approaching them; (b) a point discontinuity was not located; (c) the limiting value of a function was believed to be correct even though the capacity of the calculator to store large numbers had been exceeded; (d) horizontal asymptotes were omitted, and a function drawn to drop below its limiting value; (e) a non-existent turning point was located on an asymptote; and (f) a turning point was not located even when the question suggested one existed. Boers and Jones (1994) observed students to have similar difficulties with point discontinuities and asymptotes and Tobin (1995) discusses students' difficulties in copying calculator-generated graphs like we noted.

Other errors were incorrectly subdividing the integral associated with finding an area between two curves, and failing to round answers that were generated inaccurately on the calculator. In addition, students omitted to provide sufficient examples to support calculator-assisted numeric evaluation of limits, and to adequately label graphs when using them as reasoning for evaluating limits.

The outcomes for the last question discussed above indicate that a significant number of students (21%, n = 404) did not use their calculator for graphing even when there was a time-advantage to do so. This might suggest underutilisation of the technology as the first option for solving problems by some students, but does not preclude them from having used their calculators for checking. Underutilisation of the technology can also arise from students not knowing all the functions of their calculator appropriate for a subject, which was the case with some of the interviewed students, and from misinterpreting 'show reasoning' (see limit question) to mean 'show algebraic reasoning'. The reverse problem in questions not discussed here was of students providing graphical or numeric reasoning when an analytical approach was specified.

The results reported here suggest that calculator-based answers are not associated, in general, with higher (or lower) marks than traditional alternatives. The value of graphics calculators might therefore primarily be to develop students understanding in the teaching/learning process rather than as a tool to maximise assessment scores.

CONCLUSION

The first Calculus TEE where graphics calculators have been allowed has highlighted problem areas deriving from use of the technology. Of these the foremost are the interpretation of graphical information and students' apparent uncertainty as to when use of graphics calculators is appropriate. In this first examination with the technology, this uncertainty was accompanied by students' underutilisation of the calculators, which may cease to be a problem as calculator familiarity increases. Problems with the interpretation of graphical information indicate that instructional time needs to be spent on exploring the limitations of the calculators.

We sincerely thank the students and markers who contributed to our inquiry, and to the Curriculum Council of Western Australia for their support. Comments in this paper are not to be taken to represent the views of the Curriculum Council.

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SECONDARY MATHEMATICS TEACHERS' RESPONSES TO COMPUTERS AND THEIR BELIEFS ABOUT THE ROLE OF COMPUTERS IN THEIR TEACHING

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International studies have indicated that mathematics teachers have been slow in taking up the use of computers in their teaching even where the resources have been available. In this study, survey and case study procedures were used to determine how frequently, and for what purposes, secondary mathematics teachers used computers in some Australian schools. It also investigated teachers' beliefs about the potential of computers in mathematics teaching and learning. The findings showed that the teachers in this study rarely used the computer resources available, particularly if they taught junior (Years 8-10) and less able senior students. Only teachers with special expertise used computers regularly and, then, mostly for the teaching of calculus and statistics and usually in a calculational way.

Internationally, education researchers have expressed high expectations for the potential of computer technology to improve the teaching and learning of mathematics (Gentile, Clements & Battista, 1994; Kaput & Roschelle, 1997; Stirling & Gray, 1991). In Australia, education boards and teacher union bodies also share these high expectations (Australian Association of Mathematics Teachers Inc., 1996; Department of Education, 1995). However, studies have indicated that mathematics teachers have been slow to introduce the use of computers into their classroom activities, even when the hardware has been accessible (Becker, 1991; Marcinkiewicz, 1994; Oliver, 1994; Rosen & Weil, 1995; Zammit, 1992). As well, computer use has often been restricted to drill facts or skills rather than develop understandings (Becker, 1994).

This study examines how secondary mathematics teachers are responding to the potential of computing power in seven technology rich Queensland schools.

METHOD

This study is a series of embedded qualitative educational case studies (Stenhouse, 1990) involving a hermeneutic, interpretive and naturalistic approach (Denzin & Lincoln, 1994).

Subjects and Contexts

The study was carried out in seven Brisbane suburban schools. The schools were all considered to be technology rich in that the ratio to students to available computers was not more than four to one. The schools included both metropolitan state and private schools, but were not intended to be a representative sample of Queensland schools. Rather, they were selected because the computing coordinators and Heads of Departments (HOD) in each school indicated that access was not a major obstacle to teachers use of computers in their mathematics teaching.

Data Sources and Procedures

The data were collected over a twelve-month period through three phases. The first phase involved interviewing the computer coordinators and Mathematics HODs in each school to confirm the availability of hardware and software and to gain background information on the use of computers in each school. The responses informed the construction of survey instruments designed to probe teachers' frequency of use of computers and their beliefs about the effectiveness of computers compared to traditional instruction. The second phase involved surveying most of the teachers who taught mathematics in each school by using the instruments constructed after the first phase. The third phase was a series of ten case studies from two schools involving a hermeneutic, interpretive and naturalistic approach (Denzin & Lincoln, 1994). The purpose of this phase was to add greater insight on how teachers were responding in their classrooms to the potential of computing power, specifically their images of the roles computers could play in mathematics teaching and learning.

There were six components to the third phase. First, the teachers were interviewed about their beliefs on the use of computers in mathematics teaching. Second, at least two of their lessons were observed, teacher and student activity and black board presentations were documented with field notes, and the teachers' dialogues with their students were audio-recorded and transcribed. Most often these lessons did not involve the use of computer technology. However, each teacher was observed teaching topics of either calculus or statistics where the use of computers was possible and software were available within their school. Third, the teachers were re-interviewed for their reasons for conducting the classes in the way that they did and for their goals for able and less able students. Fourth, the researcher conducted in-service professional development for each case study teacher on the use of the mathematics software *Maths Helper* (Vaughan, 1997). The teachers were asked to evaluate the software and comment on how they might use it. Fifthly, the researcher constructed student activities based around the *Maths Helper* software, offered to teach the teachers' with this software classes, and interviewed the teachers concerning their observations. The *Maths Helper* software matched the concepts the teachers were about to teach their classes. The teachers were asked for their perceptions about how the students learned, the effectiveness of the lesson and how they would approach the teaching of the same lessons. Finally, after the data were analysed, the case study subjects were asked to comment upon the credibility of the researchers' interpretations of their beliefs about the potential of computers.

The inservice in *Maths Helper* and the subsequent questioning of teachers were designed to probe the teachers' beliefs about the potential of computers in mathematics teaching and learning. It is an indirect way of collecting data about beliefs and is considered more effective than direct questioning (Kagan, 1992). It was successful in assisting interpretation of teachers' beliefs. In particular, one case study teacher refused to allow the researcher to teach any of his classes even though the researcher had prepared specific computer based lessons on the topics his classes were studying (quadratic functions in Year ten and calculus in Year 11). This rejection was taken as a statement to indicate low evaluation of the potential of computers in mathematics teaching.

The combination use of survey instruments, interviews, classroom observations, responses to the technology intervention, and comments on the assertions, was used because it had the potential to develop a rich description and promote trustworthiness in the study.

Analysis

Analysis was emergent, contingent and cumulative; that is, constructivist (Guba & Lincoln, 1989). The survey data was analysed qualitatively; that is, inferential statistics were not used because the data did not meet the assumptions necessary for such tests to be applied. Each data collection interaction (survey instruments, interviews, observations of classes, responses to the technology intervention, and comments on the assertions) was used to inform following interactions and to construct theory. This process of building relationships between data and theory has been described as data theory boot strapping (Richards & Richards, 1994).

RESULTS AND DISCUSSION

Phase One

All the computer coordinators claimed that mathematics teachers under-used the computer resources available to them. Four of the coordinators said that teachers used the excuse of difficulty of access to computers to mask other reasons; for example; lack of knowledge of suitable software; concerns about changes to teachers' roles; lack of time to plan computer based mathematics learning; concerns about not covering the syllabus; and computer phobia.

Phases Two and Three: Frequency of Computer Use

Within the seven schools, 73% of all the mathematics staff completed the survey instruments. The staff who did not complete the instruments tended to teach subjects in addition to mathematics (e.g., science and manual arts); thus, the sample is not representative of the total school population. Table 1 summarises teachers' frequency of use of computers in their mathematics teaching. *Demonstrative* means that teachers used the software to illustrate an explanation by projecting the computer read-out onto screen for the students to observe, *Student use* means that the students used the computers, and *Communication* means that the teachers used the computers for communication (e.g., accessing the Internet).

Table 1
Percentage Frequency of Computer Use among Teachers (N=51)

Type of Use / Frequency of use	Demonstrative (Valid %)	Student use (Valid %)	Communication (Valid %)
Never used for	33	25.	33
Once or twice a year	37	31	37
Every month	18	20	18
Every week	8	14	8
Once or twice a week	2	8	2
Every day	2	2	2

The table shows that 33% of the teachers responded that they never used computers for demonstrations, 25% never had their students use computers and 33% never used computers for communication. Clearly the data indicates that computers were used sparingly by mathematics teachers in the schools. Only one (2%) of the fifty teachers sampled had students use computers every day, whilst about seven (14%) had students use computers weekly. Three of these seven teachers were clustered in the school where each student owned a lap-top computer, one teacher operated *SuccessMaker* for remedial students, and the others were subject coordinators or acting coordinators. Thus, it is apparent that not only were computers used infrequently by most teachers, but also the students who used computers in their learning of mathematics tended to be taught by subject coordinators or HODs; that is, the most experienced teachers in the schools. Very few ordinary classroom teachers frequently used or had their students frequently use computers as part of their learning. About 55% of teachers reported that they either never or hardly ever used computers in their classrooms. Fewer still used computers as a demonstrative tool.

Whether mathematics teachers used computers regularly or not tended to be a school-based decision, and this was strongly influenced by the beliefs of the senior mathematics teachers. For example, in one of the case study schools, the HOD and another senior teacher were regarded as the computer experts in mathematics. These teachers were acknowledged, by the school administration and the computer coordinator, as having great influence upon the way mathematics was taught in the school. They exercised this influence by allocating resource expenditure for software and texts book and writing most of the assessment items for mathematics. (The importance of text book selection (Carnine, 1993; Robinson, 1995) and assessment instruments (Barnes, Clarke & Stephens, 1996) has been recognised previously.) They believed that computers were not appropriate for teaching mathematics, particularly for concepts. As a consequence, except for occasional use of graphing calculators to teach statistics and calculus, computers were not used in the teaching of the senior or junior mathematics classes at the school.

As well, the use of computers was affected by the teachers' perceptions of their students. For example, in the other case-study school, computer use took two forms. In the junior part of the school, the less able junior students used computers while the able did not. The less able were given drill and practice using the software *Successmaker* once a week to help them learn basic mathematics skills and facts. In the senior school (Years 11 and 12), this was reversed; the able used computers while the less able did not. Mathematics B (Board of Senior Secondary School Studies, 1992a) students used computers in the study of calculus and statistics. For calculus, the students used graphing claculators or computers with the software *Capgraph* to plot functions and their derivatives. As a result, about half the Year 12 Mathematics B classes were competent in plotting functions and finding intersection points. For statistics, all students had some practice in summarising data using computers to calculate stem-and-leaf plots, histograms, box-and-whisker plots and measures of central tendency. Meanwhile, the teachers of senior students who did not study advanced senior mathematics (Mathematics B or Mathematics C) did not use computers in their teaching of these less able classes even though spreadsheets could have been used in the teaching of finance units and statistics software could also have been used.

Phases Two and Three: The Nature of Computer Use

In their description of teaching orientations Thompson, Phillip, Thompson, & Boyd, (1994) noted that mathematics teachers had two orientations, *calculational* and *conceptual*. These orientations can be extended to teachers' images of the potential of computers in mathematics teaching. For example, the calculational orientation can be used to describe an image of computer use where computers are used primarily as instruments to carry out calculations and procedures for deriving a numerical result, while the conceptual orientation can be used to describe an image of computer use where computers are used to focus students' attention upon the concepts underlying the mathematical procedures. This conceptual image of computer use is one where students' attention is toward rich conceptions, mathematical ideas and relationships between ideas.

Table 2
Percentage Effectiveness of Computers Compared with Traditional Instruction for Specific Objectives (N=51)

Specific Objectives	Response (Valid %)				
	1	2	3	4	5
A Makings sure that students get the right answer (Cal)	4	11	43	32	9
B Reinforcing each right answer (Cal)	2	7	43	23	25
C Providing students with practice in basic skills (Cal)	2	9	32	36	21
D Correcting student lack of understanding (Con)	14	35	25	21	5
E Helping student to construct their own representations of concepts (Con)	2	24	37	27	10
F Developing student higher order thinking skills (Con)	10	19	40	28	5
G Guiding students through prob solving process (Con)	16	21	30	30	2

Table 2 summarises teachers' responses to a survey instrument designed to probe teachers' beliefs about the effectiveness of using software compared with more traditional instruction. The items were selected from a number of scales designed by Mergendollar, Stoddart and Niederhauser (1992). Each item has been labelled either *Cal* to indicate it probes teachers' beliefs about the calculational use of computers in mathematics teaching or *Con* if it probes teachers' beliefs in the conceptual potential of computers in mathematics teaching and learning. Teachers responded on a 5 point Likert scale: 1 - "much less effective"; 2 - "somewhat less effective"; 3 - "same"; 4 - "somewhat more effective"; and 5 - "much more effective". That is, if they believed using computers was much more effective than traditional instruction in achieving an objective they responded with a "1."

One crude way of examining the data is to add the percentage of teachers who responded much less effective and somewhat less effective, thus for item A about 15% of teachers responded that using computers was less effective than traditional instruction for making sure students get the right answer, and about 41% responded that computers were more effective than traditional instruction. Thus, for the three items labelled "Cal" or calculational, between 41% and 57% of teachers believed that using computers was more effective than more traditional instructional in achieving these goals. On the other hand between 11% and 15% indicated that traditional instruction was more effective in helping students to achieve these calculational goals.

In contrast, 49% of teachers responded that traditional instruction was more effective than using computers for correcting student lack of understanding where only 26% of teachers believed using computers were more effective. Overall, between 26% and 37% of teachers believed that computers could be more effective than traditional instruction in achieving these conceptual objectives. In contrast, between 26% and 49% of the teachers responded that they believed that traditional instruction was more effective. Clearly the data shows teachers were more convinced of the potential of computers to be used to achieve calculational objectives rather conceptual objectives.

This deduction is supported by what happened in the case-study schools. Four months after the surveys were collected, several case-study teachers made the following comments supporting calculational use of computers: "good for number cruching", "taking the worry out of endless calculations", "computers are a time saver, you just plug the stuff (numbers) in", "doing things that we used to take ages to do like graphing lines and calculating standard deviations", "it's a mechanical devices that does all the work for you", "a gadget to do the work", "they can draw graphs faster than I can", and "I think it is very good for demonstrating a lot of things very quickly". The image of computers as being an instrument to take the tedium out of calculations and computations was the dominant image that the teachers saw of the potential of computers. The teachers did not mention the potential of computers to develop students conceptual understandings of mathematics concepts, nor did they use graphing calculators or computers in the observed lessons in a way that indicated they appreciated this potential. When they had an opportunity to observe a computer lesson, the teachers responded in a way which indicated that they were not inclined to use computers in a conceptual way. Rather, they made comments such as "I like the colour, it would be good on a large screen, demonstrating on a large screen". In addition, they made comments which indicated that they believed computers could hinder conceptual development: "if you let them do it on their own ... they are like parasites of (each other) rather than working themselves", "you see computers in conceptual development, I don't, the students must understand it first by doing it by hand", "technology won't help (develop understanding) it is up to the individual", "the kids won't see the processes, they will miss out on the concepts behind", "they (students) could miss out (on a concept) because they may be playing around with their computers", "students don't get the opportunity to work it out for themselves (when using computers)", and "I don't like the idea of relying upon it to teach a concept - I think it works better the other way (explain the concept first)".

Three of the teachers had both a calculational and conceptual image of the potential of computers at the beginning of the study and one teacher showed signs of appreciating their potential to be use in conceptual development at the following professional development. They showed appreciation for computers' having a role in developing student conceptual understanding by making statements such as: "I do on occasions use computers to develop concepts even if they are more time consuming", "well the good thing about computers is they are dumb ... you have got to formulate in your own mind and tell the computer how to do it ... all kids should learn programming ... I prefer kids to write their own spreadsheets", and "with the mathematics of the future, you're not going to sit down and calculate a derivative, a computer will do that for you, but you have to know why do I want to calculate this derivative, what is it going to tell me".

Thus, what teachers said supported the survey data in indicating that, although most mathematics teachers in the school had an appreciation the potential of computers to play a calculating role in student learning, few appreciated the potential of computers to play a role in student conceptual development. In fact, most teachers considered that using computers could interfere with conceptual development.

CONCLUSIONS AND IMPLICATIONS

A number of conclusions can be made from the data. First, secondary mathematics teachers in the schools surveyed have responded slowly to the potential of computers even when the resources have been available. Few teachers used computers at least weekly. When the interpreting the data from this study it must be remembered that the teachers who responded to the survey instruments were mostly specialist mathematics teachers and most of the case study teachers were very experienced senior teachers. In fact seven of the ten case study teachers were either HODs, AST (Advanced Skill Teachers) or senior coordinators. Frequency of computer use over the population of mathematics teachers in these schools would almost certainly be lower.

Second, most teachers appreciated the calculational potential of computers and rated them as equally effective or more effective than traditional instruction for doing calculations or providing practice in basic skills. Few teachers had images of computers as useful in developing students' conceptual understandings of mathematics concepts. In fact most teachers found educational explanations of how the use of computers could hinder student understanding of mathematics concepts. Thus, many of the teachers were arguing against the use of computers in the area in which they have great potential to contribute to mathematics learning, that is developing students' intuitions and understanding of concepts.

Third, none of the teachers said they used computers in the teaching of less able senior students who studied Mathematics A (Board of Senior Secondary School Studies, 1992b). These students were not only deprived of the potential of using computers to help them develop conceptual understandings, but were also deprived of the opportunity to learn industry based skills such as using spreadsheets.

These findings have implications for professional development and further research. First, professional development of secondary mathematics teachers in the use of technology needs to consider the use of computers in both calculational and conceptual contexts. To do this effectively further research is necessary to explore the relationships between teachers' images of teaching and learning, their educational goals and how these relate to computer use. That is we need to study why some teachers resist the use of computers in their teaching, particularly for helping some students to understand the concepts of mathematics and in their teaching of less able senior students.

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Acknowledgement

I would like to thank Associate Professor Tom Cooper for his assistance in the preparation of this paper.

VISUAL REPRESENTATIONS IN FIRST YEAR STATISTICS

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This paper summarises the findings of a study into the extent to which students in introductory statistics courses use diagrams when solving problems. The results of the study showed that many students did not use diagrams but those who used diagrams were more successful. Use of diagrams and university entrance score appeared to be better predictors of success than the level of mathematics studied at secondary school.

INTRODUCTION

At university many students study statistics because of its importance to disciplines such as economics, psychology, medicine, biology, and education. Statistics differs from school mathematics because of its emphasis on interpretation of data, rather than on solution of mathematical expressions. Statistical reasoning involves both visual and algebraic methods; visual or graphical methods are necessary for an initial inspection of the data to determine distribution characteristics, such as normality, dispersion, and outliers, while non-visual methods are predominantly employed in hypothesis testing and calculating confidence intervals. It is important that students can coordinate both visual and non-visual thinking when solving statistical problems.

Students who enrol in statistics with minimal secondary school mathematics may rely on learning procedures to solve statistical problems, without a sound understanding of fundamental concepts. Therefore, they may have difficulty in solving non-routine statistical problems for which they have to decide on an appropriate procedure and interpret its results. By contrast, students with strong mathematical backgrounds may be reluctant to use visual methods, such as graphing or drawing a diagram, because of an emphasis on algebraic methods in mathematics teaching. For example, Vinner (1989) found that tertiary students tended to prefer an algebraic proof to a diagrammatic one, even when the latter, as stated by students was easier to follow. He felt that this preference was affected by the method of teaching where students gained the impression that symbolic solutions were more prestigious than diagrammatic solutions.

Visual representations may be included as part of a problem information or students may produce their own representations as part of the solution process. In research on primary and secondary students' interpretations of graphs that accompanied statistical problems, Curcio (1987, 1996) identified three levels of graph comprehension: literal reading from the graph; comparisons using the graph with an emphasis on reading between the data; and reading beyond the data. If students are at the literal and comparative levels in their knowledge of graphs, they may have difficulty solving statistical problems when they are only given the raw data. However, some research suggests that students may respond at a higher level when the data is presented in raw form rather than in graphical form (Reading and Pegg, 1998). These authors analysed the responses of secondary students to two data reduction questions, one of which presented the raw data and the other a graphical representation. Other authors have questioned whether graphical ability is related to cognitive ability (Roth and McGinn, 1997). These authors suggest that graphical ability is a consequence of practice in using graphical methods in practical and social situations.

Students, who represent statistical data visually, for instance by using a stem and leaf plot or a graph, may be more successful in solving statistical problems than those who rely on algebraic methods. Shaw (1998) investigated whether the use of statistical displays by university students was associated with the use of diagrams in non-mathematical and mathematical problems. The results of her study indicated a relationship between the use of statistical displays and the use of diagrams in mathematical, but not non-mathematical situations. Moreover, students who spontaneously drew diagrams for the statistical problems were more likely to gain higher marks in the final statistical examination than students who did not.

The aim of this paper is to investigate the spontaneous use of diagrams by tertiary students as they solved statistical problems. The problems were chosen so that their solutions would be facilitated by drawing a diagram. The aims of the study were to:

- ascertain if students who drew diagrams to represent data were more successful than those who did not;
- investigate the relationship between ability, level of school mathematics, and the use of visual solution methods on students' success in solving statistical problems.

METHODOLOGY

Students in introductory statistics courses at two Australian universities in different states were given an assessment task comprising three statistical problems and one mathematical problem. At both universities the course was a large (>500) service course for first year students. Both courses covered displaying and summarising data, distributions and sampling distributions, hypothesis testing of means for one and two samples, regression and categorical data. The problems required extrapolation from and interpretation of the data, that is, responses at Curcio's third level. There were four versions of the task, the versions differing only in the order in which the problems were given. Students were asked to provide their University entrance score as an indicator of ability.

The tasks were not given at the same time, or under the same conditions at the two universities. At University 1 the assessment task was given in a lecture midway through the course when they had studied hypothesis testing with one and two samples but had not yet received any instruction on regression. Not all students were assessed; only those attending particular lectures, 186 students altogether. At University 2 the assessment task was set as a take-home assignment and the students were given 5 marks for completing it. Students were asked to spend no more than 30 minutes on the task, which was given at the end of the course, as students were preparing for their final examination. A total of 781 students completed the task.

Each student's response was given a diagram score of 0, 1 or 2 (0 = no diagram, 1 = partial diagram and 2 = correct diagram) and a score for the solution from 0 to 3 where 3 = correct solution. In addition, the students were asked to give the level of mathematics they had studied at secondary school and this was coded from 0 to 3 where 0 = no mathematics at Year 12 and 3 = the highest level of mathematics studied. However, in University 1, students with the highest level of mathematics enrol in a different statistics unit so the populations of students in the two universities were quite different.

The Assessment task

The four questions used in the assessment task are given below. The fourth question was a modification of a question used by Campbell et al (1995).

Problem 1 **Area under the normal curve**: The age of academic staff at Newport University are normally distributed with a mean of 38 years and a standard deviation of 5 years. What proportion of staff would be expected to be aged between 45 and 50 years?

Problem 2 **Linear relationship between two variables:** It has been claimed that as academic staff get older their tolerance of students decreases. A test of staff tolerance of students has been developed. Ages of a random sample of six staff at Newport University and their tolerance scores are listed below. Are the data likely to support this claim?

Name	Grey beard	Long legs	Mac	Boffin	Blondie	Shortie
Age	52	39	33	25	22	45
Tolerance	28	35	35	50	39	23

Problem 3 **Distribution of a single variable**: The employment history of a random sample of 30 academic staff at Newport University was obtained. Listed below are the number of years that they have worked at Newport University. As a person with statistical knowledge you have been asked to comment on this data.

1	7	5	2	5	6	4	5	5	6	2	2	2	6	6
3	5	6	9	5	5	1	3	4	7	5	2	2	7	6

Problem 4 **Linear algebra**: The blood alcohol readings of two lecturers from Newport University were recorded the morning after an accident. The readings were:

Alison	6 hours after accident: 5 units 8 hours after accident: 2 units	Brett	5 hours after accident: 7.5 units 9 hours after accident: 5.5 units

Assuming a linear relationship, when were their readings the same?

RESULTS

Relationship between Drawing a Diagram and Solving the Assessment Task

It is clear from Table 1 that as would be expected, the results are very different for the two universities.

Table 1.
Percentage of Responses from each University, Categorised by Diagram Usage and Solution Category for each Problem

	University 1		University 2	
	No solution, or inadequate solution	Reasonable or correct solution	No solution, or inadequate solution	Reasonable or correct solution
Problem 1	*Areas under the normal curve*			
No diagram*	48	13	16	31
Correct diagram	15	24	5	45
Problem 2	*Linear relationship between two variables*			
No diagram*	61	23	26	17
Correct diagram	3	13	8	50
Problem 3	*Distribution of a single variable*			
No diagram*	57	7	26	5
Correct diagram	17	19	24	45
Problem 4	*Linear algebra*			
No diagram*	60	19	14	42
Correct diagram	4	17	1	43

* A partial diagram was included in the "No diagram" category as such diagrams were usually inadequate as an aid to obtaining a solution.
No solution, or inadequate solution ¾ score of 0 or 1; Reasonable or correct solution ¾ score of 2 or 3.

The results for University 1 show that in general these students did not draw diagrams and the majority did not successfully solve the problems. For Problem 1, the most familiar to the students, more students drew diagrams (39%), than for Problems 2, 3, and 4 for which the percentages were 16%, 34% and 21% respectively. For this sample of students Problem 1 would have been the most familiar and in lectures and tutorials diagrams would have been emphasised for problems of this type. Problem 2 was given to students from this university before they had been given any instruction in regression and their responses reflect ideas developed at secondary school. The majority of the students (84%) did not draw a diagram and less than a quarter of these students obtained the correct answer whereas 81% of the students who drew a diagram successfully solved the problem.

The results for University 2 clearly show that this sample of students was much more successful in solving statistical problems and many more of these students drew diagrams. Nevertheless, a large proportion of them (approximately 30 to 55%) did not draw diagrams and these students were less successful in solving the problems than those who drew, particularly for Problems 2 and 3. These two problems were both open-ended and less familiar in form than Problems 1 and 4.

Table 2 shows the relationship between diagram usage and correctness; only students who obtained a maximum score of 3 for each problem are included. The results presented in Table 2 show that for all four problems those students who drew a diagram were far more likely to obtain a successful solution than those who did not, particularly for Problems 2 and 3. For Problem 2 many students who stated that there was no relationship between the two variables were influenced by their interpretations of one or two points, rather than looking at the global trend. In Problem 3 most students simply calculated measures of centre and spread without investigating the form of the distribution. Such a response was coded as 1, an inadequate solution. In fact, the data was bimodal with peaks at 2 and between 5 and 6.

Table 2
The Percentage of Students who Correctly Solved (score=3) each Problem, Categorised by Diagram Usage and University

Problem	1	2	3	4
University 1				
Correct solution, no diagram	11	14	1	21
Correct solution, diagram	46	57	18	58
University 2				
Correct solution, no diagram	62	22	0	70
Correct solution, diagram	84	87	15	95

However, it is evident from Table 2 that for University 2 many students who did not draw a diagram were successful in solving Problems 1 and 4. Problem 1 was a standard statistical problem, finding an area under the normal curve, and Problem 4 was a mathematical question, rather than a statistical one. The latter could be answered graphically, algebraically or using ratios. Students who had a higher level of mathematics at secondary school might be expected to be more successful if they chose to solve these problems algebraically. There is some evidence that this is so; 71% of the highest mathematics group were successful for Problem 1 without drawing a diagram, compared with 90% who drew a diagram. For Problem 4 the corresponding percentages are 84% and 98% respectively.

In the next section we investigate the effects of school mathematics; overall university entrance score (as an indicator of general ability); and drawing a diagram on students' success in solving these problems

Relationship between Ability, Level of School Mathematics, and use of Visual Solution Methods on Success in Solving Statistical Problems.

A regression analysis was carried out on the results for each university to determine the effect of the three factors on problem solution. Total scores for problem success and diagram usage were obtained for each student by summing the individual scores for the four problems. Additional analyses were undertaken for (a) Problems 1 and 4 and (b) Problems 2 and 3; the same procedure of adding individual scores was used to obtain scores for these problem subsets. The results for the analysis of all four problems is set out in Table 3. There was a high proportion of missing values, particularly for University 1, as not all students included information on their university entrance score when they completed the assessment task.

The results shown in Table 3 indicate that for both universities, university entrance score and drawing a diagram are significant indicators of success on the four tasks. School mathematics, while a significant predictor for University 2, is not significant for University 1. However, these three factors account for about 35% of the total variance in the students' responses. Drawing a diagram was highly correlated with total score on the four problems (0.51 for both universities) but level of school mathematics was not correlated with drawing a diagram (0.02 and 0.03). Correlations for finding a correct solution and school mathematics were 0.11 and 0.19 for University 1 and 2 respectively.

Table 3
Regression Analyses for the Effect of School Mathematics, Ability and Diagram Usage on Success in Solving the Assessment Task

University 1 (n=89)

Predictor	Constant	Ability	Diagram	School maths
Coefficient	-3.32	0.08	0.46	0.32
t-ratio	-2.26	4.56	3.98	0.94
Probability	0.026	0.000	0.000	0.348

R- square = 36.9%, F(3,90)=25.21, P=0.000

University 2 (N=656 subjects)

Predictor	Constant	Ability	Diagram	School maths
Coefficient	4.26	-0.18	0.59	0.65
t-ratio	7.77	- 6.12	16.38	4.45
Probability	0.000	0.000	0.000	0.000

R- square = 34.5%, F(3,652)=114.64, P=0.000

There are several reasons that may explain the discrepant results for school mathematics. First, the students who have the highest school mathematics results enrol in a separate course, so they were not included in the results for University 1. Second, the sample size for University 1 is small relative to University 2 and contains a high proportion of mature age students, some of whom may not have taken mathematics at secondary school and others for whom mathematics is but a distant memory. The age distributions showed that about 58% of University 1 were less than 24 years of age compared with 96% of University 2.

The data were re-analysed to investigate the possibility that students from University 2 with higher levels of mathematics might have been more likely to solve Problem 1 and 4 algebraically than students with lower levels of mathematics. The problems were combined into subsets, Problems 1 and 4, and Problems 2 and 3. Separate regression analyses were undertaken for these subsets for both universities. Table 4 shows the percentages of the variance (R-square) in scores for the problem subsets that is explained by (a) all three factors as predictors, and (b) inclusion of a diagram as the sole predictor.

Table 4
R-square Values for Regression Analyses of the Problem Subsets

Subset	University 1		University 2	
	All factors N=89	Diagram only N=186	All factors N=655	Diagram only N=780
Problems 1 and 4	30.9*	26.0	20.6***	9.8
Problems 2 and 3	56.7^{+}	50.4	42.2^{++}	41.1

* School mathematics and ability significant at $p<0.05$.
*** School mathematics and ability significant at $p<0.001$.
+ Ability was significant ($p<0.005$).
++ Neither school mathematics, nor ability significant ($p>0.05$)

The values of R-square shown in Table 4 suggest that knowledge of school mathematics contributes most to the problems that can be solved algebraically, that is, Problems 1 and 4. The values for this subset were quite different for the two universities; for University 2, which had a more mathematically competent enrolment, students were less dependent on diagrammatic methods to solve the two problems. By contrast, for Problems 2 and 3, drawing a diagram was the key predictor of success in both universities.

The means for total and diagram score for each problem subset categorised by ability level (as measured by university entrance score) are shown in Table 5. The ability levels were obtained by categorising the university entrance scores into 4 and 5 categories respectively for Universities 1 and 2, where 1 is the lowest ability category and 4 or 5 is the highest. The scores for the lowest ability categories for both universities on Problems 1 and 4 were slightly anomalous in that they are higher than those of students with a higher level of mathematics. One explanation may be that this category may include a high proportion of mature-age students or those who have not enrolled through the usual admission process. For Problems 2 and 3 this trend can be seen in the means for University 2, but not University 1.

Table 5
Mean Total and Diagram Score for Each Level of Ability. Standard Deviations are Shown in Brackets

Ability level	N*	Problems 1&4		Problems 2&3	
		Solutions	Diagram score	Solutions	Diagram score
			University 1		
1	7	2.6 (1.7)	2.3 (1.0)	1.3 (1.3)	1.1 (1.1)
2	18	1.8 (1.8)	1.4 (1.5)	1.7 (1.6)	1.0 (1.4)
3	26	2.6 (2.4)	1.9 (1.4)	2.4 (1.5)	1.3 (1.4)
4	37	3.6 (2.1)	2.4 (1.4)	2.4 (1.6)	1.4 (1.5)
			University 2		
1	34	4.3 (1.6)	2.1 (1.1)	3.4 (1.6)	2.3 (1.2)
2	103	4.2 (2.1)	2.2 (1.4)	3.3 (1.6)	2.5 (1.4)
3	126	4.7 (1.7)	2.3 (1.4)	3.4 (1.5)	2.6 (1.4)
4	185	5.1 (1.5)	2.3 (1.4)	3.6 (1.5)	2.7 (1.3)
5	221	5.5 (1.1)	2.1 (1.4)	4.0 (1.5)	2.9 (1.3)

* Number of students included in each ability level.

CONCLUSIONS

The results for these two universities clearly show that students who draw a diagram as part of the solution process were far more successful than students who do not. The majority of students in University 1 did not draw diagrams, despite being encouraged to do so throughout their course. A far higher proportion of students from University 2 drew diagrams and were overall far more successful in solving the four problems. However, as the administration of the problems were so different for the two samples, it is impossible to make comparisons. Perhaps, the students who completed the problems as a take-home task may have drawn diagrams on scrap-paper (despite having been asked to include all working). Moreover, this sample included a far higher proportion of younger students with higher levels of school mathematics as well as higher university entrance scores. These students may have been confident to change their solution strategy. Thus, they drew diagrams when they realised it was necessary, otherwise they solved the problems algebraically.

Why then do students not see that diagrams are a useful problem solving strategy? One reason that came out of interviews with a small sample of students (Shaw and Outhred, 1999) was that students were not sure that a diagram was worth the effort it took to draw it, rather than not knowing what to draw. Students seemed to want to calculate statistics, such as means and standard deviations without first obtaining a feel for the data. None of the students who were interviewed suggested a diagram might be an easier method of solving some types of problems.

We thought that students who had studied more mathematics at secondary school would be more likely to draw a diagram and also be more successful in solving the problems. However, the analysis of the data did not seem to support this hypothesis. Tertiary entrance score was found to be a better predictor of success than level of secondary school mathematics although this relationship did appear to be affected by the type of problem. Problems that were more readily solved by algebraic methods seemed to be related with level of school mathematics, whereas the problems that involved interpretation of data sets did not seem to be related. The relatively poor performance on the problems that required interpretation of data would suggest a need for the inclusion of open-ended tasks based on real or simulated data sets so that students discuss different perceptions of the same information.

Since students who draw diagrams are more successful than those who do not, greater emphasis should to be given to integrating the use of diagrams into the teaching of introductory statistics. In addition, the construction and use of diagrams may need to be incorporated into course materials and assessment. An investigation of the teaching approaches of the two universities would seem to be warrented to determine if this might contribute to the differences in the proportions of students who use diagrams.

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AN ANALYSIS OF YEAR 12 STUDENTS' PERFORMANCES ON BASIC ALGEBRA QUESTIONS

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This paper reports details concerning the performance of capable Year 12 Mathematics students in their final public examination in NSW. The focus of this analysis is on basic algebraic skills tested in the first question of the 2 unit paper. This topic has been encountered by students for a number of years and provides, potentially, an upper bound on what might be expected from such students in an examination. Success rates and the types of strategies employed by students are documented and discussed.

INTRODUCTION

The Higher School Certificate (HSC) is a critical examination for nearly 60,000 students in NSW. In Mathematics, there are five courses of study available for students to choose, and, surprisingly, when compared with other states in Australia, nearly 95% of students opt to take one of these courses. The courses are designed to cater for students of different abilities and aspirations. Of interest to this research are the performances, on a common paper, of students who choose the courses referred to as 2 unit related and 3 unit Mathematics. These two courses are designed for students who wish to take programs of study at university which require tertiary mathematics (such as economics or biology) and those who wish to pursue a program in tertiary mathematics, respectively. As such, students in this sample represent the top 50% – 95% of the student cohort in terms of overall achievement in Mathematics. (Another course referred to as 4 unit Mathematics is undertaken by the very best students in Mathematics who represent the top 5% of the student cohort in terms of mathematical ability. This group does not sit for the common 2 unit paper and is not included in the analysis below.)

Similiar to other large public examination systems, student results are reported using norm-referenced measures. These provide an indication to the public about the relative standing of students in terms of their relationship with their peers, and a mark that reflects, in a general way, students' achievements. However, there is no indication in these marks of whether the cohort has performed better or worse than previous years nor what the mark actually means. As an initial attempt to address these two issues this paper reports upon a small aspect of a large study which is designed to investigate actual student achievement in NSW Higher School Certificate examinations in Mathematics. Other reports on this investigation have been presented in other forums (Pegg, 1997; Pegg, Hadfield & Hastings, 1997). This form of information has not been available previously and represents a first attempt in NSW at identifying what students, at the end of their secondary schooling, know, understand and can do.

BACKGROUND

Student performances in Mathematics at the HSC are reported by a standardised mark usually set at a mean of 60 and standard deviation of 12.5. This, together with a percentile banding, provides an indication of a student's performance. The scaling procedure is such that the mark reported is one which the general public can interpret. This means that marks of 80 or above represent high achievement whereas marks below 50 are usually considered to represent poor achievement or failure.

However, while the marks allow a ranking of students in the State, they do not (and are not designed to) indicate a measure of what students know and understand. As a balance to this position there are plans emerging (not unique to NSW) for the reporting of the examination results to take on a more qualitative stance. However, before this can be achieved it is important that analyses are undertaken to provide some indication of the competencies shown currently by students to provide the basis for realistic criteria to be established. This process mirrors current developments in business and related fields associated with benchmarking (Bartos, 1994; Zairi, 1992).

This paper reports on an investigation into this area by documenting the performances of the selected cohort of students, over three years, on basic algebraic questions which would be seen to be among the most straight forward, familiar and less abstract of the Mathematics items tested.

DESIGN

With the support of the Board of Studies of NSW, 1000 scripts were randomly selected for each of the ten questions asked in the HSC 2 unit related examination papers for 1995, 1996 and 1997. All indications of student numbers and centres of examination were removed from the scripts. It is this sample upon which the data reported in this paper are based. Initially a small subset of papers (n=100) were categorised. A co-worker familiar with the Mathematics content was provided with the scripts and the coding categories. The agreement of categoriastion was 94%. When this was completed the full sample of scripts was analysed.

The questions chosen for the analysis are provided below. Because of space restrictions only student performances on five questions are documented here.

From 1995

Q1 (a) Factorise $9x^2 - 16$. (1 mark)

Q1 (d) Simplify ${}^{x}/_{3} + {}^{(3x - 1)}/_{2}$ (2 marks)

From 1996

Q1 (b) Factorise $2x^2 + 3x - 2$. (2 marks)

Q1 (d) Simplify ${}^{2}/_{3} - {}^{(x - 1)}/_{4}$ (2 marks)

From 1997

Q1 (b) Simplify $(2 - 3x) - (5 - 4x)$. (2 marks)

To facilitate the discussion, the analysis is considered under two broad sections.

RESULTS

Section 1

This section considers the two question parts Q1(a) (1995), Q1(b) (1996). In both cases students were expected to factorise a quadratic expression. A summary of the results is provided in Table 1.

Table 1
Percentage of Students Attaining Allocated Marks

	No attempt	0	1	2
Q1(a) (1995)	4	23	73	N/A
Q1(b) (1996)	4	17	6	73

For Q1(a) (1995), of the 73% who were awarded 1 mark (i.e., those who were correct), 3% re-interpreted the question and formed an equation (i.e., equated the expression provided with zero) and solved this to obtain $x = \pm {}^{4}/_{3}$.

Of the 23% who were awarded zero marks, five approaches were identified. In these cases students:

1. confused the expression given with a binomial squared, i.e., $(3x - 4)^2$ 4%
2. did not factorise as a difference of two squares, e.g.,
    - $9(x - 4)(x + 4)$
    - $(x - 3)(x + 4)$
    - $(9x + 4)(x - 4)$ 6%
3. attempted to solve $9x^2 - 16 = 0$, e.g.,
    - $9x^2 = 16$

      $x = \sqrt{7}$ 5%
4. did not write answer in a factorised form, e.g.,
    - $3x^2 - 4$
    - $(x - 3)(x + 3) - 16$
    - $12x$ 5%
5. found a common factor, e.g.,
    - $9x(x - {}^{16}/_{9x})$
    - $3(3x^2 - 5\,{}^{1}/_{3})$
    - $2({}^{9x^2}/_{2} - 8)$ 3%

Overall, the majority of the 23% of students who were incorrect on this question part did not appear to make errors in factorising the difference of two squares. Instead, students did not seem to recognise the expression as a difference of two squares.

For Q1(b) (1996), 73% of students were correct (were awarded 2 marks). Of these, approximately 10% of students also found a solution to $2x^2 + 3x - 2 = 0$. The most common method indicated by those who were correct was the 'cross-method' (45% of students), however, 20% of students who provided the correct answer showed no working.

In all, 23% of students were incorrect. About half of these attempted to factorise the expression, but made errors. The majority of students who provided an incorrect response used a method other than the 'cross-method'.

Of those students who were incorrect, 40%, (i.e., 9% of total), did not factorise at all. Approximately half of the students attempted to solve $2x^2 + 3x - 2 = 0$ but were unable to produce the correct answer. Others attempted to simplify the expression, however, these students demonstrated very poor algebra skills. Interestingly, one student from the sample registered five incorrect attempts to find two binomials whose product was $2x^2 + 3x - 2$. There were five incorrect strategies identified. In these cases students:

1. provided incorrect factors, such as: $(2x + 1)(x - 2)$ 12%
2. provided incorrect factors and solutions to $2x^2 + 3x - 2 = 0$ 2%
3. applied the quadratic formula to solve $2x^2 + 3x - 2 = 0$ 4%
4. applied inappropriate algebraic simplification techniques 4%
5. applied trial and error techniques by selecting two binomial products and multyplying them out, e.g., $(x + 3)(2x - 2)$ 1%

In this question, apart from the 4% who did not attempt the question, all but 8% of the students produced two binomial products for their answer. Also of interest were the attempts by 13% of the students who changed what was asked and set up and solved an equation.

These two questions, asked of different students in different years show remarkably similiar patterns of results. In both cases 73% were correct and approximately the same number of students chose to solve an equation.

Section 2.

This section considers the three question parts Q1(d) (1995), Q1(d) (1996), and Q1(b) (1997). Q1(d) (1995) and (1996) involved the simplification of algebraic fractions, whereas Q1(b) (1997) required removing brackets and collecting like terms. A summary of the results is provided in Table 2.

Table 2
Percentage of Students Attaining Allocated Marks

	No attempt	0	1	2
Q1(a) (1995)	3	17	15	65
Q1(b) (1996)	5	11	34	50
Q1(b) (1997)	0	15	10	75

For Q1(d) (1995), 65% of the students were correct and a further 15% were able to make some progress towards the solution. Five categories of incorrect responses were identified. In these cases, students:

1. manipulated correctly the fractions, but did not simplfy the resulting expression 5%
2. made a calculation error, e.g., 5%
3. experienced problems with adding fractions
   - denominator omitted 14%
   - faulty knowledge of addition of fractions demonstrated, e.g.,

     $x/3 + (3x - 1)/2 = (4x - 1)/5$

     $x/3 + (3x - 1)/2 = 3/x \times (3x - 1)/2$ 5%
4. made a cancelling error simplifying the answer, e.g.,

   $(11x - 3)/6 = (11x - 1)/2$ 1%
5. attempted to solve an equation by making the expression equal to zero 2%

Overall, 32% of students were incorrect. About half of these students showed very little knowledge of adding fractions, especially algebraic fractions, e.g., many had no common denominator in their answers, or added the given numerators.

Q1(d) (1996) was similiar to the previous question, discussed above, except that this question required the subtraction of two algebraic expressions and not the addition. As expected the number of students correct was much lower here than for the previous question, i.e., 50% compared with 65%. There were five categories of incorrect responses identified. In these cases students:

1. manipulated correctly the fractions, but did not simplfy the resulting expression 4%

2. made a calculation error, e.g.,
   $(8 - 3x + 3)/12 = (12 - 3x)/12$ 9%
3. ignored the effect of the minus sign on the second term, e.g.,
   $8/12 - (3x - 3)/12 = (8 - 3x + 3)/12 = (5 - 3x)/12$ 12%
4. experienced problems with subtracting fractions
   - denominator omitted, e.g.,
     $8/12 - (3x - 3)/12 = 8 - 3x + 3 = 11 - 3x$ 6%
   - faulty knowledge of subtraction of fractions demonstrated, e.g.,
     $8/12 - (3x - 3)/12 = 8(3x - 3)/12$ 12%
5. attempted to solve an equation by making the expression equal to zero 2%

The large difference in performance on this question compared with the previous question is now able to be explained. The presence of the minus sign meant that 12% of the students made a specific error (see strategy 3 above). When this is taken into account the performance of students on this question across both years is remarkably similiar.

For Q1(b) (1997), 75% of students were correct. This is the best result of all the items reported in this paper.

About one third of those incorrect (i.e., 9% of total) simplified (correctly or incorrectly) then solved for x. Others, 40% of incorrect students (10% of total), calculated the expression as a product, i.e., $(2 - 3x)(-5 + 4x)$.

Most of the remaining errors occurred because students did not change the sign when removing the brackets. Considering that this aspect of simplifying is emphasised in class from the time students do this topic in Year 9, it is surprising that approximately 10% of students still make this error.

In all there were four categories of errors identified. Students:

1. simplified correctly and solved for x 7%
2. simplified incorrectly and solved for x 2%
3. calculated the binomial product $(2 - 3x)(-5 + 4x)$ 10%
4. made errors in collecting like terms, e.g.,
   $(2 - 3x) - (5 - 4x) = 2 - 3x - 5 - 4x = -3 - 7x$ 6%

Overall, 82% of students were able to successfully simplify the expression. However, 7% of students went further and equated the expression to zero and solved an equation. Given that this form of question would have been encountered in Year 8, and students would have completed many examples of a similiar nature over the next four years this result is disappointing.

DISCUSSION

In summary, these data suggest that approximately:

- three quarters of the cohort can successfully factorise a binomial or trinomial quadratic;
- two thirds of the cohort can add two algebraic fractions but when a subtraction is introduced it drops the success rate down to about 50%; and
- three quarters of the cohort can successfully expand out brackets involving a minus sign and collect like terms.

Significant problems identified in student responses include:

- an inability to recognise the difference of two squares;
- a tendency to want to reinterpret questions so that they became equations to be solved:
- an inability to understand the place of numerators and denominators in fractions; and
- difficulties in monitoring basic algebraic and arithmetic operations while undertaking the solution process.

The cohort of students in this sample is drawn from the top 50% of the Year 12 ability range in Mathematics in NSW. It is from this group of students that future mathematics teachers, engineers, and the like are predominantly drawn. As such this group represents an important, even significant, body of students to a society in terms of future responsibilities. Significantly for many of these students, competence at Mathematics is going to be an important pre-requisite.

The questions reported are among the most straight forward and least abstract of all the questions posed in the two courses. However, with overall success rates of between 75% and 50% it is clear that this perception of low difficulty is not a view that students would seem to share. A confounding variable is that the ideas tested and the very question types are ones that all students would have encountered on many occasions over the previous four years. Why, after all this, the students' performances are as low as they are is difficult to explain.

When these tasks are considered from a cognitive perspective the results are also disconcerting. One would expect the working memory demands of such questions to be relatively light. Because of their preponderance in other years and overall familiarity, one would expect that all students would have developed routines and procedures for attempting such questions. In addition, the 'traps' or typical mistakes made in questions with minus signs must have been encountered many hundreds of times and should come as no surprise to students of this calibre. However, the results show that for many of these students these assumptions are not true and that there responses are no better than would be expected of many Year 9 students. It is as if these students have not progressed despite four years of continuing exposure to attempting such questions.

CONCLUSION AND IMPLICATIONS

The results summarised in this paper are taken from students' attempts at undertaking an external examination. Because of this, it is difficult to pass off the results in terms of students not wanting or choosing to give their best. Instead, it could be argued that given the importance of the examination and the preparation time allocated (in terms of trial papers and extensive revision over several weeks) that the responses provided by students should represent an optimum performance. This being the case, then there is little opportunity for complacency in the findings for Mathematics educators, parents or students.

Given that these questions were selected because of their familiarity and ease, it will be interesting to identify the trends in other topics tested in which the material is more abstract and the time exposure to the ideas is considerably less than was the case with the questions surveyed here. It is only by such analysis that a clear picture can be developed of the true state of Mathematics learning in our senior years. However, the process of collecting these data is time consuming and expensive.

This research does highlight the need to go beyond reporting student performance at the HSC in terms of marks which allow norm-based referencing. It is also clear that procedures which are currently in place actually mask the real extent or weaknesses of student understandings. While it is not a panacea, it does seem, from our perspective, that if teachers were made aware of their students' performance in terms of what they actually know and understand, then a serious commencement could be made in helping students reach their true potential.

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Acknowledgment

The support received from the Australian Research Council (A.R.C. Ref. No. A79231258) and the Board of Studies in NSW towards carrying out this research is appreciated. The views expressed are those of the authors.

SOME LANGUAGE ISSUES IN THE ASSESSMENT OF SECONDARY MATHEMATICS

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Many students have difficulty with the language used in both the learning and assessment of mathematics. This paper investigates the use of English in the assessment of senior secondary mathematics in New South Wales. Using input from students, along with a linguistic perspective, it analyses a question from a recent NSW Higher School Certificate examination and suggests how the language of the assessment might be changed in order to make the mathematics in it more accessible to the students.

BACKGROUND

Part of the rationale underpinning the NSW Higher School Certificate Mathematics courses, *Mathematics in Society* and *Mathematics in Practice*[1], is that they are studied "... by beginning with situations which involve mathematics rather than by beginning with the mathematics itself." ..." (NSW Board of Studies, 1989, p.2). These courses attempt to make the mathematics studied more relevant, more accessible and less abstract than other mathematics courses offered at Higher School Certificate level. This is reflected in topics such as consumerism, transport, accommodation, chance and gambling and measuring which students might expect to encounter in their everyday lives. However, these contexts need to be examined in the light of the language that is used to make them meaningful.

While, over recent years, there has been a great deal of attention paid to the role of language in mathematics learning and teaching, much of this has concentrated on the discourse in mathematics classrooms, either between learners or between teachers and learners (Bickmore-Brand, 1990; Chapman, 1997; Ellerton & Clements, 1991; Mousley & Marks, 1991; Mousley & Sullivan, 1996) and very little on written language which might be used in texts or assessment items (Love & Pimm, 1996; Newman, 1983; Veel, 1995). Using student voices as a data source, this paper considers a linguistic analysis of a *Mathematics in Society* question, and leads to some suggestions for making the mathematics more accessible to the students.

METHOD

In the 1996 *Mathematics in Society* Higher School Certificate examination (NSW Board of Studies, 1996), Question 21(c) is presented as an advertisement for a clothing store-a real life situation to which students are assumed to be able to relate. The text of this question is presented in Figure 1. An illustration accompanied the text in the examination paper but it has been omitted here, following the advice of the students involved in this study who suggested strongly that they ignored the illustration while reading and trying to solve the question.

Student Voices

Input to the analysis of questions from the 1996 Higher School Certificate *Mathematics in Society* examination paper was obtained from a small group of five Year 11 students who attended school in the South West of Sydney. The students (3 male, 2 female from a variety of ethnic backgrounds) were asked to analyse this question, among others, by :

1) attempting to solve the question individually;
2) in gender groups and then as one group, discussing the question in terms of how they feel about it, what it is asking them to do, how they would go about doing that, what words they focus on, whether the description of the situation and the illustrations are helpful, whether they have to read all of the information and what the phrases *greatest percentage discount* and *uppermost numbers* mean to them.
3) in gender groups, rewrite the question so that the mathematics is more accessible.

While no claims can be made about the generalisability of the results from this preliminary study, they do point to some concerns which need to be addressed by mathematics educators, both in secondary schools and beyond.

Figure 1
1996 Higher School Certificate, Mathematics in Society, Question 21 (c)

Jo's Clothes is having a sale. To find the percentage discount on an item, a customer rolls three dice and adds the uppermost numbers.

(i) What is the greatest percentage discount that could be received?
(ii) For an item marked $100, how much would be paid if the smallest possible discount is rolled?
(iii) In how many ways can three dice be rolled to obtain a total of 5?
(iv) Calculate the probability of rolling a 5% discount.
(v) What is the probability of paying $95 or more for an item marked $100?

RESULTS

The students were fairly relaxed about the actual question although one did suggest that he "hated probability". Others suggested that the way in which the question was written and the perceived difficulty of the words used could influence how they felt about it. The word *uppermost* in Question 21 (c) was mentioned as being unnecessarily complicated while one student asked about the noun group *greatest percentage discount*, "Is it like a hundred or the highest you can get in that question?" This suggests a level of confusion which may make it impossible for the student to deal with the rest of the question.

When the students were asked about how they go about doing questions like these, they suggested that they: read the question thoroughly; made sure they understood it and underlined the important words.

The students focussed on the words *how much*, *what is the greatest percentage discount*, *calculate* and *what is*. They were of the opinion that if the questions started off with what you had to do - *calculate, multiply, find*, for example - then that would make it easier to get to the mathematics because "it tells you straight away that you have to do this. It's not you have to read the whole thing then at the end it tells you what to do and you have to go back to the beginning of the question then go straight through it again and then you work out the answer. This way it will tell you at the beginning and you can work it out straight away." The need for an active voice in the language used can be critical for some speakers of English as a second language whose original language may not deal with passivity of voice in the same way as English.

The students reported that the description of the situations - the contextualisation - was useful in some questions but not in others. In some cases, it only provided a distraction from the mathematics. They felt that the sort of questions in their text where the mathematics is explicit and there is little attempt to contextualise the situations were easier - "It saves time and gets to the point." "Sometimes the reading can be confusing and you can sit staring at it for too long." Similarly, diagrams and other illustrations could be useful. However, in Question 21 (c), the artwork was seen to be useless in terms of helping the students.

The students all claimed to understand the specific terms listed, although they suggested that *uppermost* could well be replaced by *top number*. There was agreement among the students that the language used in the question was not "everyday language" which might be used in real-life situations. When they were asked what their teacher did if a number of students were stuck with the language in a class exercise, they agreed that "[s]he explains that easier. She does it step by step. She doesn't use the same words as that, she tries to, like, break it down into smaller words and then probably give us an example some times."

DISCUSSION

Experiential Distance Continuum

Generally, to make meaning in 'real' situations, either in 'real life' or in the mathematics classroom, a more 'spoken-like' language mode is used because of the interpersonal nature of the transactions implicit in the text. For example, if a situation existed where a person was able to obtain a discount by rolling dice, as suggested in this question, the following language might be typical of the actual, as against the written, situation.

Figure 2
Spoken-like Language for Question 21 (c)

Customer:	So I've got to roll them to see what I get?
Proprietor:	Yeah. Just add 'em up.
Customer:	O.K. Here goes.
Proprietor:	That's 13%.
Customer:	Great.

In this verbal exchange there are social interactions occurring and the language is almost a kind of action in itself. (Eggins, 1994, p.54). For example, the language indicates turn taking, rolling the dice and adding up. In this case, the language and the actions that accompany it are virtually one in the same. On the other hand, in the written text of the question there is no social action: the "language is in fact creating and therefore constituting, the social process.... language is being used to reflect on experience rather than to enact it." (Eggins, 1994, p.54) The differences between these two texts can be represented on a continuum, which Martin (1984, p.27) calls the *experiential distance continuum*. When closer to the action, the language will be more 'spoken-like' because of the immediacy, spontaneity and dynamism of the social interaction involved. When reflecting on experience, the language will tend to the more 'written-like' because of the distance between the language and the situation.

This particular dimension of language has important implications for the two Higher School Certificate mathematics syllabuses under consideration here. While they are predicated on establishing everyday contexts for mathematics, it would appear that the language used in the Higher School Certificate examination questions may not be the language of the everyday. There is clearly a tension between the intention of some of the questions, including the example given above, and the language used in them.

Context Dependency

Spoken language is often context dependent because the understandings that people get depend on 'being there'. On the other hand, written language needs to be context independent so that the reader can understand what is going on. The writer needs to build the context for the reader so that the messages being communicated can make independent sense. The text in Figure 1 is designed to provide this information. However, the very building of this context may cause problems for the students attempting the task.

Lexical Density

The Higher School Certificate courses under discussion attempt to provide real life scenarios for the mathematics being learned or assessed. Hence, they are compelled to provide context independent examples which, in turn, result in much greater reading demands for the students. The greater reading demands are created because of the 'conceptual baggage' that formal writing can pack into relatively short text. One way to measure this is to calculate the *lexical density* of the two texts by dividing the number of content words by the number of clauses used (Eggins, 1994, p.60). Figure 3 shows the written text of part of Question 21 (c) with the content words - the main parts of verbs, the nouns, adverbs and adjectives - underlined and the clauses separated by double slashed lines, in preparation for the calculation, while Figure 4 shows the spoken-like equivalent with similar preparation.

Figure 3
Written Text and Lexical Density

" Jo's Clothes is having a sale.// To find the percentage discount on an item.// a customer rolls three dice// and adds the uppermost numbers."

In this particular text there are 14 content words and 4 clauses, giving a lexical density of 3.5.

Figure 4
Spoken-like Text and Lexical Density

Customer:	So I've got// to roll them // to see what // I get?//
Proprietor:	Yeah. Just add 'em up.//
Customer:	O.K. Here goes.//
Proprietor:	That's 13 %. //
Customer:	(That is) Great.//

In this text there are 10 content words and 8 clauses (if some of the ellipsis is replaced), giving a lexical density of 1.2.

The written text has nearly three times the lexical density of the spoken-like text. It incorporates many more concepts that the students have to unpack if they are going to understand what the question is asking them to do. By recreating the everyday experience through dense, context independent text, the question places much greater demands on the readers, sometimes to the extent that the meaning is hard to retrieve.

Passive Voice

In Question 21(c) (i), the students are asked, "What is the greatest percentage discount that *could be received*?" The passive voice used in this question can cloud the meaning beyond the understanding of many students. It can remove the action from the everyday. It is highly unlikely that the customer would ask the question of the shopkeeper using that exact wording. Text written in the passive voice is generally considered to be more difficult to comprehend than text written in the active. The passive voice is not the preferred method of communication among teenagers but it is used extensively in mathematics learning and assessment.

Nominalisation

In Question 21(c) (i), there is another challenge arising from the noun group, *the greatest percentage discount*. This group has three distinct concepts packed into one naming or nominal group. To unpack this complex nominal group the student would have to gloss it as:

- *the greatest:* largest number that can possibly be added from the rolled dice;
- *percentage:* the proportion of occurrences of that added number out of all possible occurrences, expressed out of one hundred;
- *discount:* the amount that can be subtracted from the full price.

The single nominal group has the effect of hiding the complexity of the concepts represented by it. Grammatically, the actions of adding, proportioning and subtracting have been changed into a single 'thing'-a noun or nominal group. This process of changing verbs (actions) into 'things' is called *nominalisation* and is a common practice in mathematics (Veel, 1995). One of the effects of nominalising parts of text can be to the text much more concise. Another effect, however, is that it makes the text much more abstract.

Nominalisation can result in abstracted concepts which, in themselves, have "no unique, unambiguous representations in the real world" (Veel, 1995, p.5). The formality of the nominalisation process is directly at odds with the professed syllabus rationale to enable students to use "mathematics successfully and appropriately in everyday situations" (NSW Board of Studies, 1989, p. 2). There is a clear contradiction between the spirit of the syllabus and the way it is assessed. The net effect of using dense, nominalised texts can be to make their meanings inaccessible to all but the experts (Collerson, 1994, p.82).

CONCLUSION

The exercise of rewriting Question 21 (c) provided confirmation of the analysis discussed above. The students removed the passive voice and introduced action and personalisation. Both nominalisation and lexical density were reduced through a simplification of the sentence structures. The students were unanimous that there was no need to include the information in the 'illustration' which was part of the original question as it did not help them understand what was asked to be done. In fact, many of the students felt that the 'illustration' really only had the potential to confuse, rather than assist them in getting to the mathematics required to answer the question. The rewritten questions are shown in Figure 5

Figure 5
1996 Higher School Certificate, Mathematics in Society, Question 21 (c) - Rewritten by students

Group 1 (Female)

(i) What is the highest percentage discount you can receive after rolling the three dice and adding them together?

(ii) Find the lowest percentage discount you can receive after rolling the three dice. For an item marked $100, use the lowest percentage discount to find the new price.

Group 2 (Male)

Jo's Clothes is giving you the chance to get a discount. All you have to do is buy something and roll three dice. The numbers facing up when added determine the percentage discount.

(i) What is the biggest percentage discount that you can receive?

(ii) If an item costs $100, what is the amount paid when the smallest discount is rolled?

(iii) How many different ways can you roll three dice to obtain a total of 5?

(iv) Calculate the probability of rolling a 5% discount.

(v) What is the probability of paying $95 or more for something which costs $100

While there are many criticisms which could be raised about the students' rewritten questions from both mathematical and linguistic points of view, the overall message is clear. The students have confirmed much of the analysis discussed earlier in this paper and have shown that, in order for them to feel part of the 'everyday and real-life mathematics' supported by the syllabuses and reflected in the examination paper, accessible language must be used. This language must be active, personalised and not so dense that it makes it difficult for the students to reach the mathematics required of them.

In many ways, what the students have done in their rewritten question is precisely what their teacher is reported to do for them: "[s]he explains that easier. She does it step by step. She doesn't use the same words as that, she tries to, like, break it down into smaller words and then probably give us an example some times." In the examination, there is no teacher to "explain that easier". The students have to do it themselves and it appears from the comments of this small sample of students that they have trouble doing so. Not only does there appear to be a mismatch between the written syllabus aims and the assessment practices but also between the teaching and assessment practices. Even though the two courses: *Mathematics in Society* and *Mathematics in Practice* have been consigned to history, the challenge still remains: how do we use language in mathematics to provide a contextual, relevant, accessible and rigorous mathematics challenge for our senior secondary students without making such a challenge incomprehensible to the students?

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COMPUTER ALGEBRA SYSTEMS FACILITATE POSITIVE LEARNING STRATEGIES

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Refining mathematical understanding through the use of multiple representations and negotiation of meaning has been considered important for students. This paper reports on an observational study of students using a Computer Algebra System in an introductory calculus unit at undergraduate level. There was strong evidence that the use of this technology was a catalyst in students using these positive learning methods. However students felt that the use of this technology aided but did not underpin their learning of mathematics.

BACKGROUND

Computer Algebra Systems (CAS) provide symbolic manipulation, graphs and tables at the press of a button. It has been suggested (Mayes, 1993, Heid and Zbiek, 1995) that such packages may be used in the teaching and learning of mathematics to move the focus of learning from procedures to concepts. Day (1993) encouraged us that 'The power and flexibility of technology can help change the focus of school algebra from students becoming mediocre manipulators to their becoming accomplished analysts' (p30). Over the last decade, as this technology has become less expensive and more accessible, mathematics educators have started to explore the possibilities CAS may offer students.

Experimental studies have suggested that the use of CAS improved learning outcomes for mathematics students. Heid (1988) reported on a study undertaken with undergraduates studying an introductory calculus course. One group was taught using CAS to present and explore concepts while a comparison group was taught in a traditional manner without CAS. When assessed, both groups showed similar results on a skills test but the CAS group showed greater understanding of concepts and the ability to use different representations. Palmiter (1991) reported that an experimental CAS group had covered the same calculus course as a traditional group but with fewer hours teaching. The students in the experimental group outperformed the traditional group on both conceptual and computational examinations. These positive outcomes might be related to the learning methods fostered by the use of CAS.

A feature offered by Computer Algebra Systems is the ability to swap quickly and correctly between different representations ie between algebra, graphs and tables. Dreyfus (1991) wrote:

> To be successful in mathematics, it is desirable to have rich mental representations of concepts. A representation is rich if it contains many linked aspects of that concept...One does not get the support that is needed to successfully manage the information used in solving a problem unless the various representations are correctly and strongly linked. One needs the possibility to switch from one representation to another one, whenever the other one is more efficient for the next step one wants to take....Teaching and learning this process of switching is not easy.(p32)

Using a CAS to produce different representations requires very little effort; this encourages students to swap between representations in order to find the information they require. This method of learning may help them develop Dreyfus' 'rich mental representations of concepts'.

Although there is not complete agreement about how mathematical concepts are learned, contemporary thought supports a view of 'constructivism'. This model recognises that all learning builds on experience. The mind naturally organises repeated experiences into complex networks of concepts, rules, and strategies referred to as schema. These schema are not fixed but continually change over time as students are exposed to examples and counter-examples (Romberg, 1993). One of the strengths of CAS is that they allow students to look at many examples (or non examples), represented algebraically, graphically, or numerically, in a short space of time.

A constructivist view of the learning of mathematics also emphasises that a student's understanding of concepts needs to undergo a process of personal and social negotiation before it is internalised (Vygotsky, 1978). Negotiation with peers, the interplay of ideas backwards and forwards, allows each student to refine their thinking based on the language and experience of others. Such a process may lead to a student 'assimilating' new information into their current schema or force a change as they accommodate discrepant experiences (Romberg, 1993).

Tall (1989) identified four types of environments within which students may build and test such schema. Inanimate: where the stimuli come from objects manipulated by the student. Cybernetic: where the system responds to the student according to pre-ordained rules. Interpersonal: where the stimuli come from other people and Personal: within the student themselves. A CAS adds a cybernetic environment to the traditional classroom where students may explore mathematical ideas.

THIS STUDY

The purpose of this study was to monitor students' responses to the use of CAS and to determine if its use did indeed lead to students adopting positive learning strategies including using multiple representations and negotiation of meaning. The research was undertaken in 1998, at the University of Ballarat. This observational study involved a group of 30 undergraduate students who used the computer algebra system, DERIVE 2.55, to assist in both their learning and assessment for an introductory calculus unit

METHOD

The students were taught in two groups, one of 20 and one of 10 students. This grouping was determined by timetable constraints and experienced teachers taught both groups. Students had 4 hours of formal tuition per week with 2 of these hours being held in computer labs. Data was collected throughout the semester from a range of sources: questionnaires, observation and assessed work as detailed in Table 1. These instruments focused on how students used DERIVE, their learning strategies and how they felt about such a use of technology.

Table 1
Summary of Nature and Focus of Data Collected

Nature of Data Collected	Focus of information
Background questionnaire n=28	Previous maths experience Previous technology history Initial response to DERIVE
Lab Classes 10 reports of observations by researcher	Student engagement Student learning methods Student understanding
Solutions submitted by students weeks 1,4,7: total 27	Interesting approaches / errors
Questionnaires: using different questions	Response to DERIVE
Week 3 n=24, Week 5 n=16	Perceived learning
Week 7 n=25, Week 10 n=15	Use of DERIVE
Assignments Marks, n=29	Measure of learning
Representative assignments, n=6	Interesting approaches / errors
Observations by researcher during tests	Use of DERIVE
Test results and scripts	Interesting approaches/ errors
Test 1, n=30, Test 2, n=28	Use of DERIVE
Unit evaluations, n=18	Response to use of DERIVE

Table 2
Percentage of Students using Graphs, Tables or Algebra to deal with Functions

	Never	Occasionally	Half of the time	Very often	Total
Q1 Look at Graph	0	12	46	42	100
Q2 Look at Table	13	58	25	4	100
Q3 Use Algebra	0	43	22	35	100

RESULTS AND DISCUSSION

Students Used Multiple Representations

Class observations, made by the researcher, indicated that all students swapped freely between algebraic and graphical representations of functions but they seldom used tables unless specifically directed to do so.

This observation was confirmed by the questionnaire results. In week 3, students were asked to respond to three statements 'If a problem involves a function I …look at the graph…look at the table…use algebra'. The results, summarised in Table 2 emphasise that students do make use of multiple representations but show a preference for graphs.

Students' lack of use of tables to solve problems may reflect the use of different representations in the teaching of the course. In classroom examples, done by either teacher or students, tables were almost exclusively used in the developmental phase of conceptual understanding and rarely in problem solving.

In addition it may be that students did not find the tabular representations helpful to their understanding of functions. The use of patterns illustrated in tables was been put forward as a good tool for developing algebraic thinking (Australian Education Council, 1990).

However MacGregor & Stacey, (1992), found that there is a 'tendency for students to focus on the differences between successive values of the dependent variable, rather than on the relationship connecting dependent and independent variables' (p366). In the context of functions this means that many students do not focus on the patterns which could help them establish the function's rule and when looking at limits they may not link the change in the dependent variable to the change in the independent variable.

Students Consulted Each Other as they Negotiated Mathematical Meaning

Students were timetabled for two hours per week in computer laboratories. They had worksheets to guide them and questions from the textbook to complete. There were always some questions that encouraged the students to explore patterns and use inductive logic. Observations made by the researcher, during these computer laboratory classes, recorded that, from the beginning of semester, students worked together and discussed their work. In week 9 the following note was made:

> Most students now work in pairs or threes. Some share a computer while others choose to work side by side. They compare and discuss their results. They compare and discuss what is on their screen and how this applies to the problem. Occasionally the discussion widens with students standing to view screens in rows ahead or rotating the screen for others to see.

The students were also questioned about the discussions that took place in the labs. For example, in week 3, students were asked to respond to statements about their shared computer use. Their responses, summarised in table 3, confirm the observation that most talking taking place in the computer laboratory sessions related to consultations about mathematics rather than their social lives.

Table 3
Percentage of students discussing Mathematics while using CAS

	Never	Occasionally	Half the time	Very often	Always
When we share a computer we discuss what is on the screen	0	9	16	46	27
When we share computers we don't talk about maths	19	52	29	0	0

Since computers were introduced to classrooms back in the early 1980's it has been reported (for example Phillips, 1984) that students exploring mathematical ideas on computer terminals were eager to share their work and learn ideas and strategies from each other. In 1984 students' enthusiasm could have been related to a novelty effect however this study's results concur with Heid & Zbiek (1995) and Thomas, Tyrrell & Bullock (1996) suggesting that while computers are now commonplace they still act as a catalyst to focus discussion.

Students Consulted the Computer as they Negotiated Mathematical Meaning

Students not only used the CAS to find answers to specific set questions but also to their own questions. The CAS was used both as a tool and an independent expert. The tool enabled students to look at a lot of examples quickly and know that these results were correct within the limits of the technology. It also allowed students to look at the same example represented in different ways or to swap between these displays. The independent expert allowed students to test ideas. For most topics, students were observed to effectively use the CAS to change values of coefficients and exponents in order to make and test conjectures. This process was used to establish patterns and develop rules inductively. While students sometimes became frustrated when their conjectures failed they seemed content to keep trying until a correct rule was established.

Unit evaluations confirmed that most students had deliberately interacted with the CAS to extend and clarify their mathematical ideas. Students were asked to respond to statements about their use of DERIVE and these results are summarised in table 4.

Table 4
Percentage of Students Responding Positively to the Use of DERIVE

Questionnaire statement	% positive responses
DERIVE had helped me to see patterns	71
I tried out ideas using DERIVE	70
I used DERIVE to try changing functions to see what happened.	76
I find using DERIVE helps me understand mathematics	65

These questionnaire responses demonstrated that most students responded positively to the use of CAS in a cycle of student input – computer feedback – student reflection – leading to new student input. The 'reflection' stage usually involved group discussion. In this negotiation the students consulted the computer as an impartial expert.

The technology had some limitations. Observation of students in laboratory classes and analysis of solutions to the first problems sets showed that the limitations of the graph plots, on the machines being used, caused some confusion. For example, in graph windows, axes were not labelled, piece-wise defined functions, with discontinuity, appeared with vertical lines joining the 'jump' in the graph and straight lines were sometimes distorted. In the algebra window the package gave complex as well as real solutions to equation. Most of these students had no experience of complex numbers and so did not recognise the symbols. In these situations the computer was not a helpful expert in terms of providing clear simple solutions for the student..However these facets of the CAS prompted important discussions between students and with the teacher. It drew attention to features of the functions that some students may have overlooked. The constraints of the package presented fewer problems later in the semester as the students rapidly gained familiarity with both the concepts and the technology.

Just as interpersonal communication does not always leave all the people involved with the same understanding, reading a computer screen can result in different perceptions by different users. Each student's response was influenced by their previous experience of algebra, graphing and their visual perception. Tufte (1990), noted that there may be visual perception problems associated with figures which have several lines close together. Analysis of students' worked solutions to questions in week 1 showed an example of this. Students were asked to sketch, on one set of axes, x^n for increasing powers of n. A number of students drew graphs that doubled back or even became 'horseshoe' shaped. These students believed that they had carefully copied what they saw on the screen.. When the background surveys of the students who made these errors were checked it was found that they had a limited school mathematics background in algebra and calculus.

A key difference between using a computer package and working with pen and paper is that the CAS responds to the student's input, thus providing feedback for the ideas the student is exploring. In some senses negotiation takes place as the student works with the CAS. A student, for example, may make a conjecture about the salient features of a particular gradient function. This idea may be tested immediately by finding the derivative algebraically, and then graphing. The student quickly refines their understanding as they recognise examples that do and do not belong to the pattern they are seeking. A cycle of 'student input → computer feedback → student reflection → new student input' repeats until the student feels that they have successfully understood the concept involved. In this way consultation with the impartial computer aids students as they build and refine their understanding of concepts.

Students' Preferred Learning Strategies

As part of the Week 5 questionnaire students were asked to identify the tools which they would select for speed, confidence, and understanding in response to a range of mathematical questions. The most common choices of the 16 students who responded are summarised in table 5. For each category, the table indicates the tool chosen by the highest percentage of students and the percentage making that choice.

Table 5
The Tool Most Frequently Selected for Speed, Confidence and Learning

For this type of problem	Speed	Confidence	Learning
1. Solve for x : x 2x + 1 = 5	PP (75%)	PP (75%)	PP (100%)
2. Solve for x: (x-4)(x+1)=0	DA (56%)	PP (50%)	PP (83%)
3. Find the intercepts and shape of a y=2x+1	DG (56%)	DG (56%)	PP (63%)
4. Find the x and y intercepts for $y=x^2+5x+6$	DG (38%)	PP (47%)	PP (83%)
5. Find the turning point for $y=x^2+5x+6$	DG (47%)	DG (47%)	PP (44%)

Legend: PP (pen/paper) DA (Derive-Algebra) DG (Derive Graph) CC (Calculator- computation) CG (calculator- graphs) OT (Other –please specify)

These results indicate that while students often felt that using CAS was fastest, and they might be more confident of obtaining a correct result with DERIVE, they commonly felt that they learnt most when they worked by hand with pen and paper.

This result was reinforced in the unit evaluations when students were asked if they felt that 'CAS might offer fresh hope to students who had previously experienced difficulty with algebra'.

> Student A: The use of a CAS would, in some cases, give fresh hope but only if there is still a grounding in the concepts behind the problem.
>
> Student B: I don't think it helped my understanding, because while the computer does the work for you it is too easy not to fully understand what is going on.
>
> Students C: CAS offers the student the answers without really needing the basics. I personally would have enjoyed more time spent on the basics.
> Student D: It does assist, however a basic understanding of the maths is required first. You can not expect the computer to do everything for you. Personally I would rather do theoretical algebra and calculus, it enables me to understand it better and I have more confidence.
>
> Student E: It is a good idea, however although the use of computers will give you the correct answer (as long as entered correctly) it does not let the user know what they have done or how they would go about getting the answer if they were to do it manually.
>
> Student F: CAS in this unit has been a great tool I have learnt to use. I have found it very useful and it is a great piece of knowledge I could use in later life.

This sample is representative of the responses. It indicates that most students felt that the use of CAS contributed to their learning but that pen and paper experience was more important. This result is not surprising as the previous experience of these students in school mathematics focused on pen and paper routines. This was the skill valued by school assessment and therefore highly valued by the students. Background information collected showed that only 11% had used a graphical calculator at school, 37% had used a computer for some maths related purpose and 11% had used both a graphical calculator and a computer at school. Using technology in learning and assessing mathematics was a new experience for most of this group and so many were unsure whether this was legitimate mathematics

CONCLUSIONS

This paper has discussed three positive learning strategies that have been adopted by students using CAS and the their response to this use of technology. The results confirm past findings about the way students learn mathematics and also affirm the place of CAS as a facilitator of positive learning strategies.

Students frequently made use the CAS to explore mathematical ideas by swapping between representations. They showed a preference for graphs and algebra while making little use of tables. Future refinement of both software interface and teaching materials may extend the use of multiple representations.

Students working with CAS discussed their findings with each other and treated the CAS as an independent expert. These interpersonal and cybernetic stimuli helped to extend their mathematical thinking and modify their mathematical schema. Sometimes the limitations of the technology resulted in to confusing responses. Again future development in both technology and teaching materials could solve these problems.

Students expressed reservations about using CAS to learn mathematics. They expressed a view that real mathematics was that done with pen and paper. The availability of CAS raises the question of what mathematics should be taught and assessed at school and undergraduate level. This challenge has been recognised for nearly two decades but it is yet to be resolved.

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THE CONFLICT BETWEEN TEACHERS' BELIEFS AND CLASSROOM PRACTICES

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The relationships between the beliefs and practices of mathematics teachers have been shown to be complex. In this case study of one mathematics teacher, it was found that while the teacher had clear and quite strongly held beliefs about mathematics and its teaching and learning, his classroom practices did not always reflect these beliefs. In some lessons other factors meant that the lessons was not conducted in a way which reflected the teacher's beliefs. However, the teacher was aware of this contradiction.

Many factors influence the classroom practices of mathematics teachers. Thompson (1992) noted that not only do teachers' beliefs about mathematics and its teaching and learning appear to have a significant effect on their approaches to teaching, but also that the relationship between beliefs and practice is complex. Studies have demonstrated "varying degrees of consistency between teacher's professed beliefs about the nature of mathematics and the teachers' instructional practices" (Thompson, 1992, p. 134). The aim of this case study was to explore the relationships among beliefs, context and the mathematics teaching practices of one year 8 mathematics teacher. The case study was part of a larger study into the use of writing activities in mathematics classrooms.

BELIEFS AND PRACTICE

Lerman (1983) described mathematics teachers' beliefs as a continuum from "absolutist" at one end to "fallibilist" at the other extreme. Ernest (1989) proposed three descriptions as follows: the "problem-solving view" in which mathematics is seen as an expanding field in which the process of inquiry is central; the "Platonist view" in which mathematics is seen as a static, unified body of interrelated structures; and the "instrumentalist view" in which mathematics is seen as a set of unrelated facts, rules and procedures which can be used in specific situations. Thompson (1992) noted that research has shown that "the beliefs professed by individual teachers concerning the nature of mathematics have been found to be generally consistent" (p. 134).

Teachers' practices, which are a function of their beliefs, have been characterised in various ways. In this study a set of four descriptions by Kuhs and Ball (cited in Thompson, 1992) were used.

1. *Learner-focused*: Mathematics teaching that focuses on the learner's personal construction of mathematical knowledge;

2. *Content-focused with an emphasis on conceptual understanding*: mathematics teaching that is driven by the content itself but emphasises conceptual understanding;

3. *Content-focused with an emphasis on performance*. mathematics teaching that emphasises student performance and mastery of mathematical rules and procedures; and

4. *Classroom-focused*: mathematics teaching based on knowledge about effective classrooms. (p. 136)

Thompson noted that teachers are often not able to clearly enunciate their approaches to teaching and that there has been considerable inconsistency in the results of studies aimed at linking teachers' beliefs about teaching and their instructional practices.

Ernest (1988) described three factors that appear to influence the practices of mathematics teachers, the teacher's beliefs, the social context and the "teacher's level of thought processes and reflection" (p. 1). He discussed the interrelatedness of beliefs about mathematics, approaches to teaching, ideas of learning and the thought processes of the teacher and associated his three conceptions of mathematics, the instrumentalist, Platonist and problem-solving views, with three teacher roles, namely instructor, explainer and facilitator respectively. Using data from the *Second International Mathematics Study*, Sosniak, Ethington and Verelas (1991) found that teachers did not appear to hold coherent points of view about the nature of mathematics and its teaching. They concluded that the apparent inconsistencies might be due to the need for teachers to contend with many different views and pressures in the day-to-day act of classroom teaching.

METHODOLOGY

An ethnographic case study was adopted for this research. The researcher became immersed in the group so as to present an "holistic depiction of an uncontrived group interaction over a period of time, faithfully representing participant views and meanings" (Goetz & LeCompte, 1984, p. 51). This was an interpretive study that sought to interpret the data at two levels, firstly from the viewpoint of those being studied and secondly "from the perspective of someone who is aware of other systems and of theoretical perspectives on sociocultural systems" (Eisenhart, 1988, p. 105), that is, from the perspective of the researcher. The case for this six-month study was one experienced mathematics teacher and his two year 8 mathematics classes.

A range of data collection methods was utilised in this study. Nickson (1992) described the four typical methodologies used for data collection in such studies, these being "participant observation, ethnographic interviewing, a search for artefacts (available written or graphic materials related to the topic of study), and researcher introspection" (p. 107). She noted that other data could be obtained by surveys and questionnaires. All of these data collection methods were applied to the study.

The researcher engaged in participant observation as a regular visitor to the classroom (at least once each week over a six month period, sometimes without notice), becoming closely acquainted with the teacher and students. During visits, some lessons were audio-recorded and field notes were kept. Informal discussions also took place with the teacher and individual students, the researcher acting as an additional tutor when the students worked in groups or individually. The teacher was interviewed at length at the completion of the classroom phase of the study. By this time the teacher and researcher were well acquainted and it was more like a conversation which was audio-taped and the researcher's used an "interview guide approach" (Best & Kahn, 1993, p. 201). A series of general questions was used as a basis for the discussion.

Two students were also interviewed for the study. They were chosen as "key informants" (Goetz & LeCompte, 1984, p. 119) and were interviewed three times outside the classroom as well as interacting with the author during normal class time. The aim was to gain further insights of an "insider's" view of what was taking place. Samples of writing were collected from all of the students at the commencement of the study, at intervals during the study, and in the last week of the study. The writing samples were analysed by a previously described method (Shield, 1995) and were used to support conclusions regarding students' apparent views of mathematics. In order to clearly establish the overall context of the study, several other documents were collected. These included syllabus documents, the school's work-program in mathematics, other planning documents written by the teacher, and the textbook used by the classes.

The mathematical beliefs of students and the teacher were examined using an existing questionnaire based on an instrument developed by Schoenfeld (1989) and modified for use in Australia by Southwell and Khamis (1992). The version used in this study had 46 items in common with the Southwell and Khamis version, 40 of these being common with the Schoenfeld version. A version of the instrument was also prepared for the teacher.

The extensive, mostly qualitative data were examined in order to develop detailed descriptions of the context in which the teacher worked and of the teacher himself, including characterisations of his beliefs and his classroom practices. These descriptions were grounded in the multiple data sources that were searched for examples which either supported or conflicted with ideas expressed verbally by the teacher.

RESULTS

While extensive written descriptions of the teacher and his working context were developed, space limitations here allow only for samples of the descriptions to be presented. The samples have been chosen to represent key concepts in the teacher's beliefs and practices.

The Context

The school at which the case study was conducted is a Catholic co-educational high school in an outer Brisbane suburb with approximately 600 students in years eight to twelve. In the interview, the teacher involved in the study (referred to forthwith as Ken) noted that this principal treated teachers as professionals and had good relationships with teachers and students. There is a wide range of abilities apparent in the students as well as a range of other needs, especially associated with language difficulties. The school provides one full-time and one part-time English as a Second Language (ESL) teacher with specific language skills in Spanish and Vietnamese respectively. There is also a full-time student counsellor and a full-time remedial teacher.

The year 8 mathematics curriculum is based on the Years 1 to 10 Mathematics Syllabus (Department of Education Queensland, 1987). The syllabus represents a significant departure from previous syllabuses in its intent, although some schools' interpretations have adhered to past approaches. The school involved in this study has implemented the syllabus in line with its intentions, at least at the year 8 level. The syllabus requires a broadening of approaches to the teaching of mathematics including the use of activity based learning, discussion, problem solving and investigation. Language use is also given prominence in the syllabus.

In year 8 mathematics, a textbook (Shield & Wallace, 1988) was used for most of the topics and was supplemented by a range of other materials, some developed by Ken. The school has a formal work program, which is not often referred to by the classroom teachers. There is also a "working" document, much smaller than the work program and written mostly in terms of the textbook. Ken was the leader for year 8 mathematics and took two of the four classes himself, working closely with the other two teachers involved. The teachers mostly used the same classroom activities as Ken although there were occasions when they adhered more closely to the textbook presentations than he did.

The students were allocated randomly to the four year 8 mathematics classes and records from primary schools showed a very wide range of mathematical abilities in each class. A survey of students' beliefs about mathematics was administered to the classes two weeks after the commencement of the school year. The beliefs being expressed would therefore be based on their seven years of primary school and other personal experience. The results on most of the 46 items were very similar to those recorded by Schoenfeld (1989) with tenth and eleventh grade students in the United States and to those of Southwell and Khamis (1992) with both primary (grades 4 to 6) and secondary (grades 7 to 10) students in Australia. Characteristics included: school mathematics consists mainly of facts and procedures; some people are good at mathematics and some are not; teacher questions usually require memory of the right answers; and good teachers show students lots of ways to look at the same questions and show students how to answer the questions on tests. The examples of student writing about mathematics collected on five occasions during the study demonstrated that the students saw mathematics as mostly involving the reproduction of set procedures.

The Teacher

At the time of the study, Ken had a total of 22 years of teaching experience. He commenced teaching in primary schools. In his sixth year, Ken transferred to secondary teaching in mathematics and physics, having studied pure mathematics to third-year university level. Ken had taught at the school involved in this study for 14 years. In addition, he lectured part-time to pre-service teachers and had completed a Master of Education degree.

When asked in the interview for his view of mathematics, Ken had some difficulty providing an answer. He firstly talked about the importance of students liking mathematics. Later he stated the following.

> I don't see mathematics as just computing. . . . algorithms, starting and following a procedure, that sort of thing. So what I'm really thinking of is mathematics is a much broader area than that. I'd like to say it's certainly got a problem-solving focus.

In the questionnaire, Ken answered "very true" to the items about mathematics being creative and problem solving being important and answered "not true at all" to items relating to mathematics already being known by mathematicians and problems only being able to be done one way. These items helped to elucidate the ideas which Ken had difficulty expressing in the interview. He appeared to hold a view of mathematics that could be classified generally as a fallibilist or problem-solving one. However, in the interview, Ken several times gave a pragmatic qualification to his views. For example, having just mentioned the problem-solving focus, he stated the following.

> That's my ideal. But I think when we come back to the real, I've got a syllabus dictating what I've got to do. I've got a school program that's dictating what I have to do. . . . I can do a little bit of work around it. . . . it's just a matter of survival almost. You've got so many deadlines to meet.

Later in the interview Ken also mentioned:

> so-called recipe or rote learning . . . there's a certain amount of that, it's going to happen.

The transcripts of the recorded lessons were searched to identify examples which would either confirm or refute his apparent problem-solving, fallibilist view. In the lesson on 5 March, Ken introduced an activity based on a diagram from a previous lesson. The class had been working on the geometry of plane shapes for several lessons. The diagram consisted of a circle with three diameters at 60 degrees to each other. The following instruction was issued.

> I want you to write down below it anything you can tell me about that final diagram. Write down all the things you can observe in it.

Within a short time, students had written up to ten observations. Ken responded with enthusiasm to responses such as: the lines intersect; 6 angles; 6 parts, like fractions (a reference to the type of diagram used in fraction work); 6 sectors; acute angles; all the same size. He particularly complimented the student who used the word sectors. After the lesson, Ken remarked that he had simply wanted to have the students think more widely and possibly associate other ideas with the work they had been doing.

In the lesson on 11 March, the class was again working in the topic of geometry, this time on ruler and compass constructions. Two students provided alternative methods for making an angle of 120 degrees. The teacher responded with the following statements.

> I like that. That's really good thinking. . . . What do you think? Wasn't that good. That's exciting, one of those different methods you can think of.

The two examples from the lessons reported above support the earlier conclusions about Ken's view of mathematics and its teaching and learning. He was stimulating creative approaches from the students and fostering a problem-solving climate in which alternative methods are valued.

In the lesson on 18 May, Ken used what could be described as a problem-solving approach in developing the procedure for multiplying two common fractions. While he led the discussion, students were involved in answering questions. Ken used a 10 by 10 grid and worked through three examples using the lengths of the sides of rectangles to represent the fractions and an area model to represent the product. Students then had to state a general rule for finding the product of two common fractions. A similar approach was used on 24 May to develop the idea of multiplying and dividing numbers by 10, 100, 1000 and so on.

Not every lesson observed could be classified as being supportive of an active, problem-solving view of mathematics. For example, in the lesson on 31 March, students were learning how to find the highest common factor and lowest common multiple of pairs of numbers. This time Ken stated and explained the methods with an emphasis on setting out and the use of language. He later stated that he needed to achieve a particular end required by the program and was aware of the instrumentalist approach he was using. In the interview, Ken described how in his teaching he attempts to use an investigative approach and tries to encourage student contributions to the discussion. However, he further acknowledged accepting that, in some circumstances, telling the students the way to do it may be necessary.

One of the strongest points about teaching mathematics made by Ken in the interview was the need to foster positive attitudes in students. He repeated several times how much he strives to ensure that his grade 8 students finish the year liking mathematics. He mentioned a number of aspects of his teaching that might contribute to this including the use of concrete materials, group work and cooperative learning, and encouraging students to discuss answers. Further insights into this were provided by the students. One key informant (Alison) expressed her lack of interest in some of the concrete activities, preferring to move on with doing the algorithms. The other key informant (Jenny) demonstrated in one interview her difficulty with the concrete model used by Ken in the previous lesson. The general views of mathematics and its teaching and learning expressed by the group on the questionnaire saw it as a set of facts and procedures to be learnt. The students' writing, which did not change in style during the study, also demonstrated a view of mathematics as set procedures. Their difficulties in talking about the mathematics they were learning indicated a lack of experience with this approach.

A survey of the transcripts of the recorded lessons revealed that in these lessons, Ken did most of the talking. The student responses were quite brief. Ken did attempt to have students explain as on 11 March, when he asked the class:

> Okay I'm going to ask you then to try to describe in your own words, best of all come to the board and use my compass and ruler and other things and tell how we go about some of these.

The first student chosen appeared to have some idea of the mathematical method but experienced great difficulty in articulating this. Ken needed to prompt her throughout the explanation. A similar pattern occurred in other lessons. The students' use of terminology in the brief responses was mostly accurate. However, they generally had difficulty expressing their understandings.

In the interview, Ken also identified writing as an element of his teaching. This was an activity that he used from time to time and in various forms. Sometimes students created their own exercises and problems in written form. At other times they were asked to write an explanation in prose, as occurred in the lesson on 18 May.

> Right, I want you now, in your own words, to write down what you think the pattern of doing the multiplication of common fractions is, ... write down in your own words what you think will be the way we do the multiplication.

In the interview, Ken lamented the fact that he hadn't used writing as much as he would have wished, once again the practicalities of the situation coming into play.

> And I would have liked to have done a heck of a lot more actual writing where they write in their own words, but I didn't get organised enough to get more of it done.

DISCUSSION

Ken was a teacher who reflected on his teaching practices. His stated beliefs about the nature of mathematics were generally in line with Ernest's (1988) problem-solving view. There was considerable evidence of practices that mirrored this view. Alternative solutions to problems were encouraged and rewarded and the relationships among mathematical ideas were emphasised. Activities in lessons often involved the use of concrete materials and discussion. Investigations were used and students' errors were capitalised on for the purpose of learning. Ken's teaching could be described as "content-focused with an emphasis on conceptual understanding" but tending towards "learner-focused". While it was apparent that most students enjoyed the lessons and participated willingly in the classroom activities, there were also evidently a number of conflicts between the students' expectations and the classroom practices.

There were some lessons in which the practices were directed much more towards a "content-focused with an emphasis on performance" approach reflecting an instrumentalist view of mathematics. It was clear in the interview that Ken was well aware of the differences in approach which he used, and he appeared to be a little frustrated by this. In explaining these contradictions, Ken noted various pressures related to the context in which he worked. In the interview he expressed strongly how at times he felt the demands of the syllabus and the school's program limiting what he could do in the classroom. However, with the year 8's Ken enjoyed a high degree of autonomy and his preferred approach was very much in line with the requirements of the syllabus. He was in charge of running the program and he set the tests. Perhaps the perceived pressure was not a result of the immediate needs of the year 8 program but more a response to the known expectations of the other teachers who would teach these students in year 9 and beyond. While Ken wanted his students to learn their mathematics with understanding, he was conscious of the demand for student proficiency with mathematical procedures as a requirement for later learning. There is also the possibility that, as a teacher of 4 classes other than the two grade 8 groups, at times Ken was simply too busy to devote the time to planning to teach in the way that he preferred. As he stated in the interview: *It's just a matter of survival almost. You've got so many deadlines to meet.*

CONCLUSION

This case study of the beliefs and practices of one mathematics teacher provides some insights into the complex interactions involved. Conflicts can arise between what a teacher believes is best for students and what can actually be achieved in the classroom. This could indicate a lack of a coherent view of mathematics as noted by Sosniak et al. (1991). However, other evidence may indicate strongly held beliefs that are reflected in practice. It is likely that the day-to-day pressures of teaching mean that at times teachers may need to make curriculum decisions that do not meet their ideals.

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UNDERSTANDING DECIMALS: THE PATH TO EXPERTISE

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This paper reports a longitudinal study of children's understanding of decimal notation. The understandings of 3211 students were classified into four categories and changes over long periods were tracked. The progress of sixty-four students over about three years is reported. When expertise is attained it is generally retained. Over about a year, many students remain in the same misconception category, but in the longer term they move between misconceptions. Improvements and hypotheses to be investigated in the future are noted.

INTRODUCTION AND BACKGROUND

Many students throughout schooling and indeed many adults have difficulty understanding decimal notation. In this paper, we present preliminary results from a longitudinal study that is tracing the development of students' thinking about this challenging topic. As well as being of interest in its own right, this is a case study of how students' understanding develops and/or stays the same with progress through school and in the context of various types of instruction.

There are several ways of classifying the erroneous rules that students may apply when ordering decimals (Resnick, Nesher, Leonard, Magone, Omanson & Peled, 1989; Sackur-Grisvard & Leonard, 1985). The coarsest classification is that some students select "longer is larger" (e.g. deciding 0.125 is larger than 0.3) whilst others select "shorter is larger" (e.g. deciding 0.3 is larger than 0.496). Stacey and Steinle (1998), working with interview and written data, traced the various ideas behind these erroneous rules, identified further misconceptions and developed a diagnostic test. This Decimal Comparison Test takes about five minutes and asks students to select the larger decimal from 30 carefully chosen decimal pairs. It enables ten patterns of thinking to be diagnosed. Some of these patterns of thinking are "longer-is-larger" misconceptions, some are "shorter-is-larger" misconceptions and others belong to neither of these. Although future analyses will use the ten patterns of thinking, this paper reports student progress in terms of four major categories (A, L, S and U):

- apparent-experts (A) who may possess excellent understanding or may apply correct rules not understood or may have one identified incorrect patternof thinkin(Steinle et al (1998)),
- longer-is-larger misconceptions (L) which result from any of five identified patterns of thinking and possibly others,
- shorter-is-larger misconceptions (S) which result from three identified patterns of thinking and possibly others,
- unclassified (U) which is generally a large group since the criteria for classification are quite stringent. It includes students thinking about decimals in unknown ways and others who are inconsistent.

The incidence by age of the various misconceptions about decimal notation is reported in Steinle and Stacey (1998) using cross-sectional data. The longer-is-larger category decreases from Grade 5 (32%) to Year 10 (5%), the trend suggesting that it is unlikely to be common in adult life. The shorter-is-larger category is consistently between 10% and 15% which suggests that this general belief may continue into adulthood. The percentage of apparent-experts also plateaus to about 60% in Year 10, which suggests that there are many adults who have difficulty understanding decimal notation. This paper moves from the cross-sectional analysis to the beginning of a longitudinal analysis, which traces the movement of individuals in the overall data and reports on two questions:

- do students stay in the same category or move frequently from one to the other?
- what are the common paths through the misconceptions to attaining expertise?

Much of this paper provides a similar analysis to that in Stacey and Steinle (submitted), but is based upon a substantially enlarged data set (8708 instead of 5383 tests).

The Longitudinal Sample and Testing

This section presents preliminary results of the longitudinal study from 1995 to 1998. The sample was originally selected to contain a good mix of schools and to maximize the possibility of following students when they changed from primary (Years 0 to 6) to secondary school (Years 7 to 12) at approximately 12 years of age. It consists of classes from:

- one state secondary school in a low socio-economic area and its three "feeder"primary schools,
- one church secondary school in an middle socio-economic area and its main feeder primary school,
- one private girls school in a high socio-economic area with both primary and secondary students,
- two large state primary schools situated in the same middle socio-economic area and the three high schools to which their students mainly progress and
- one church girls' secondary school in a high-middle socio-economic area.

Students were tested with the Decimal Comparison Test at most once every six months, making a total of seven testing times in the data under consideration, from the end of 1995 to the end of 1998. However, schools tested less often than this, for various reasons including different dates for joining the program. The year level distribution of students is shown in Table 1. Note that many students are counted more than once, some up to six times each. In total, 8708 tests are analysed for this paper, although there is no longitudinal data yet for 806 students.

Table 1

Year Level Distribution of Students Completing Tests 1995-1998

Year level	Year 4	Year 5	Year 6	Year 7	Year 8	Year 9	Year 10	Total
Number of students	336	965	1468	2114	1917	1067	841	8708

The number of students who have completed the Decimal Comparison Test exactly one, two, three, four, five and six times is shown in Table 2. The students who have completed only one test play no further role in the analysis in this paper, as it aims to track change of individuals from one test to others. For this analysis, the first time an individual undertook the test will be called Test 1, the second time will be called Test 2 and so on. The tests are numbered for the individual, rather than by the date administered. Therefore for some students Test 1 was in 1995 while for others it was in 1996, 1997 or 1998. Because we test intact classes to maximise the usefulness of feedback to teachers, we always have new students entering the testing program. For some students, Tests 1 and 2 have been taken six months apart, whereas for others they may be one year or even 18 months apart if the student was absent on some testing days. This unsatisfactory feature of this preliminary analysis will be addressed in subsequent work.

Table 2
Number of Students by Number of Tests Completed

Exact number of tests completed	One test	Two tests	Three tests	Four tests	Five tests	Six tests
Number of students (N=3211)	806	639	776	718	208	64

RESULTS

Changes of Classification over about Six Months

Table 3 illustrates the changes in classification that occur over consecutive tests. The abbreviations A, L, S and U refer to apparent-expert, longer-is-larger, shorter-is-larger and unclassified groups, respectively. (The classification M is discussed below.) The numbers are amalgamated from all tests. The data is therefore from Test 1 followed by Test 2, from Test 2 followed by Test 3 up to Test 5 followed by Test 6. As noted earlier, these changes are mostly over a period of about six months, but will also include changes over longer periods where students missed out on intermediate testing. This anomaly in the data will be eliminated when the final analysis is done, to give a better measure of change over six months. Another anomaly of the data is that those 1766 students who have been tested more than twice contribute several times to the data. The 64 students who have been tested 6 times, for example, each contribute 5 times to the data in Table 3.

Table 3
Changes in Classification over Consecutive Tests (N = 5497)

Earlier Classification	Later Classification A	L	S	U	M*
A (N= 2851)	2560 (90%)	41 (1%)	69 (2%)	181 (6%)	39 (1%)
L (N= 1179)	295 (25%)	519 (44%)	139 (12%)	226 (19%)	5 (0%)
S (N = 739)	268 (36%)	72 (10%)	265 (36%)	134 (18%)	11 (1%)
U (N = 728)	347 (48%)	78 (11%)	102 (14%)	201 (28%)	18 (2%)
M *(N = 48)	32 (67%)	1 (2%)	3 (6%)	12 (25%)	8 (17%)

* Note M (standing for "misread") is a subset of U (Unclassified) and is therefore shaded.

Table 3 shows that from one test to the next, almost all of the apparent-experts (A) stayed as apparent-experts and approximately one third of other students became apparent-experts. There is a clear tendency for students not becoming experts to retest in the same category. Nearly half of the longer-is-larger (L) students (and nearly two-thirds of those who did not become experts) retested as L. The shorter-is-larger (S) students moved more than the L students, but still about one third stayed in the same category. Amongst the S students who did not become experts, about half remained as S. It will be important to repeat this analysis separating students by age group, as age is likely to be an important determinant of the speed and direction of change.

For interest, Table 3 also shows the numbers of students in the Unclassified category who consistently selected the smaller instead of the larger decimal in the comparison test. These students are labelled M for "misread". They may be A students who have misread the test instructions or they may think that decimal numbers are on the negative side of zero and are therefore ordered in a reverse way. Most of these students retest as A, but the other one third may have a substantial misunderstanding.

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Changes of Classification over about Two Years

Table 4 is similar to the Table 3 except that it shows the changes in classification that occurred from Test 1 to Test 4, Test 2 to Test 5 and Test 3 to Test 6. All these students have therefore been tested at least 4 times. Students may have taken these tests only 18 months apart, but since most students missed some testing, for convenience, we have labelled this "changes over about two years". None of the students who were initially in the misread (M) category were tested, although some students have moved into this category.

Table 4
Changes in Classification over about Two Years (N = 1326)

Earlier classification	Later classification A	L	S	U	M
A (N= 505)	471 (93%)	2 (0%)	10 (2%)	22 (4%)	**7 (1%)**
L (N= 486)	256 (53%)	104 (21%)	35 (7%)	91 (18%)	**6 (1%)**
S (N = 190)	120 (63%)	8(4%)	38 (20%)	24 (13%)	**3(2%)**
U (N = 145)	95(66%)	18 (12%)	12 (8%)	20 (13%)	**2 (1%)**

Over about two years, more than half the students in L, S and U move to A. However, about 20% of students retest in L or S, which probably indicates that school has made little difference to the way they think about decimals, even though in these middle years of school they would be encountered on almost a daily basis.

The fact that 63% of S students move to A, whereas only 53% of L students do, supports the longstanding observation of Sackur-Grivard and Leonard (1985) and others that S is a more sophisticated understanding of decimals than is L. We had previously expected that it would therefore be more likely that L students would become S than vice versa. We had hypothesised that students may move into the more sophisticated misconception on the way to expertise. There is some support for this hypothesis in both Tables 3 and 4, as the percentage of students moving from L to S is greater than the percentage moving from S to L. (It should be noted that the likelihood of movements between L and S are small in comparison with the likelihood of movements to other categories.) The larger number of L students, however, results in a net flow of 67 students from L to S in Table 3, with a corresponding number of 27 in Table 4.

Changes of Classification over about Three Years

A group of 64 students who had reached year 7 (45 students) or Year 8 (19 students) have been tested 6 times, over a period of about three years. Their categories at each of the six tests is shown in Figure 1. How representative is this group of students who have done the test six times? Comparing Figure 1 with cross-sectional results from Steinle and Stacey (1998) discussed earlier, indicates that they are probably better. By Test 6 in Years 7 or 8, there are 48 (75%) apparent-experts compared to about 50% reported previously. There are several reasons why this may be so. Firstly, to be present on the six days of testing indicates that are likely to be regular school attenders with relatively stable schooling. Secondly, the fact that their teachers have made the effort to test six times indicates commitment on their part. This bias will require careful treatment in future analysis.

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Figure 1
The Progress of 64 Students over 6 Tests

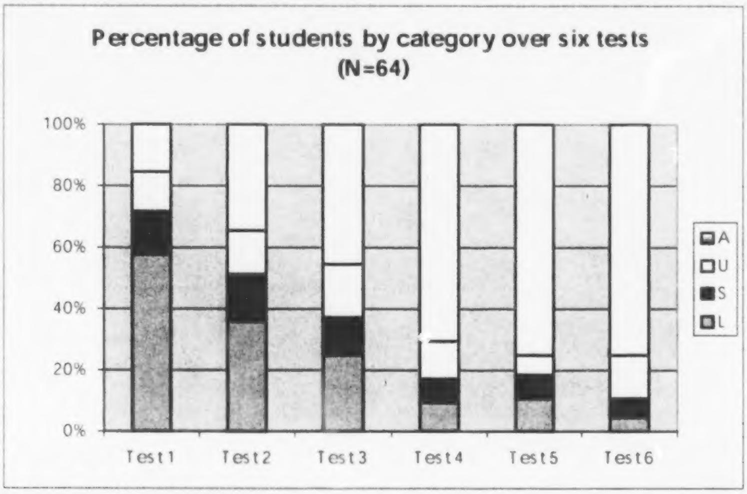


Table 5 shows the initial classification of these 64 students at Test 1 as well as their classification at Test 6. All 10 students who were first tested as A retested as A in Test 6. These students are likely to have been among the most able students (having achieved early expertise in Year 4 or Year 5) so it is not surprising that they test again as experts about three years later. Most of the other students have also become A over the three years, as would be expected. However, 16 students (25%) have not yet reached expertise. There is clearly still work for their Year 8 and 9 teachers to do on fundamental understanding of decimals.

Table 5
Changes in Classification from Test 1 to Test 6 (N = 64)

Earlier classification	Later classification			
	A	L	S	U
A (N= 10)	10 (100%)	0	0	0
L (N= 37)	25 (68%)	2 (5%)	3 (8%)	7 (19%)
S (N = 9)	6 (67%)	1 (11%)	1 (11%)	1 (11%)
U (N = 8)	7 (88%)	0	0	1 (13%)

A First Attempt at Following Individuals to Expertise

This section gives an initial view of the paths that students take through the categories on a series of tests. It is concerned here only with data from the 64 students who completed the test six times (the same data set as Table 5). As noted above, these students entered the study in either Year 4 or 5; their sixth test occurring in Year 7 or 8. Table 5 shows, for example, that of the 37 students who tested as L at Test 1, 25 moved to A at Test 6; 2 retested as L at Test 6, while another 10 moved to S or U at Test 6. This information is displayed as 37 (25,2,10) in the first cell of the second row of Table 6. Thus the first column in Table 6 can be constructed from Table 5. Note that the first cell in the first row of the table, i.e. 10 (10,10,0) indicates that all 10 of the students first tested as A, retested as A in Test 6. In this row, the first two numbers in the brackets must be the same.

Table 6
Numbers of Individuals in each Category at each of the Six Testing Times and, in Brackets, Numbers Indicating Test 6 results. (Sample of students who have been tested 6 times, N=64)

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6
A	10 (10,10,0)	22 (21,21,1)	29 (27,27,2)	45 (42,42,3)	48 (43,43,5)	48
L	37 (25,2,10)	23 (17,2,4)	16 (10,1,5)	6 (2,1,3)	7 (2,1,4)	3
S	9 (6,1,2)	10 (4,3,3)	8 (5,3,0)	5 (1,1,3)	5 (1,2,2)	4
U	8 (7,1,0)	9 (6,1,2)	11 (6,3,2)	8 (3,3,2)	4 (2,1,1)	9

Key: (i,j,k) indicates i students tested as A at Test 6; j students retested at Test 6 in the same category as Test 1, and k students moved to a non-expert category.

Looking along the first row of Table 6 shows that almost all A students retain their knowledge of how to complete the test, although the proportion decreases from Test 1 to Test 5. This probably indicates that students who recently become apparent-experts are the ones most likely to "regress". Those who tested as A at Test 2, for example, have had a long time to consolidate their correct ideas.

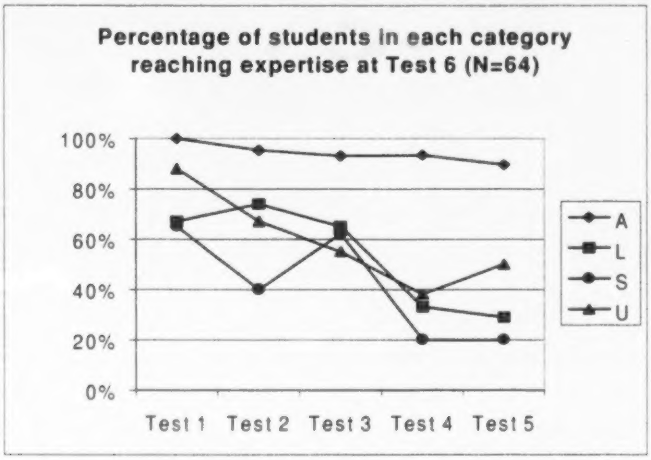
The remaining rows of Table 6 gives information about the movements in and out of the L, S and U categories. By comparing the last number in the brackets (k) in each of these rows, we can see that more students leave the L category and move to a non-expert category, than students in either S or U. So not only to the numbers in the L category decrease, the students who leave this category contribute to the group of students wandering between the S and U categories.

While the numbers of students in the S and U categories seem reasonably stable over the six tests, it should be noted that there is considerable movement in and out of these categories. For example, at Test 2 there are 9 unclassified students, but only 1 of these retests as U in Test 6. Likewise, at Test 4 there are 5 S students; only 1 moves to A in Test 6, another 1 retests as S in Test 6, while 3 students move to either L or U at Test 6. As the number in L is constantly decreasing, it is more likely that these 3 students end up as U at Test 6.

The numbers in S and U stay surprisingly steady, but students move in and out of them over time. This is consistent with our experiences when we interviewed students from a class of Grade 5 and 6 students (not in this sample). These students had recently been studying fractions and so it seemed that the number of shorter-is-larger students who interpreted decimals as fractions (reciprocal thinkers) was inflated by the recent experience.

As mentioned above, the proportion of students in A who retest in A at Test 6 decreases across the table. Figure 2 illustrates this decrease, as well as a similar decrease in the other categories. Students in a category at a later test are less likely to reach expertise (A) by Test 6 than students who have been in that category earlier.

Figure 2
Percentage of Those Students who Tested in a Given Category at a Given Test who Reach Apparent Expertise at Test 6



DISCUSSION

The purpose of this paper was to report a preliminary analysis of data showing students' progress in their understanding of decimal notation. Several ways in which the analysis can be made more revealing are evident: by following individuals, by following classes so that teaching effects can be observed, by separating the analysis by age group and by using the finer classification system. This analysis is now being planned.

The preliminary analysis has provided the following results to be confirmed later. There is quite marked stability of classification. As we would all hope, there is a general trend towards expertise, and this is showing most clearly in the longest data runs. In the short term those students who do not achieve expertise tend to remain in the same category. After about six months to a year, about a third of students in non-expert categories retested in the same way. Even after a passage of about two years, about 20% of the students with a misconception are classified in the same way. This is an important result given the stringency of the classification criteria and it confirms our informal data (from interviewing previously classified students) that the test, although taking only a few minutes, is highly reliable.

Students in different classifications behave differently. Apparent-experts nearly always stay in this category. This would be expected of students who "really understand" decimals. However, at least in the context of this test, the skill of decimal comparison is well retained even by those who use a rote-learned rule (e.g. compare digits from left to right or add zeros). Those who attain expertise the earliest retain it the best. Following individual paths will help in unravelling students' thinking further and eventually providing better guidelines for teachers.

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Acknowledgement

This work was carried out with a grant to from the Australian Research Council. We thank the many teachers and students who have participated.

ACCESSIBILITY OF APPLICATIONS TASKS

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This study identified the conditions that impeded or facilitated students' access to applications tasks at the senior secondary level. Impeding conditions comprised language related, representational, memory related, organisational and task specific conditions. Facilitating conditions comprised memory related, perceptual and engagement conditions. Some conditions were the result of the complexity inherent within the task whilst others arose from students' resource levels and as such acted as intervening conditions between task complexity and the difficulty the students experienced in accessing the tasks.

APPLICATIONS TASKS IN LEARNING AND ASSESSMENT

In Mathematics B, the main pre-tertiary mathematics subject in Queensland, the use of applications is advocated in order to develop positive attitudes towards mathematics and to increase mathematical competence by being concerned with "the modes of thinking which provide ways of modelling situations in order to explore, describe or control our social, biological and physical environment" (QBSSSS, 1992, p. 1). These reasons for advocating the use of applications in classrooms are based on widely held, though not universal (Verstappen, 1994), beliefs about the motivational powers of contexts and their use enhancing the links between the abstractness of school mathematics and the use of mathematics in the real world (NCTM, 1991; Niss, 1987). The use of contexts, while possibly addressing these concerns, presents other dilemmas, however.

The location of mathematical tasks in meaningful contexts for both teaching and assessment purposes is considered by many authors to be problematic (e.g., Amit, 1994; Patronis, 1997). As Boaler (1993), Clarke and Helme (1993) and Noss and Hoyles (1996) point out, contexts are not immutable and interpreted in some universal fashion by all students. Furthermore, Clarke and Helme (1993) suggest that even though a student may find a particular context interesting and engaging the demands of language and contextual detail may constitute an excessive cognitive load impeding progress towards a solution. Stillman and Galbraith (1998) infer from their study of applications tasks, that the focus of applications teaching should be on reducing the time students spend on orientation activities by "developing cognitive skills that facilitate more effective problem representation and analysis, and by promoting the development of metacognitive strategy knowledge" (p. 185) to facilitate appropriate decision making during orientation. It is imperative then that the conditions affecting students' access to applications tasks and the strategies they employ to take advantage of facilitating conditions or successfully overcome impeding conditions be identified. These strategies will be the subject of a future paper.

THE STUDY

This paper reports part of a much larger study which investigated the effects of context on students' approaches to, and performance on, applications tasks in Mathematics B. Specifically, this part of the study addressed the following research question: What conditions facilitate or impede students' access to applications tasks? These conditions may be (a) inherent in the nature of the task or (b) conditioned by the local context.

The study was conducted in two public high schools in a large provincial city. Data collection occurred over a six month period although contact with both schools was continuous during this time. Forty-three students in the last two years of high school (Years 11 and 12) participated in the study.

METHOD

Administration of the Tasks

All students were videotaped as they attempted to solve applications tasks individually with each student completing up to four tasks. An example is included in an appendix to this paper. These tasks differed in terms of familiarity, complexity and degree of contextualisation. The term *degree of contextualisation* is meant to convey the range of embeddedness that exists between the mathematics that can be used to model a situation and the description of the situation. This can range from the simple case where there is virtually no integration of the model within the context, the context merely acting as a border surrounding the mathematics which can be readily removed without loss of meaning, to the situation where the two are totally integrated and separation becomes difficult as the mathematics derives its meaning from the context. Although the majority of the tasks used in the study were closed rather than open-ended, the nature of some tasks provided scope for students to move in a direction appropriate to their perception of the problem. This occurred when tasks were more like true modelling tasks than mere applications or a fair degree of leeway was allowed in students' interpretation of the task.

Student Interviews

Sixty-four semi-structured stimulated recall interviews were conducted and recorded immediately following completion of the tasks. The videotape of their task solving sessions were reviewed by students in conjunction with the script of the task during the interview. Students were asked to draw diagrams, where appropriate, to illustrate their understanding of the problem context and what it was they had to do. The use of the videotaped task solving sessions as a visual stimulus throughout the interviews allowed both the interviewer and the interviewee to track the student's developing understanding of the task context by discussing their changing perception of the task as observed in their changing use of diagrams throughout the review of the task solving session.

Analysis

All interviews and task solving sessions were transcribed. The interviews were analysed using the qualitative data analysis software QSR NUD.IST (Qualitative Solutions and Research, 1997). NUD.IST facilitates "grounded theory" construction (Tesch, 1990) which attempts to capture and interrogate the meanings emerging from data. This is achieved by constructing and exploring new categories and themes as they arise from the data then refining these through a "process of progressive category clarification and definition" (Tesch, 1990, p. 86). This was done using a variety of matrix displays (Miles & Huberman, 1994).

RESULTS

Conditions Impeding Task Accessibility

Thirteen conditions were confirmed as impeding task accessibility at both sites. These were clustered into five groups in descending frequency of appearance: (a) language related conditions, (b) representational conditions, (c) memory related impeding conditions, (d) organisational conditions and (e) task specific conditions. These will now be examined in more detail.

Language Related Impeding Conditions

Language related conditions accounted for 62 of the instances of conditions impeding task accessibility. Language problems that stemmed from the use of technical language to describe the context, comprehension difficulties and unusually worded tasks were reported in 37 of the 64 cases of the study. This was by far the most frequently reported problematic condition. In addition, cuing words in the problem statement that were of either a mathematical or contextual nature were reported as not being salient for the task solver in 19 cases. Lengthy problem statements were also seen as a problem in seven cases.

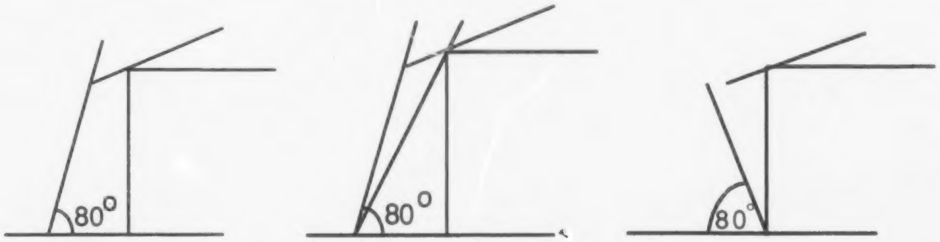
The nature of the context in which a task is set may entail the use of technical vocabulary associated with the context. Ideally, the task will be worded in such a way as to minimise the use of such vocabulary and to explain where necessary. However, even then some students will have difficulty. However, even normal everyday language or the language typically associated with particular mathematical contexts can cause difficulties which erode students' confidence in their ability to tackle the task at hand. In addition, students become very used to the wording of tasks from textbooks and the classroom. If the wording is not what is expected, this will just be a distraction for some students, whilst for others it can present an insurmountable difficulty.

Mathematical words and terms are often used in textual descriptions of tasks to clarify particular mathematical features of the task but these can impede rather than facilitate task access when their salience for the student is so low that they are ignored as is shown in the following exchange. The task statement defined the pitch of the roof as "the angle the slope of the roof makes with the *horizontal*" [italics added] but the student labelled this angle as being between the roof and vertical wall in her diagram.

> I: What are those two things?
> B3(TASK 18): That's the roof and the vertical line.
> I: Is that what it says?
> B3(TASK 18): The angle of the slope of the roof with the horizontal. The pitch of the roof is 12 degrees. So, like...this is like 12 degrees (draws over line of roof).
> I: And where would the horizontal be?
> B3(TASK 18): Here (draws in horizontal line marking top of wall).
> I: Yes, so if it is there, where would the 12 degrees be?
> B3(TASK 18): In here (marks in 12 degrees between roof line and horizontal).

Everyday language used to describe the context can also involve critical cues which students do not notice until their salience is raised as is shown by this further exchange. The student rereads the task statement and then attempts to mark in the angle of elevation of the sun (see Figure 1a). In this case the observer would be out from the building not at the base of the wall as implied by the fact that the wall is "*just* in shade" [italics added].

Figure 1
First (a) and Final (b) Version of B3's Third Diagram and her Fourth Diagram (c).



(a) (b) (c)

> I: Umm, that is one possibility, isn't it? Now what else does it tell you, you have to have? ... [Student rereads problem.] So where is your shade going to be?
> B3(TASK 18): Awh, umm, in here, this area here, down the bottom (indicating the ground beside the shed).
> I: Did it say that? If you were talking about that how would you say that? Would you be talking about the wall?

> B3(TASK 18): The walls of the shed are in shade.
> I: Yes.
> B3(TASK 18): So, this is in the shade. (Draws down line representing external wall.)
> I: Uhuh, but it has got one word...it is very critical if you have a look at the word about the walls in shade.
> B3(TASK 18): The external walls...from the roof...from the roof overhang are just in the shade from the roof overhang.
> I: What does the word 'just' mean here?
> B3(TASK 18): Just sort of really close.
> I: The wall is just in shade.
> B3(TASK 18): Awh, so like, *I thought it meant that it was in the shade from the...I thought the overhang was in the shade as well.* So it's not that...*It's the wall is just in the shade* so it is like this.

At this point the student not only has to realise the significance of the word "just" but also she is now forced to be more flexible in her thinking and look at the situation from another angle. As she has focussed her attention on one particular visual representation for so long it takes her quite an effort to break her train of thought, being happy to accept an obvious incongruency in the situation before being forced to make the effort to think of other possibilities (See Figures 1b and 1c).

Seven students commented on the length of some of the tasks as being "off-putting". However, all but two of these students were able to overcome any difficulties this may have caused. They appeared to be speaking on behalf of others (that is, how they perceived other students without their skills might react). However, not all students, whom these students would have expected to have this reaction, agreed. Student A90 was one of the students in the study achieving at the Very Limited Achievement level in her school program. She felt the length of the Ecosystem Problem (one A4 page) was an advantage as it was "very descriptive".

Representational Conditions

Representational conditions accounted for 40 of the reported instances of conditions impeding task accessibility. These consisted of difficulties mathematising the context (18), integrating given or derived contextual information (12) and difficulties extracting the mathematical information from the context (10).

At times, students will understand the context fully and know what it is that they want to do but they are unable to do this as they can not mathematise the situation. Often this was demonstrated by their inability to draw a diagram representing the problem situation despite readily acknowledging the need to produce a visual representation. At other times, it was the extraction of the mathematical details from the context that was the problem.

Sometimes, students did not perceive in a situation that which was quite obvious to others because they were unable to integrate cues. They had a tendency to focus attention on parts of the situation rather than the whole. In the Road Construction Problem, B4 was so focussed on the angles made by the slope of the bank that it didn't occur to her that they formed parts of triangles so in her diagram she failed to join up the lines representing the sloping bank with the top of the bank. Another reason students had difficulty in this task with formulating a representation of the problem situation was that they were unable to integrate all the separate pieces of information from the context with their encyclopaedic knowledge of the world. Several students mentioned how they had to realise the dirt had to be removed to form the new lane before they could mentally integrate all the requirements of the task. In a richly elaborated context the task of integration becomes overwhelming for some students.

Memory Related Impeding Conditions

Memory related impeding conditions were problematic in 32 instances. These resulted from recall difficulties (15), the recall of metacognitive person or task knowledge that discouraged a student from accessing the task (8), the fact that the mathematics involved in particular tasks was not studied recently (6) and interference from recalled prior knowledge of the situational context (3).

Recall difficulties mainly involved the recall of a procedural skill that the student felt would assist in solving the task. There were examples, however, where the problem was the recall of a formula or concept. On two occasions prior knowledge which would have facilitated student's accessing and solving the task at hand was not recalled.

Recall of metacognitive knowledge and beliefs about themselves as task solvers can affect the outcome of students' cognitive activities by discouraging them from accessing a task because they believe it will be difficult for them or their chance of success is low. Similarly, beliefs or preferences for particular types of tasks can affect whether or not a student will access a task. Student A18's initial reaction on seeing the Shaft Problem was to think that it would be difficult to solve because it may involve a derivative and he did not consider himself to be "very good at derivative questions". These beliefs or preferences have developed from students' past experiences with similar tasks.

Recency of study of the mathematics involved appears to be a two-edged sword. If students have consolidated the skill or knowledge it is a simple matter of recall with the more recent the topic has been studied the more likely it will be readily recalled. Students at both sites suggested that by about a month without revision the memory trace is beginning to fade whilst longer periods certainly lead to forgetting. If, however, the skill or knowledge has not had time to be consolidated because it has been studied too recently students will still have difficulty applying it. Attempts were made to avoid this occurring in this study by pre-testing students for mathematical skills and then using tasks where they had demonstrated facility with the skills involved. Even then, there were isolated cases where students who had demonstrated that they could use the required skill on the pre-test were unable to recall it during the task-solving session three days later.

There were three cases where student's prior knowledge interfered with their development of a solution, however, in two of these cases the students involved were able to overcome this. In the Tide Problem, for example, student B7's belief that tides repeated every 12 hours interfered with the data given in the problem but she was able to recall the problem conditions and correct her error.

Organisational Conditions

Organisational conditions consisted of difficulties formulating a plan of attack (15) or establishing a goal (9). Inability to formulate a plan of attack was instrumental in several students' failure to access the task at hand as they did not know where to begin. Both comprehension difficulties which are purely language related and contextual demand can lead to a student not being able to decide what it is they are expected to do in a task or students can misinterpret what it is they are expected to find. Student A61 struggled to establish any meaning in the context of the Road Construction Problem where she was expected to check the angle of the bank beside the road. Instead, she misinterpreted her goal as finding the best angle at which to have the road inclined.

Task Specific Conditions

Only one of the task specific conditions was confirmed at both sites. It was the situation where the mathematics required was not obvious in a task which was reported as problematic in six cases. In the Ecosystem Problem and the Road Accident Problem, for example, the mathematics involved was considered by some students to be rather elusive as several options occurred to them but none of these stood out from the others.

Conditions Facilitating Task Accessibility

Conditions facilitating task accessibility which occurred at both sites were grouped into three clusters in descending order of frequency of appearance: (a) memory related conditions; (b) perceptual conditions, and (c) engagement conditions.

Memory Related Facilitating Conditions

Memory related conditions were reported as facilitating task accessibility in 45 instances. The majority of these were almost evenly divided between the recall of a similar or parallel task (19) and the recall of prior knowledge (18). The fact that the mathematics involved was well rehearsed was reported as making the task more accessible in six cases. Metacognitive person or strategy knowledge encouraged engagement with the task in two cases, one from either site.

Some students were able to make use of the similarity between the task they were attempting to solve and previous tasks they had solved. These students capitalised on the transfer between mathematically similar problems even though the contexts were different to facilitate their access to the problems in the study.

Relevant prior knowledge can reduce both the contextual and mathematical demands of the task. Relevant declarative or procedural knowledge from other subject areas such as Physics can often be of use also. The recall of relevant episodic or encyclopaedic knowledge facilitated the student's understanding of the task (Stillman, 1998). The prior experience of having seen a particular mathematical model used to represent a physical phenomenon was sufficient to facilitate access to a less constrained applications task such as the Tide Problem.

Although information stored in Long Term Memory does not decay, it does become more difficult to retrieve if it is not too frequently recalled (Best, 1995). Information which is well rehearsed is much more easily retrieved. Consequently, when students were set tasks that involved well rehearsed mathematics they found this helped them access the task.

As was indicated in the impeding conditions, past experience can have an effect on students' personal metacognitive knowledge. This is not always negative. Repeated success with particular strategies boosts a student's confidence in taking risks and using these strategies on future tasks.

Perceptual Conditions

Perceptual conditions facilitating task accessibility consisted of the recognition of trigger words and visual features in the task statement or any visual representation accompanying the task statement in 14 cases, being able to visualise the situation so as to understand it fully in 10 cases, and the instantaneous recognition of mathematical cues in three cases.

Readily identifiable task characteristics which include specialist vocabulary such as "incline", "angle of elevation", "rate of growth" or "slope" and typical diagrams or graphs make tasks more accessible to those who recognise them and realise their significance.

Several students believed that it was necessary for them to visualise the situation described in a task before they could solve it. Students who used this technique successfully spoke of acquiring it from teachers who taught them in the Junior secondary school. When this technique was used, the point of understanding when all the pieces fell into place was quite sudden and observable on the videotapes of the task-solving sessions.

Sometimes the recognition of mathematical cues in a context is instantaneous and the person doing the task is unable to articulate why or how they recognised these so quickly. They are unaware of even having thought about it. It is possible that the mathematical cues are just so explicit they are instantly extractable or, alternatively, that such thinking is unavailable for conscious introspection.

Engagement Condition

There was just the one engagement condition confirmed at both sites. In 14 cases, engagement with the task context was seen as facilitating a student's access to the task. This was evidenced by students being able to integrate various combinations of the following: (a) contextual information from the task statement, (b) prior knowledge of the situational context, (c) mathematical information given in the task statement or (d) mathematical information derived from information given in the task statement.

DISCUSSION

Some of these conditions which facilitate or impede students' access to tasks are the result of the inherent complexity of the task whilst others arise from students' resource levels. When the conditions facilitating task access are examined it is apparent that some conditions appear to reduce the difficulty level of a task for particular students whilst others contribute to reducing the complexity of the task. The conditions that reduce difficulty are either personal in nature (e.g., being able to visualise or possessing prior knowledge of the situational context) or attributes of the task that are susceptible to individual variation when a particular student interacts with the task (e.g., how recent a particular piece of mathematics required in the task has been studied). On the other hand, the complexity of the task appears to be reduced by particular attributes of the task (e.g., the presence of trigger words or visual features). An examination of the impeding conditions confirms this separation of task difficulty from complexity on the basis of task difficulty resulting from students' personal attributes and their interaction with the task. Students' resource levels thus act as intervening conditions between the complexity of a task and the difficulty level the students experience in attempting to access the task. These intervening conditions explain the different consequences that occur (e.g., different difficulty levels or whether impeding conditions are overcome) when different students attempt tasks of the same complexity.

CONCLUSION

This paper sought to identify the conditions affecting students' access to applications tasks as contexts can facilitate access for some, but not all, students. By documenting conditions that facilitate or impede task access and the strategies that students use to take advantage of facilitating conditions or successfully overcoming impeding conditions, teachers are provided with the means to develop teaching activities which focus on reducing the time students spend on orientation activities during the solution of these tasks. Such activities should promote the development of cognitive skills that ensure more effective problem representation and analysis and metacognitive strategy knowledge which facilitates appropriate decision making during orientation.

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APPENDIX

The Shed Problem (Task 18.1): An owner-builder is designing a shed for her backyard in an area that is affected by cyclonic winds. The length of the overhang of the roof is to be such that in mid-summer, when the angle of elevation of the sun is 75°, the 2 metre external vertical walls of the shed are just in shade from the roof overhang. She wishes to use Trimdeck roof sheeting. Because of the cyclonic wind classification for the site, council regulations restrict the overhang width (the horizontal distance from the outside edge of the roof to the wall) for the particular design to a maximum of 520 mm. If the pitch of the roof (the angle the slope of the roof makes with the horizontal) is 12°, is it possible for the owner-builder to meet council regulations and keep the walls shaded as stated above?

BEST COPY AVAILABLE

CHOICES OF COMPUTATIONAL STRATEGIES

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This study was designed to explore the computational choices made by 25 students in Years 5-7. The data was collected in individual interviews and the results showed that many students have a very limited repertoire of methods at their disposal. Students often tended to lack confidence in their approach and often it was only then that they reached for a calculator. A number of students used two or more computational methods for some items.

BACKGROUND

There is much debate around the following issues.

- What is the place of standard written algorithms in the classroom?
- What computational choices are sensible in today's calculator age?
- What does it mean to make efficient and effective computational choices?

A National Statement on Mathematics for Australian Schools (1991) indicated that a goal for students in both Band A and Band B was to be able to "choose computational methods (mental, paper and pencil, calculator) and check the reasonableness of results" (pp. 115, 121). This raises the question as to how adept students are at choosing an appropriate computational method with which to solve a problem.

There does seem to be general agreement that students should be taught to make sensible choices among mental, written and calculator methods of computation. This study was designed to investigate the choices made by students in Years 5, 6 and 7 when given a variety of computation tasks. In particular the researchers were interested to discover whether students actually had a choice of computation method and, if so, how they exercised that choice.

Previous studies (Reys, Reys & Hope, 1993; Price, 1997; Carraher, Carraher & Schlieman, 1985) have also focussed on the choices made by students, but not on whether the students had any alternative strategies at their disposal.

Major policy documents and curricula in recent times have promoted the goal that students should learn to make sensible computational choices (AEC, 1990; NCTM, 1989). The issue of computational choice is a complex one. Coburn (1989) has suggested that the notion of computation needs to be broadened to better reflect the variety of choices available to students.

Price (1997) studied teacher presence as a variable in computational choice and found that students favoured the use of paper and pencil over other methods of calculation. It is very interesting to note that with the teacher present, students chose written methods 56 percent of the time, calculators 26 percent, and mental methods 19 percent of the time. Yet a 10 percent swing from written to calculator methods occurred when the teacher left the room. An earlier study (Reys, Reys & Hope, 1993) found similar trends in terms of written algorithms dominating computational choice, but found a higher incidence of mental computation and less use of calculators. They also noted that computational choices varied according to the nature of the numbers used in each computation item.

The preference for choosing written algorithms is probably related to the amount of time spent in school classrooms teaching standard written algorithms (Porter, 1989). Price's (1997) findings that students choose to use written algorithms more often when teachers are nearby than when they are not present, again supports the influence of the teacher on the students' choices here. Carraher, Carraher and Schlieman (1985) found that the setting affected computational choice. In an out-of-school setting the students used self-taught methods whereas in the school setting they chose to use school-taught methods.

Ruthven (1995) has argued that attitudes and beliefs play a role in student computational choice, in particularly the propensity to use a calculator. He notes that "... their degree of scepticism about the legitimacy and beneficence of the calculator and of lack of confidence in the calculator mode of calculation affect students' computational choices" (p. 229). To suggest that there is a clearcut choice between the various computational alternatives of mental, paper and pencil and calculator is too simplistic. Ruthven (1998) who studied the use of mental, written and calculator strategies by upper primary students noted that "... a refinement of the common-sense trichotomy between mental, written and calculator methods was necessary ..." (pp. 29-30). Several models have been proposed to describe the computation process (NCTM, 1989; Coburn, 1989; Trafton, 1994). However, these are linear in structure and imply that students tend to focus on only one method of computation at any time, such as calculator use for example. Swan and Bana (1998) developed a model to represent situations where students use a combination of two or three strategies for a particular computation.

THE STUDY

This study investigated computational methods and computational choice among 25 Year 5, 6 and 7 students spread across two classes in a primary school in Western Australia. Each student was asked to undertake a series of eighteen computation exercises which were presented both in oral and written form (in horizontal format). This was carried out in an individual interview situation, and the student was free to work mentally, or use the pencil and paper or calculator provided. For each item the initial choice of computation method was noted, along with whether or not the student obtained the correct solution. The student was then asked if he/she could answer the question using another method. If the student indicated he/she could solve the problem in another way then the interviewer requested he/she demonstrate this. Records were kept of successful and unsuccessful attempts, and of computational preference. Previous studies have focussed on computational choice but not on whether students had alternative choices at their disposal nor whether they used a combination of methods to solve a particular problem. Students were questioned to determine whether their initial choice was their only choice, and they were also asked to explain their methods of solution.

The Instrument

The instrument used to determine students' computational preferences was developed as a result of a pilot study (Swan and Bana, 1998) and also from previous studies relating to computational preference (Reys, Reys & Hope, 1993; Price, 1995). The instrument is shown in Table 1 and consisted of 18 computation items. All but two of these were presented out of context in order to focus on the computation by eliminating associated extraneous variables.

Table 1
The Eighteen-Item Test Instrument

Number	Item	Number	Item
1	28 + 37	10	14 x 9 ÷ 6
2	74 – 36	11	$^1/_2 + ^3/_4$
3	369 ÷ 3	12	$10 - 4^3/_4$
4	36 x 25*	13	$^2/_3$ of 45
5	70 x 600*	14	$1.99 + $1.99**
6	29 x 31*	15	$4.93 + 39c**
7	33 x 88*	16	7.41 – 2·5
8	1000 x 945	17	0.25 x 800
9	10% of 750	18	3.5 ÷ 0·5

* Items used by Reys, Reys & Hope (1993) and by Price (1997).
** Items were presented in a shopping context.

Results

The analysis of the data collected in this study is not yet complete. However, some preliminary results are presented here. Firstly, the computational preferences of students in four items used in two previous studies are examined. Then several selected interview extracts are presented to illustrate instances of how computational preferences are exercised by students, with particular attention to the use of a mix of methods.

Table 2
Percentage Distributions of Computational Choices Over Mental, Written and Calculator Use for Four Multiplication Items in Three Studies

Item	Swan & Bana n = 25			Reys, Reys & Hope n = 250			Price n = 18		
	M	W	C	M	W	C	M	W	C
36 x 25	20	40	40	20	71	9	3	70	27
70 x 600	48	8	44	39	45	16	41	33	26
29 x 31	12	64	24	29	63	7	5	56	39
33 x 88	12	56	32	13	73	14	3	66	31
All 4 items	23	42	35	25	63	12	13	56	31

Table 2 compares students' computational choices in the Reys, Reys and Hope (1993) study, the Price (1995) study, and the present study for four multiplication items. It should be noted that only broad comparisons can be made, as the items were administered differently in the three studies and the ages of the students also differed slightly. Also, only the US study by Reys et al used a large sample. It is interesting to note that the US study showed limited calculator use compared to the other two studies undertaken in Australian schools. The US students favoured written methods in all four items. However, this difference may simply be due to recency and sampling factors.

Further analysis of the results is still to be carried out but it should be noted there were several instances where students chose a particular method but were unsuccessful, then when asked if they could use another method they did so and found the correct solution.

Sample of Interview Extracts

The extracts which follow have been selected to illustrate the computational choices that students make and to show that these seem to fit a model of computation which includes a mixture of types, rather than a situation where students follow one particular method.

The first extract shows that Noel in Year 7 used a blend of calculator and mental methods to solve the problem.

Interviewer: Two thirds of 45
Noel: 15
I: 15 is it?
N: No it would be 30.
I: How did you do it?
N: I did 45 divided by three on the calculator to find one third and then doubled it in my head.

Several students adopted a similar approach The same item evoked a different response from Tim in Year 7 who completed the calculation 45 ÷ 3 on paper and then completed the calculation mentally and wrote 15 = 30, indicating the mental leap to the solution once he had established what one-third of 45 was by using the written algorithm.

The following student, Kylie from Year 7, demonstrated an ability to complete the same item in number of ways. For example, the item '369 ÷ 3' produced two different responses from her.

I: Three hundred and sixty-nine divided by three.
K: One hundred and twenty-three.
I: I noticed that you used a calculator, why?
K: Because it was easier.
I: Could you do it another way?
K: Yes — written

She then went on to demonstrate a perfectly executed short division algorithm. Clearly this student had at least two methods at her disposal and made the choice to use a calculator. She also demonstrated this choice in other items. For example, when confronted with the in-context question, "I went to the shop and bought two bottles of Pepsi at $1.99 each, so how much did they cost?", Kylie gave the following explanation.

K: $3.98 (after obviously undertaking mental calculations, then using a calculator)
I: I noticed that you tried to do it in your head to start with, and then you used your calculator. Can you tell me why?
K: I was getting confused with the numbers. I had put 2 and 2 together and got 4, and took 2 away and got 3.98, so I just checked it on the calculator.

The researcher had made the assumption that the student could not perform the calculation mentally and so had reached for the calculator. Asking the student to explain what she had done revealed a different picture.

Some students appear to like the security or familiarity of the written algorithm. In response to the item 369 ÷ 3 Linda in Year 7 wrote the algorithm down and the answer almost simultaneously.

I: That was pretty quick. I noticed that you wrote it down, but what was going on in your head?
L: As soon as I saw 369, well 1 x 3 is 3, 2 x 3 is 6 and 3 x 3 is 9.

It appears that some students do monitor what is happening in a calculation and adopt different approaches accordingly. The item '33 x 88' drew the following interesting response. A student wrote down 33 x 88 in the traditional format for the multiplication algorithm and ended with an answer of 44. He then changed his mind, reached for a calculator and computed the correct answer. When asked to explain why he had done this he had difficulty explaining his action and eventually replied, "because it was just quicker".

At times students seemed reluctant to express their thoughts for fear of giving an incorrect answer. In response to the item '10% of 750' the student referred to above used a calculator but did not make use of the percentage key. He entered 750 ÷ 10 into the calculator. When asked about this he said, "I did it in my head, but then I just checked it on the calculator".

The interviews revealed some gaps in student understanding. Sharon in Year 6 experienced a great deal of difficulty with subtraction. In her case she probably did not have any computational choice other than to use a calculator. Her comments below indicate that she does not have any mental or written strategies, other than an inefficient counting method, and is therefore restricted in her computational choice.

I: I noticed that you started writing strokes on paper and then you changed your mind and went for the calculator, why?
S: It just takes too long. I was going to write down 74 and then strike out 36 and then count.

The following extract from an interview with Alex in Year 6 highlights a mix between mental computation and his informal jottings on paper when the burdens on his short-term memory become too great.

I: Thirty-six times twenty-five
A: Nine hundred
I: Now you seemed to do most of it mentally there, but I notice that you wrote two things down. Tell me what happened.
A: I was a bit confused, there was too much to remember. First I started off with 2, 2 times 6 equals 12, and 1 is left over, so 2 times 3 equals 6, add 1 equals 7, so that's 72, so I added a 0, that's 720 (which he wrote down), and then I times, 5 times 6 equals 30, then 0 take 3, put the three where the other 3 is, then 5 times 3 equals 15, and the 3 is 18, so that's 180 (which he wrote down) and add it up.

Alex used a similar method when tackling the item '29 x 31'.

I: What were you doing it in your head when you wrote down 870 and 29?
A: 3 times 9 is 27, then you leave the 7, take the other 2, and 3 times 2 equals 6, add 2 equals 8, that's 87, and I added a zero because that's in the tens column, so I wrote it down there, so 1 times 29 is 29.

Likewise, Steven in Year 6 mixed mental methods and informal jottings to complete the item '28 + 37'.

I: Okay, you wrote down 50, then you wrote down 65, what were you doing?
S: I just added 20 and 30 together, and then I added 7 and 8 together.

A mix of calculator and written algorithm approaches was noted in the item '33 x 88'. A few students used the calculator to multiply three by eight, which is of concern, and then they went on to complete the written algorithm from that point. It should be noted that students sometimes chose unsuccessful methods. For example one student used mental computation to determine the answer to 369 ÷ 3 and gave an answer of 70. When asked whether she could do the calculation another way she used a calculator and achieved the correct result.

It appears that in some cases when the students lose confidence they reach for a calculator. For example one boy in response to the item '70 x 600' wrote down 600 x 70, decided he couldn't do it and then reached for a calculator. He could not explain why he couldn't do it on paper but was convinced that he couldn't. Perhaps he wasn't sure what to do with all the zeroes. Another student tackling the same problem tried using a mental method but ended up using a written algorithm. When asked about this his response was "I was just thinking, times that by 100, and then taking a couple off that, but I kept getting mixed up".

CONCLUSIONS AND RECOMMENDATIONS

As stated above, the data analysis for this study is still on-going. However, the initial findings based on the preliminary data from the 25 students drawn from Years 5, 6 and 7 attending a single primary school suggest the following tentative conclusions.

- Students often do not have a range of computational methods at their disposal. Some students do not have the luxury of a choice because they only know one method and therefore that method is their only choice. For example many students did not know how to perform a percentage calculation using a calculator and therefore this method of solution was not available to them.
- Students often start performing a calculation in one way using a particular computational method and then change part way through the calculation.
- When students lose confidence or realise they have made a mistake they will often use a calculator to complete the calculation or to check their working.
- Students seem to use the method that they feel most safe with. They might know how to do it another way but prefer to take the safe route.
- Some students make inappropriate choices because the method they use is unsuccessful in that they cannot complete the computation or it leads to an incorrect result.

These initial findings give a glimpse of the choices students do make and indicate that these choices are not simply split between three alternatives, but in some cases involve a combination of methods. The chosen methods are often idiosyncratic, depending on the item and on the individual. This suggests that the vast amount of time spent on teaching particular computational procedures in classrooms — whether mental, written or calculator methods — is likely to be of very limited use in the promotion of computational facility or number sense in students. It is suggested that exposing students to a variety of approaches and encouraging them to take responsibility for their own learning will be more beneficial. This could enhance students' choices of methods as well as increase their confidence in the computation process.

More research is needed to determine what factors affect students' computational preferences. There is evidence that the teaching program has some effect, particularly in reference to using a calculator. The magnitude of such an effect should be investigated through longitudinal studies.

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REDUCING THE INCIDENCE OF MATHEMATICAL MISCONCEPTIONS IN 'MIDDLE BAND' STUDENTS

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Previous studies conducted collaboratively by the author have shown that a strategy based on Piaget's notion of cognitive conflict was very successful in causing a reduction in the frequency of mathematical misconceptions exhibited by a group of very bright tertiary students and that much of this improvement persisted over time. This study investigates whether the same method effectively reduces the frequency of mathematical misconceptions exhibited by average or 'middle band' first year university students.

INTRODUCTION

Several researchers (Bell, 1982; Davis, 1984; Farrell, 1992; Margulies, 1993; Perso, 1992; Swedosh, 1996) have studied various aspects relating to the mathematical misconceptions exhibited. These include the variety of misconceptions, their frequencies of occurrence, and their importance to the student's future learning of mathematics. The number and range of different types of mathematical misconceptions is enormous, "and a complete list may not even be practical" (Davis, 1984, p. 335). Swedosh (1996) discussed the types, frequencies, and possible reasons for the misconceptions commonly exhibited by mathematics students on entering tertiary mathematics subjects at the University of Melbourne (U. of M.) and at LaTrobe University (LaTrobe).

The importance of understanding basic concepts, especially if continuing one's study of mathematics, is well documented. Eliminating misconceptions, where students have developed them, is therefore also of great importance. Government bodies have recognised the need for a sound preparation for further studies and for increasing the participation rate in post-secondary mathematics education (Ministry of Education Victoria, 1984; Victorian Government, 1987; Australian Education Council, 1990). In A National Statement on Mathematics for Australian Schools (Australian Education Council, 1990), a major goal is that "as a result of learning mathematics in school, all students should possess sufficient command of mathematical expressions, representations and technology to continue to learn mathematics independently and collaboratively" (p. 18).

In several areas of mathematics, success at a particular level depends heavily on previous mastery of basic concepts and also on being able to confidently use certain skills (Swedosh, 1996). Swedosh also argues that the level of understanding of some of the basic mathematical concepts which are expected to be acquired in secondary school determine, to a large extent, the preparedness of students to study tertiary mathematics. As the mastery of pre-requisite knowledge has such a profound influence on subsequent learning of mathematics, there has been great interest about mathematical misconceptions, the frequencies with which they occur, and any method which has the potential to reduce these frequencies, amongst a large number of mathematics teachers, lecturers, and tutors.

Preparedness of students to undertake tertiary mathematics is an issue of great significance in the light of statements made by Blyth and Calegari (1985) that "contrary to a widely held belief, 90% of all HSC students do apply for tertiary entrance. It is reasonable to infer from this that most students see HSC as a preparation for tertiary studies" (p. 312) and also, "statistics collected by the Mathematical Association of Victoria (MAV) show that 75% of all tertiary courses require a pass in HSC mathematics" (p. 312). These statements were made some years ago but there is evidence to suggest that this situation has not changed dramatically since then.

Having made pronouncements about the influence of misconceptions to future learning, several researchers (Bell, 1982; Davis, 1984; Farrell, 1992; Margulies, 1993; Perso, 1992; Swedosh, 1996) have studied ways to reduce or eliminate misconceptions. Many of them believe that teaching techniques can be developed which aim at diagnosing and eliminating those misconceptions by considering the misconceptions exhibited by students. A variety of attempts have been made to eliminate students' misconceptions in a diverse range of areas. Of these, the "conflict teaching approach", based on Piaget's notion of cognitive conflict, has met with some success. In this approach, a teacher and a learner discuss inconsistencies in the learner's thinking so that the learner is able to realise that the conceptions exhibited were inadequate or faulty and needed modification (Tirosh, 1990). Vinner (1990) supports this premise and states that "there is no doubt that if inconsistencies in the students' thinking are drawn to their attention, it will help some of them to resolve some inconsistencies in a desirable way" (p. 97). Several authors have found the conflict teaching approach to be an effective method to successfully resolve a range of misconceptions relating to aspects of mathematics and physics (Stavy and Berkovitz, 1980; Strauss, 1972; Swan, 1983; Swedosh and Clark, 1997).

Swedosh and Clark (1997) used the conflict teaching strategy in a first year mathematics class at the U. of M which comprised very bright and mathematically strong students (as indicated by their very high Year 12 scores) and found that the frequency of mathematical misconceptions was greatly reduced after the strategy had been employed.

> It is clear that by first challenging or undermining the misconception held by the students by showing the ridiculous outcomes which can flow from such 'rules', and then replacing the 'damaged' concept with the correct one, mathematical misconceptions can be, to a great extent, eliminated (p. 498).

The study by Swedosh and Clark (1997) found the strategy to be effective in significantly reducing mathematical misconceptions. Despite this success, several questions remained including whether the improvement seen would persist, and whether the method would be as effective with students who were not as able. There was some concern that the improvement might be a short term phenomenon and that students might revert to the conceptions which they previously held. Swedosh and Clark (1998) found that

> The results show that while a small proportion of the improvement diminished, a large improvement was still evident one year later and most of the benefit to students had been retained. (p. 595).

There was also concern that the results may not be replicated with students who were not as able as the original group. Swedosh and Clark (1998) stated that

> The improvement for students whose backgrounds were not as good may not be as dramatic or may not occur. The reason for this conjecture is two-fold: the students involved in this study were capable of recognising the inconsistencies in their previous thinking and then learning the correct concepts quickly; and these students demonstrated during discussions with the authors that they were embarrassed by making the errors that they did on what they considered to be material of such an elementary nature, and therefore had a strong desire to remedy the situation. Both of these qualities are likely to be more pronounced with better students (p. 594).

The goal in this study is to ascertain how generalisable the conflict teaching approach is in terms of its efficacy with students of different ability levels. Having determined that this strategy is extremely effective with very able students, and that the benefit persists over time, it is important to learn whether the strategy can be successfully employed to significantly decrease the frequency of misconceptions exhibited by average students thereby improving their chances of future success in their study of mathematics.

METHODOLOGY

The experiment was set up with a treatment group who would be subjected to the conflict teaching approach and a control group who would not. Each of the groups was comprised of students who were studying the same first year university mathematics subject at the U. of M. in Semester 1, 1998. The subject, which is known as Introductory Mathematics, is the easiest first year university mathematics subject which is offered at the U. of M. Both of the groups were taught by a lecturer who had consistently received high ratings from the students for their teaching.

It was considered important that the backgrounds of the students be comparable so that the results of this study could be put into context and not be distorted by inherent differences between the groups. The students reported on in this study are those students who sat both of the tests, and who had completed Mathematics Methods 3/4 (MM) but not Specialist Mathematics as part of their Victorian Certificate of Education (V.C.E.) in 1997 prior to enrolling at the U. of M. in 1998. There were 90 students in this category in the treatment group and 50 students in this category in the control group. All students who did not fit into this category have been excluded from this study. Those excluded from this study include the few students who studied Introductory Mathematics having completed Specialist Mathematics (the most difficult V.C.E. mathematics subject), mature age students, as well as interstate and overseas students each of whose backgrounds were quite different to the students considered herein.

In each of the three Common Assessment Tasks (CATs) in a V.C.E. subject, students are awarded a grade from E to A+ corresponding to a ten point scale from one to ten. Using this scale, Table 1 shows the average mark for each group for MM CAT 1 (the challenging problem), for MM CAT 2 (facts and skills), for MM CAT 3 (the analysis task). Table 1 also shows the average Tertiary Entrance Ranking (a percentile with the highest possible ranking of 99.95 and with about 23 students for each .05 — .05% of students in the state is about 23) for each group.

Table 1
Average Scores for the Two Groups

Students in Each Group	TER	Average Score Maths Methods CAT 1	CAT 2	CAT 3
90 students in the treatment group:	82.82	8.26	7.53	7.20
50 students in the control group:	79.00	7.84	6.94	6.56

There appears to be a small difference in the background levels of the two groups, but, as the statistical analysis to be used is one which compares the relative extent of improvement in each group, the difference is not large enough to cause concern and the two groups can be considered to have similar backgrounds.

Both groups were taught the same material at the same time and both groups were given the same tests at the same times. An internal control mechanism was also set up by having some questions on the test which neither group received any treatment. The objective was to control extraneous influences so that if a significant difference was found between the performances of students in the two groups, we would have eliminated the effect of the teacher (as much as is practicable), the effect of being taught the material, and the effect of the testing, on the improvement made. We would therefore be able to attribute any difference in the improvement of the two groups to the treatment.

The test consisted of nine questions and was a subset of the seventeen questions in the test used in Swedosh and Clark (1997). These nine questions were selected as they were considered to be at an appropriate level for these students.

The original seventeen questions consisted of questions which were similar to the questions posed in earlier tests at the U. of M. and at LaTrobe (Worley, 1993) and each of which had previously had a high frequency of misconceptions exhibited by students (Swedosh, 1996). Some of the questions were similar to those in the list of questions provided in 'Algebraic Atrocities' (Margulies, 1993, p. 41). Each of the questions on the test was designed so that if a student had a particular misconception, this fact would be apparent when the response of the student to that question was considered.

The test was first administered to both groups in mid-March, a couple of weeks into the semester. Students were given ten minutes to complete the test. This was found to be ample for students to complete the test and did not impinge too greatly on the lecturer's class time. Every student's test was carefully examined on a question by question basis to gain information on any misconceptions which had been exhibited. The number of students who had attempted each question, how many had answered the question correctly, how many had exhibited a misconception, and how many had given another wrong answer were recorded so that comparisons could be made later.

During the semester, both Introductory Mathematics classes met incidentally the concepts and techniques embodied in questions two to seven but did not meet anything relevant to questions one, eight or nine. Questions two to seven therefore become the focus of this study. For the six focus questions, both groups were taught the necessary concepts as they arose in the course. The treatment group were additionally taught using the conflict teaching approach in that they were also shown the absurdities which would arise if the 'rule' was interpreted incorrectly. In other words, at the time of needing to understand the concept, in addition to teaching the concept, the conflict teaching approach was used in an attempt to eliminate or reduce the frequencies with which these misconceptions would occur. The fact that the conflict teaching approach was integrated into the regular teaching programme and was taught incrementally, avoids the Hawthorne effect and is educationally more sound. The misconceptions which were exhibited, having previously been identified and discussed in the literature, were specifically targeted for treatment. An example of this is that 37.8% of students exhibited the misconception that given $^1/_x$ - $^1/_b$ = $^1/_a$, then $x - b = a$ and therefore $x = a + b$. The students were shown a similar equation to which the result was self-evident, such as $2 + 3 = 5$. Using the misconception shown above leads to the absurd conclusion that $^1/_2 + {}^1/_3 = {}^1/_5$. The correct method, using a common denominator, was also taught. The rationale is that having seen that the concept, when used in that way, leads to a silly conclusion, the student will avoid that misconception and would be more likely to use the concept correctly and not be distracted by alternate approaches or tempted to try an illegal 'short-cut'. The other three questions on the test would serve as another control. Neither group would meet concepts relating to these questions during the course of this subject. Even so, the exposure to various mathematical topics could indirectly cause some improvement, but the absence of any related instruction should mean that any improvement is comparable if the two groups are comparable.

In mid-May, about nine weeks after the original test, students in both groups were given the same test as they had sat in March. They were not informed that there would be a second test and therefore would not attempt to prepare for it which would have the potential of affecting the results of the second test. Again the test of each student was carefully examined on a question by question basis and the number of students who had attempted each question, how many had answered the question correctly, how many had exhibited a misconception, and how many had given another wrong answer were recorded. A statistical comparison was then able to be made between the frequency of the misconceptions on each question in the first test and the second test for each group.

After the second test, two of the students who made the greatest improvement were contacted and asked to come in and discuss their performance in the tests. These interviews will be discussed in the results section of this paper.

THE TEST

The nine questions in the test are shown below. The most common misconception(s) are shown to the right of each question :

Simplify expressions 1-3 as fully as possible.

1.	$100!/98!$	$2!$
2.	$3^x \times 3^x$	9^{2x}; 3^{x^2}
3.	$2^x + 2^x$	4^x

Solve for x equations 4 - 7:

4.	$x^2 = 81$	$x = 9$
5	$x^2 - 4x = 0$	$x = 4$
6.	$x^2 = x$	$x = 1$
7.	$1/x - 1/b = 1/a$	$x = a + b$
8.	Solve for $x : 2x + 4 < 5x + 10$	$x < -2$; $x > 2$
9.	Factorise $(2x + y)^2 - x^2$	$3x^2 + 4xy + y^2$

THE TEACHING STRATEGY

The concepts targeted were those required to correctly solve the focus questions. The conflict teaching approach was used on questions two to seven for the treatment group. It was hoped that by demonstrating to the students that if this misconception was employed it would lead to a patently ridiculous conclusion, that as a result, students having this conception would be willing to discard it and replace it with the correct concept. Using this method, it was anticipated (based on Swedosh and Clark (1997)) that the frequency of misconceptions exhibited in the first test would be reduced.

The examples used to show the absurdity caused by the misconception and also to demonstrate the correct concept in each case were mainly numerical. They were slightly different to the questions posed on the test (the numbers were changed slightly, etc.). By doing this, it was thought to be unlikely that students would remember the examples they had been shown and would need to use each concept correctly to answer the respective question. Some of the examples which were shown to the students to demonstrate either the correct concept or the absurdity caused by the damaged concept are shown below.

$2^3 \times 2^3 = 8 \times 8 = 64 = 2^6 = 4^3$; $2^3 \times 2^3 \neq 2^9$; $2^3 \times 2^3 \neq 4^6$; $2^3 \times 2^3 \neq 4^9$

$2^3 + 2^3 = 8 + 8 = 16$; $4^3 = 4 \times 4 \times 4 = 64 \neq 16$

$x^2 = 16, x^2 - 16 = 0, (x-4)(x+4) = 0, x = \pm 4$ also, for a quadratic, we generally expect two solutions (one may be repeated);

$ax^2 + bx + c = 0$; $x = (-b \pm \sqrt{(b^2 - 4ac)})/2a$

$2 + 3 = 5$; $1/2 + 1/3 \neq 1/5$

RESULTS

In this section, information will be provided about the responses to the six focus questions and confidence intervals will be used to determine whether there was a statistically significant difference between the change in the proportion of misconceptions made by the treatment group and that of the control group. In fact, 95% confidence intervals were calculated for the differences in the population proportions of misconceptions between the two groups. If such an interval includes zero, the difference between the groups is not significant. The formula for the confidence intervals is

$$p_1 - p_2 \pm 1.96\sqrt{\frac{p_1(1 - p_1)}{n_1} + \frac{p_2(1 - p_2)}{n_2}}$$

where p_i is the proportion of misconceptions exhibited by group i and n_i is the number of subjects in group i. The same statistical technique was used on the three questions which were not met in the course, and none of the differences was found to be significant.

Tables 2 and 3 show the frequency of each category of answer given in the first and second tests for the control group and the treatment group respectively. There were 50 students in the control group and 90 students in the treatment group. 'Av. per st.' is the average per student.

Table 2
The Control Group

Question	Correct		Misconception		Other wrong		No attempt	
	Before	After	Before	After	Before	After	Before	After
2	27	27	18	21	2	2	3	0
3	8	4	30	42	5	3	7	1
4	0	18	48	30	1	2	1	0
5	24	30	13	14	12	6	1	0
6	16	23	23	16	8	9	3	2
7	9	9	19	17	15	15	7	9
Total	84	111	151	140	43	37	22	12
Av. per st.	1.68	2.22	3.02	2.80	0.86	0.74	0.44	0.24

Table 3
The Treatment Group

Question	Correct		Misconception		Other wrong		No attempt	
	Before	After	Before	After	Before	After	Before	After
2	48	77	32	9	8	4	2	0
3	7	27	60	48	14	13	9	2
4	6	49	84	41	0	0	0	0
5	51	73	24	12	15	4	0	1
6	34	45	46	29	7	16	3	0
7	9	19	34	24	27	32	20	15
Total	155	290	280	163	71	69	34	18
Av. per st.	1.72	3.22	3.11	1.81	0.79	0.77	0.38	0.20

Comparing the results from the first test, we see that the two groups are very similar: an average of 1.68 correct compared with 1.72, 3.02 misconceptions compared with 3.11, and so on. In fact, none of the differences between the groups was found to be significant on any question or on the total. It is interesting to note that the average frequency of 'other wrong answers' given (which were not specifically targeted) was much the same for both groups and showed little change from before treatment to after.

Table 4 shows the percentages of each category of answer given in the second test for each group. In Table 4 'Q' is the question number. The last three columns show the results of the confidence interval which compares the proportion of misconceptions exhibited by each group. The first two of these columns give an estimate at the 95% confidence level for the interval in which the difference between the population proportions would lie. As previously stated, if the interval includes zero, the difference is not significant. In the last column, * indicates those differences which are significant.

Table 4

	Of those who attempted the question on the Test 2						Significance test		
Q	% Correct		% Misconception		% Other error		(misconceptions)		Sig. Diff.
	Control	Treat.	Control	Treat.	Control	Treat.	Test 2		
2	54.0	85.6	42.0	10.0	4.0	4.4	0.1698	0.4702	*
3	8.2	30.7	85.7	54.5	6.1	14.8	0.1703	0.4531	*
4	36.0	54.4	60.0	45.6	4.0	0.0	-0.0259	0.3148	
5	60.0	82.0	28.0	13.5	12.0	4.5	0.0021	0.2882	*
6	47.9	50.0	33.3	32.2	18.8	17.8	-0.1514	0.1736	
7	22.0	25.3	41.5	32.0	36.6	42.7	-0.0725	0.2618	
Av.	38.0	54.7	48.4	31.3	13.6	14.0	0.0027	0.3396	*

The results indicate that there is a significant difference between the overall proportions of misconceptions exhibited by the two groups. The proportions of misconceptions for the control group dropped from 54.3% to 48.4% from the first to the second test while those of the treatment group dropped from 55.3% to 31.3%. There is a corresponding increase in the proportion of correct answers given by the treatment group relative to the control group. Five of the six questions show an appreciable difference between the two groups, with three of these differences significant. Question six was the exception.

Soon after the second test had been administered, two of the students from the treatment group who made the greatest improvement from the first test to the second were contacted and invited to an interview to discuss their performance in the tests. The two students were interviewed separately. Both students offered similar explanations for their improvement in that each stated, without prompting, that there were occasions in the second test when they were about to write down an answer, but they realised that using the method which they had, would lead to a silly answer. Both mentioned that they had come to this decision by remembering what they had been shown in class in terms of the 'silliness' which could result from some strategies, and by substituting in numbers to see whether the answer might be reasonable. They then considered the need for an alternate method and remembered the correct one.

CONCLUSIONS

This study set out to ascertain how generalisable were the benefits of the teaching strategy used, based on the notion of cognitive conflict, when considering students of different ability levels. The strategy had previously been found to be extremely effective with very able students, and it had been found that the benefit persisted over time. To establish that this method was useful beyond the scope of a very bright and somewhat atypical group and that it could be successfully employed when dealing with an average group of students, would make it a much more useful strategy. For this reason, it was considered to be very important to learn whether the success of the strategy could be generalised to that extent.

The experiment conducted provides strong evidence that the conflict teaching strategy can be successfully employed to significantly decrease the frequency of misconceptions exhibited by average students. The specifically targeted misconceptions were decreased for the treatment group by a significantly greater amount than for the control group, whereas the 'other wrong answers' (which were not specifically targeted) were not, and neither was the frequency of misconceptions on the three questions which were not taught during the course. The major benefit to be gained from incorporating this strategy, which is extremely simple to implement, into one's teaching, is that not only is it likely that fewer students will have these misconceptions, but, as a result of this, many will directly improve their chances of being successful in their future studies of mathematics. It is suggested that the use of this strategy, especially with regard to the concepts herein, is particularly applicable to senior secondary mathematics as well as to beginning tertiary mathematics.

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Acknowledgements

I would like to acknowledge the cooperation of Frank Barrington and Angie Byrne and the advice provided by David Clarke. Their efforts are sincerely appreciated.

CHILDREN'S UNDERSTANDING OF THE NUMBER SYSTEM

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A cross-sectional study of 132 NSW rural children from grades K-6 assessed counting, number sense, grouping/partitioning, regrouping, place value and structure of the number system. Task-based interview data exhibited lack of understanding of the base ten system, with little progress made during Grades 5 and 6. Few Grade 6 children used holistic strategies or generalised the structure of the number system. Grouping strategies were not well linked to formation of multiunits; additive rather than multiplicative relations dominated the interpretation of multidigit numbers.

Understanding the multiplicative nature of the base 10 system is critical to the development of numeration, place value and number sense. A critical problem is that children do not recognise that the numbers they use are part of a system, and thus they do not have the multiunit structures to understand how the numbers are regrouped in mental and written algorithms. Further, understanding of the use of powers of ten is needed in order to construct the multiunit conceptual structures for multidigit numbers.

RESEARCH ON NUMERATION AND PLACE VALUE

There has been continuing strong research interest in children's development of numeration and place value. Several studies have described the development of 'ten as a unit' (Boulton-Lewis & Halford, 1992; Cobb & Wheatley, 1988; Steffe & Cobb, 1988; Thompson, 1982), and the understanding of place value (Bednarz & Janvier, 1988; Kamii, M., 1982; Ross, 1989, 1990; Thompson, 1992). It is suggested that the construction of new conceptual multiunit structures is an ongoing process that occurs within the classroom environment that includes many elements other than just the representational objects for number (Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter, & Fennema 1997).

The importance of mental models that reflect conceptual structure of numeration has been highlighted by Boulton-Lewis and Halford (1992). From an information processing model, Boulton-Lewis and Halford (1992) pursued the question of how children were influenced by concrete analogs in their developing numeration and place value skills, particularly in relation to counting and place value. At the same time, Hiebert and Wearne (1992) investigated children's representations of numeration including their understanding of decimals. Other studies endeavoured to categorise in developmental levels, key aspects of place value knowledge (Jones, Thornton & Putt, 1994; Resnick, 1983; Ross, 1990). However, few studies have consistently examined the relative influence of these key elements of children's understanding of the structure of number system.

Classroom-based Studies

Much of the groundwork focusing on children's difficulties with numeration and place value has given rise to classroom-based studies (Bednarz & Janvier, 1988; Carpenter, Fennema & Romberg, 1993; Fuson, Fraivig & Burghardt, 1992; Hiebert & Wearne, 1992) and learning frameworks (Denvir & Brown, 1986a, b; Fuson et al., 1997; Jones, Thornton, Putt, Hill, Mogill, Rich & Van Zoest, 1996; Resnick, 1983) aimed at improving the teaching and learning of numeration, place value and multidigit operations. Case studies and teaching experiments have investigated young children's development of two and three-digit numeration (Cobb & Bauersfeld, 1995; Jones et al., 1994).

New Directions for Research on Numeration and Place Value

From the developing body of research on numeration and place value we know that a child's understanding of the numeration system is complex, is not necessarily lock-step, and develops over many years. There is also a fundamental change in the way a child understands number from the early notion of number as a counting unit, to the construction of composite units (Steffe, 1994) and the reinitialising of units (Confrey, 1994). The process that starts with treating a collection as a whole and then develops as a system that is built on the iteration of grouping collections, requires significant cognitive reorientations.

Research on how children extend their early number understanding and skills to cater for the expanding system of generating number names and symbols that we know as the Hindu-Arabic numeration system has been far less comprehensive than the work on early number. In the early 1990's there seemed a need to understand more fully why children failed to develop a structure for the numeration system at a time when concrete materials and more conceptually-based teaching approaches had been advocated for over a decade in Australia (Booker, Irons & Jones, 1980; Dienes, 1960; 1964).

Despite much research on counting and place value in the 1980's, by the 1990's researchers could not assert any firm explanations about why children fail to grasp the structure of the number system. Sinclair, Garin, & Tieche-Christinat (1992) make the crucial point that:

> Understanding place value is not a matter of simply 'cracking' an arbitrary written code following adult explanation or some degree of exposure to computation. It is indissolubly linked to understanding the number system itself. Grasping it implies understanding a multiplicative recursive structure. (p. 93)

This study investigated which aspects of developing number knowledge contribute to the apparent failure of children to make sense of numeration as a number system. Two broad research questions were addressed. What strategies do children use in solving numeration tasks involving the elements of counting, grouping, and structuring place value? How are critical aspects of counting and grouping related to understanding the base ten structure of the numeration system?

These questions are critical because children need structural flexibility in counting and grouping in order to operate meaningfully with the number system. The role of visualisation of the counting sequence was also examined in view of children's representations of the numeration system which has been reported elsewhere (Thomas & Mulligan, 1995; Thomas, Mulligan & Goldin; 1996)

METHOD

The study was designed as a broad exploratory investigation employing task-based interviews and quantitative and qualitative methods of analysis. A descriptive approach was used to provide evidence of qualitative differences in the way children use their strategies and relate key elements of the numeration system.

Sample: A cross-sectional sample of 132 children from Grades K to 6 was randomly selected from six Government schools in the Western Region of New South Wales. Five of the schools were from three large regional towns and one was the only school in a small rural town. The sample was representative of a wide range of socio-economic backgrounds.

Interview tasks: A total of eighty-nine tasks were incorporated after trialling in a pilot study. The tasks were designed to probe understanding of numeration through: counting; grouping/partitioning; regrouping, place value; structure of numeration and number sense. Many of the tasks were refined from those used by previous researchers (Bednarz & Janvier, 1988; Cobb & Wheatley, 1988; Davydov, 1982; Denvir & Brown, 1986; Labinowicz, 1985; Mulligan, 1992; Ross, 1986; Steffe & Cobb, 1988; Wright, 1991). The tasks were graded by level of difficulty and different subsets of tasks were given to each grade cohort.

Interview Procedures: All interviews were conducted by the researcher in October and November of the school year and these were carried out in a small room separate from the classroom at the schools. The interviewer told the children that he was interested in how they worked out the answers to some questions. The tasks were presented verbally to the children, with concrete or pictorial material. The tasks were re-read to the child as often as necessary to assist him/her in understanding the intention of the question. Concrete materials and pencil and paper were available on the table. The interviewer explained to the child that this material was available for use if required and that he/she was to 'think aloud' as the 'maths activities' were being done. The child wase praised for their attempts, but no feedback was given as to the correctness of their responses. When a response was unclear, follow-up neutral questions were asked by the interviewer such as: "Can you tell me how you did that?"; "Can you describe what you did there?" and "Did you see anything in your mind when you did that?". The interviews were audio-taped and the length of time for each interview ranged from 25 to 65 minutes.

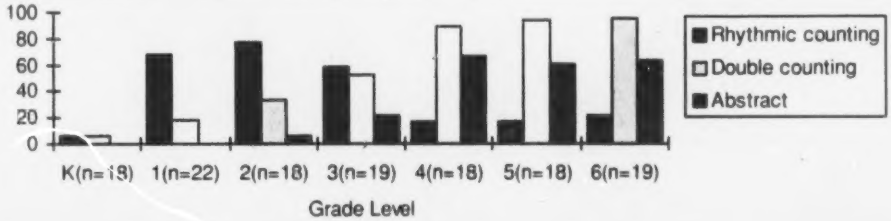
Analysis of Data: Item Response Analysis using Student-Problem curve theory (Hamish, 1983) and the Rasch model (Rasch, 1980) were used initially to obtain some overall measure of student performance in Grades 4 to 6. The main analysis of results involved coding responses for student performance and strategy use (which is reported partially in this paper) across tasks and grades. The coding of responses was trialled in a pilot study which was devised to indicate correct, incorrect or non-response to the tasks. Strategies used for both incorrect and correct responses to tasks were coded in order to classify the range of numeration skills and understandings. Re-coding was conducted by two independent coders for 20% of responses which established a high level of intercoder reliability (0.92).

ANALYSIS OF RESULTS

The discussion will provide an overview of key findings based on children's performance and an analysis of their strategies. The majority of children across Grades 1-6 recognised and used concrete materials to represent grouping of numbers, could identify place values of digits in numerals, and could successfully carry out algorithmic procedures. However, many relied on unitary counting in mental calculations and they did not necessarily use structured materials meaningfully. Children may have shown good performance on 2-digit mental calculations, but generally the use of unitary counting methods prevailed and so many children could not extend their successful use of small numbers to larger numbers. There was, in general, a weak awareness of structure and, in particular, of the multiplicative nature of this structure. It appears from the data that additive relationships within the number system are better understood and used than multiplicative relationships. The lack of conceptual understanding of the tens and hundreds structure of number means that the knowledge of ones, tens and hundreds that exists is not connected and so ability to work with larger numbers is restricted. On the other hand, some young children acquired elements of understanding of place value, represented number in ways that reflected elements of structure and developed their own efficient mental strategies.

Children's counting abilities were shown to be of fundamental importance to developing understanding of the number system. Figure 1 shows that by the end of Grade 2, most children still had a strong reliance on rhythmic counting but other children had developed double counting skills. For Grade 5 and 6 children there was little discernible progress with rhythmic or double counting.

Figure 1 *Counting Task 5: Counting by Threes*
Solution Strategies for Counting Task 5, Counting in Threes: Percentage of Sample Giving Correct Responses, by Strategy use.



At Grade 2, Figure 2 shows the majority of children (67%) were undertaking mental calculations using unitary counting methods for tasks involving 1-digit numbers. At Grade 4 there were still 11% of children using counting-on by ones.

Figure 2 Regrouping Task :1 Mental addition (43+8) represented by Pregrouped Material Solution Strategies for Regrouping Task 1, Addition: Percentage of Sample Giving Correct Responses, by Strategy use.

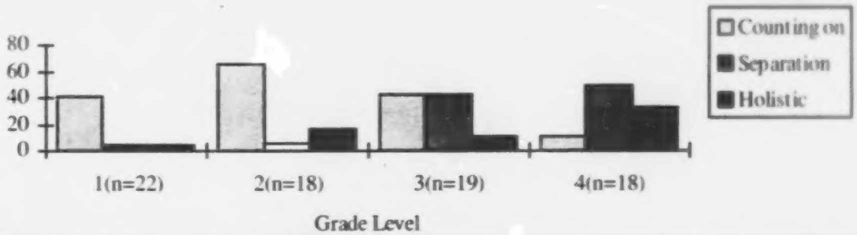
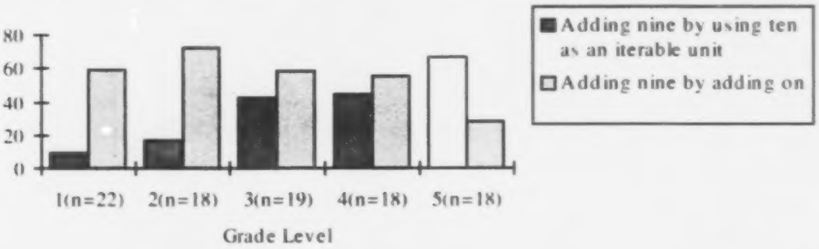


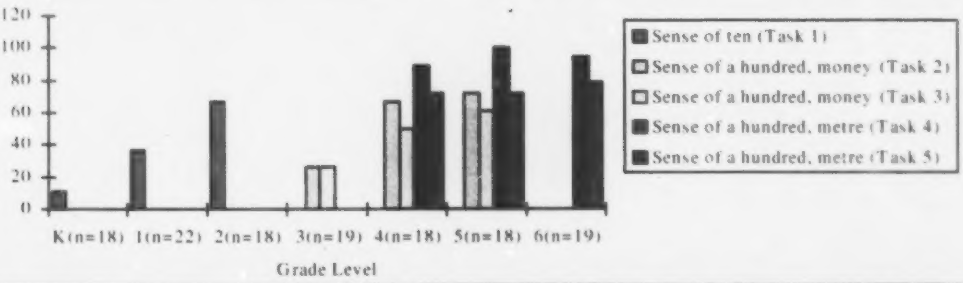
Figure 3 shows 33% of Grade 5 children still using unitary counting for Regrouping Task 2.

Figure 3 Regrouping Task 2: Addition with Concrete Material, add 9 to a Representation of 52 using bags of 10 shells and single shells. Solution strategies for Regrouping Task 2, Addition: Percentage of sample giving correct responses, by strategy use.



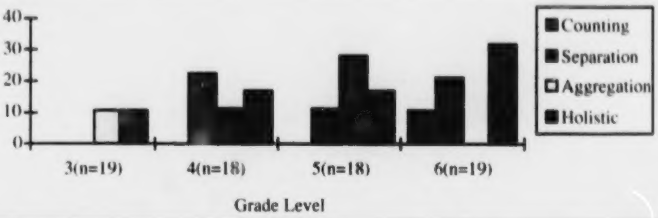
The results of the number sense tasks compared in Figure 4 also showed that children in Grades 2 and 3 exhibited low performance on using the part-whole relationships with ten and one hundred.

Figure 4 Number Sense Tasks 1to 5: Addition to Ten and Applications Performance on Number Sense Tasks 1 to 5: Percentage of Sample Partitioning Tens and Hundreds, on Separate Tasks. Number Sense Tasks 1 to 5: Addition to ten and applications



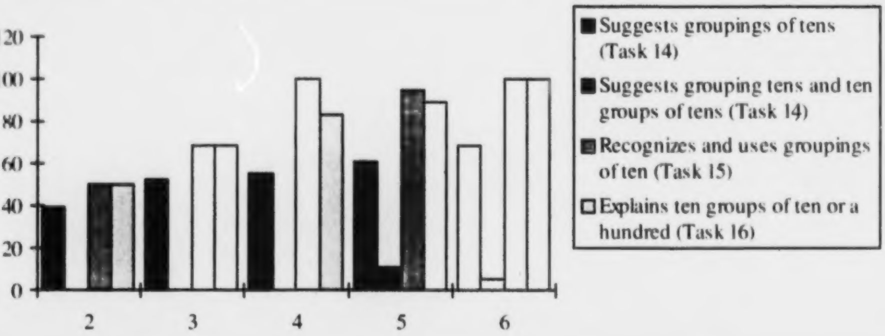
A child who uses ten as a singleton unit might be able to recite the decade numbers but makes no sense of the increments of ten - the units of one and ten co-exist but are not coordinated. The results of this study (Figure 5) show that 22% of Grade 5 children could not deal with two different units simultaneously. Approximately a third of Grade 6 children could not successfully add two 2-digit numbers mentally, where the first number was represented with pregrouped material (Regrouping Task 7).

Figure 5 *Regrouping Task 7: Addition involving ones, tens and hundreds (245 + 98), the first addend shown as pregrouped material. Solution strategies for Regrouping Task 7, Addition: Percentage of sample giving correct responses, by strategy use.*



Another third of the Grade 6 children used counting or separation strategies. The existence of a significant number of children using counting and separation strategies could be explained by their strategies reflecting classroom instruction that commonly emphasises unitary counting in the early years and written procedures for algorithms in the later grades. There was a substantial number of students in Grades 2 and 3 (50% and 32% respectively) who were not successful in recognising and using groupings of ten to quantify a collection of objects shown in Figure 6.

Figure 6
Structure Tasks 14 to 16: Find the Number of Marks in a picture(144 marks randomly drawn).Performance on Structure Tasks 14 to 16, Groupings: Percentage of sample suggesting and recognising someone else's groupings of ten as a grouping number and as a grouping of groupings number.



Item response analysis also demonstrated the lack of progress made by children on many tasks over Grades 4 to 6. The difficulties that children experience with understanding the structure of the number system was further highlighted by the decline in performance shown by Grade 6 children when counting using groups of 10 x 10, repeated use of groupings of 10, suggesting the use of 10 groups of ten, and interpreting zero as a place holder.

DISCUSSION OF RESULTS

It appears that many children can only recognise numbers in terms of additive properties rather than a combination of additive and multiplicative properties. Many children in this study did not realise that multiunits are related through multiplication or that they can be exchanged. In a recent study, Clark and Kamii (1996) reported that although some children develop multiplicative thinking as early as Grade 2, most children still could not demonstrate consistent multiplicative thinking in Grade 5. The present study confirms these results. A substantial minority (about 20%) of the Grade 3 students had developed such an intuitive understanding of powers of ten that they could use the recursive multiplicative structure of the array of 10,000 dots to count the number of dots successfully. By Grade 6, however, there was still a significant number who could not quantify 10 groups of 10 groups of 10 (10x10x10).

Children appeared to have had little experience recognising or using arrays. There was over a third of Grade 6 children who could not use their recognition of the pattern of hundreds in an array of dots to quantify the whole collection. Many children did not know how to use recognised groups of one hundred objects to quantify a collection of 10 000 objects. They did not make the connection that there was a need for equal grouping or multiplication in order to quantify the collection.

The idea that the numeration system is additive in the simplest way (multi-digits represent the total of the face value of the digits) is very strong among Grade 1 children (95.5%) and persists with some children until Grade 4 (22%) and probably beyond. A surprisingly large number of children (61% in Grade 2 through to 31% in Grade 6) was not able to suggest grouping by tens as a means of quantifying a collection. Many children seem slow to grasp these basic elements of grouping tens, formation of multiunits and the way the position of digits plays a role in terms of quantity.

CONCLUSIONS

This study showed that understanding of numeration developed slowly over the Kindergarten to Grade 6 period and that very few children were able to generalise the multiplicative structure of the system. There was evidence of the use of abstract counting strategies by some Kindergarten children, but the performance on counting tasks in the upper grades was still poor for many children. Although the performance on the estimation task was consistently good across the grades, there was lower than expected performance on many number sense tasks because of a high reliance on counting strategies rather than holistic strategies. Although there was good performance on using grouping in quantifying and building grouped material, there were indications that children did not understand the significance of ten in the number system. This understanding is critical to their further development of understanding and use of the numeration system.

The study highlights the difficulties that primary school children have in understanding the complex nature of the number system. Children did not understand the multiplicative relationships within the system that are the basis of place value structure and the patterns in the counting sequence. Children could count and group in tens but did not relate these processes to a base ten structure.

Implications

Many children were shown not to have developed a basic knowledge of the numeration system outside the ones to thousands range by the end of Grade 6. As Dienes' blocks are used widely in New South Wales classrooms, it seems that there might be a connection between children's limited number knowledge and their experiences with the standard representations and interpretations of the blocks. Boulton-Lewis & Halford (1992) argued that concrete materials are only useful if "children clearly recognize the correspondence between the structure of the material and the structure of the concept" (p. 21). Although it was not a specific focus of this study, it appears that children who are familiar with Dienes' blocks may have a limited understanding of the structure inherent in the blocks.

There is an emphasis in schools on multiplication tables and algorithms, but children are not sufficiently exposed to the idea that the base ten system is based on multiplication by 10. Multiplication and division need to be more closely linked, and more experiences bringing out the recursive nature of repeated groupings needs to be provided.

Attempts to develop professional support for teaching and learning to improve the understanding of numeration are currently in progress both in Australia and overseas. The NSW Department of Education and Training has two school-based professional development initiatives focused on numeracy. Both the *Count Me In Too* (CMIT) Project for K-3 grade teachers and the *Counting On* Project for Year 7/8 Secondary teachers (for less able students) aim to facilitate better understanding of students' mathematical strategies when using the number system.

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DIFFERENCES BETWEEN THE SEXES IN MATHEMATICS ACHIEVEMENT IN AUSTRALIA

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In this paper an investigation is reported on whether the trend of sex differences in mathematics achievement of students at the lower secondary school level changed over time by bringing the mathematics achievement of the students to a common scale, which is independent of both the samples of students tested and the samples of items employed. The scale is used to examine the differences in mathematics achievement between the sexes in Australia over a 30 year period. Conclusions are drawn about such differences in mathematics achievement over time in Australia.

INTRODUCTION

During the past 30 years there is probably no other topic in Australian education that has attracted more discussion and more research, including widespread action research programs, than the issue of differences between the sexes in mathematics achievement in Australian schools. Reviews of research conducted in Australia by Leder and Forgasz (1992) and Barnes and Horne (1996) into gender and mathematics refer to "gender" rather than "sex" differences, and follow Megarry (1984) who argues for the use of "gender" to denote "the set of meanings, expectations and roles that a particular society ascribes to sex". Thus "gender" is seen to be a social construct, while "sex" refers to the biological category to which a person belongs. In this paper interest is primarily in recorded differences in mathematics achievement between males and females, the distinction being based on observable biological differences.

Leder (1992) has also reviewed changing perspectives in the area of gender as well as implicit sex differences in mathematics learning from an international viewpoint. However, this review does not examine factors that would account for gender differences both across countries and over time. Articles by Keeves (1973) and Baker and Jones (1993) address aspects of such differences which provide evidence that the measured differences between the sexes in mathematics achievement are at least in part socially based. It is only by the examination of such differences in naturally occurring circumstances that explanation can be advanced, since the conduct of randomized experimental trials is not possible.

Unfortunately much of the research into differences in mathematics achievement between the sexes in Australia has suffered from several serious shortcomings: (a) selection bias, since comparisons are made using groups that are not representative of a target population which is complete and has not suffered from self selection; (b) the sampling procedures employed are inadequate because, while large numbers of students are tested, they are drawn from too few schools; and (c) the estimation of error for significance testing fails to take into consideration the use of a cluster sample design. If sound comparisons are to be made for the detection of differences between the sexes in mathematics achievement, then large representative and random samples of students must be tested and significance tests with appropriate estimates of error must be employed.

The problems have been recognized for over 30 years of the inappropriate use in educational research of significance tests in situations where schools are selected as the primary sampling units and students are selected from within schools. Sometimes very crude estimates of error have been calculated which seek to make allowance for the use of a cluster sample design (for example, Keeves, 1968), although, in general, the problem has been ignored, because appropriate computer programs have not been available for the accurate estimation of sampling and measurement errors. In the past few years computer programs have become available, for example WesVarPC (Brick et al. 1997) and HLM (Bryk and Raundenbush, 1992), that can be readily used to estimate errors which would permit the testing of such data for statistical significance. Under these circumstances it would seem important for analyses to be undertaken of three data sets that involve large representative samples of Australian students assembled at different times over a period of 30 years in order to test whether there are significant differences between the sexes in mathematics achievement, and whether these differences have changed in magnitude and direction over time.

The aim of this paper is then to report whether differences can be detected between the sexes in mathematics achievement in Australia and whether such differences have changed in magnitude and direction over time as might be expected from the widespread programs that have been introduced in Australia to remove all aspects of discrimination associated with gender in schools particularly in the area of mathematics learning.

It is necessary to emphasize that several questions must be addressed before these analyses can be undertaken.

1 The samples selected must not be confounded by the effects of selection bias. This requires that the samples are drawn from schools across Australia, at a stage of schooling where the study of mathematics is compulsory and prior to the period where some students have dropped out from school or where the study of mathematics is optional.

2. It is necessary to establish that the instruments employed to measure mathematics achievement are assessing a single dimension and that measurement is not confounded by the presence of separate components of arithmetic, algebra and geometry. This requires verification that a single dimension of mathematics achievement is being assessed by the instruments employed.

3. In order to make meaningful comparisons it is desirable to bring the different measures of mathematics achievement on the different occasions to a common scale. This can be done by the use of Rasch scaling of the measures and the use of concurrent equating procedures, provided that items and persons satisfy the requirement of unidimensionality, and there are common items in the tests across the different occasions.

4. The issue of whether the differences in mathematics achievement that might be observed should be classified as "sex" or "gender" differences can only be considered after such differences have been shown to be present in the sets of scores under consideration. If no differences in mathematics achievement are detected after replication then no hypotheses can be meaningfully advanced as to the origin of such differences between the sexes. It is important to recognize that a failure to detect differences which can be considered as other than occurring by chance, should not be taken to indicate that no differences between the sexes occur in mathematics education, or that there are not differences between the sexes with respect to those aspects of mathematics achievement that have not been examined in this study.

THE DATA EMPLOYED IN THIS STUDY

The data employed in this study were obtained from the First, Second and Third International Mathematics Studies conducted in Australia and reported by Keeves (1968), Rosier (1980) and Lokan, Ford and Greenwood (1996) respectively. These are all studies that have used large random and representative samples of students from schools across Australia, prior to the stage of schooling where the study of mathematics is not obligatory, and where attendance at school ceases to be compulsory.

In the First International Mathematics Study (FIMS), conducted in 1964, two groups of students participated, namely 13-year-old students in Years 7, 8 and 9 and students in Year 8 of schooling. In total, 4320 students took part in this study. In FIMS only government schools in New South Wales (NSW), Victoria (VIC), Queensland (QLD), Western Australia (WA) and Tasmania (TAS) participated. In the Second International Mathematics Study (SIMS), which was administered in 1978, nongovernment schools and the Australian Capital Territory (ACT) and South Australia (SA) were involved as well as those states that participated in FIMS. In this study 5120 students were tested.

Meanwhile, in the Third International Mathematics Study (TIMS), which was conducted in 1994, government and nongovernment school students in all states and territories including the Northern Territory were involved. In the TIMS study 12,850 students were tested at the lower secondary school level.

In 1964 and 1978, the samples were age samples and included students from Years 7, 8 and 9 in all participating states and territories, although in FIMS a Year 8 sample was also tested. In TIMS the samples were grade samples drawn from Years 7 and 8 or Years 8 and 9. In ACT, NSW, VIC and TAS students in Years 7 and 8 were selected while in QLD, SA, WA and NT samples were drawn from Years 8 and 9, enabling a Year 8 Australia wide sample to be derived for purposes of comparison with the 1964 data.

In order to make meaningful comparisons of mathematics achievement of boys and girls over time by using the 1964, 1978 and 1994 data sets, the following steps were taken. The 1964 students were divided into two groups 13-year-old students in one group (FIMSA) and all Year 8 students including 13-year-old students at that year level as the second group (FIMSB) since in addition to an age sample, a grade sample had also been drawn. It is important to observe that 13-year-old students in Year 8 were considered as members of both groups. In the first group, students were chosen for their age and in the second group for their year level. The 1978 students were chosen as an age sample and included students from both government and nongovernment schools. In order to make valid comparisons between the 1978 sample and the 1964 sample, the 1978 government school students were divided into two groups. The first group included all government students who participated in the study (SIMSG), and the second group included all government school students in the five states excluding students from SA and ACT (SIMSR). Meanwhile, in TIMS the students were chosen as a grade sample. The common sample group for all states and territories was the Year 8 students. In order to make the TIMS samples comparable with the other samples, only Year 8 government school students in the five states that participated in FIMS and SIMS are considered as the TIMSR data set in this study. While the samples employed in these three studies were large and the response rates were, in general, adequate, being in excess of 80 per cent for the FIMS and SIMS studies, there was some evidence of selection bias or response bias with respect to sex, in so far as male students were over represented in the FIMS and SIMS studies and under represented in the TIMS study.

MEASUREMENT PROCEDURES EMPLOYED IN THE STUDY

In this study the procedures employed to compare the achievement differences between male and female students on the three occasions involved the use of the Rasch model to scale students' responses to the mathematics test items. The Rasch model has been shown to be the most robust of the item response models (Sontag, 1984), and was used in this study primarily to equate students' performance in mathematics on a common scale for the Australian investigations conducted in FIMS, SIMS and TIMS.

Unidimensionality

In order to employ the Rasch model for calibrating the items in the mathematics tests it was necessary to examine whether or not the items satisfied the requirement of unidimensionality (Hambleton and Cook, 1977). If the items were found not to satisfy the condition of unidimensionality, it would not be possible to employ the Rasch procedures for the calibration of the tests. Furthermore, it would not be meaningful to compare differences between the sexes with respect to mathematics achievement since a total score would be meaningless.

Consequently, confirmatory factor analysis procedures were employed using LISREL 8 (Jöreskog and Sörbom, 1993) to examine the unidimensionality of the mathematics test items. Confirmatory factor analysis is a statistical procedure employed for investigating relations between a set of observed variables and the underlying latent variables (Byrne, 1989; Spearritt, 1997). The results of the confirmatory factor analyses of FIMS and SIMS data sets revealed that a nested model in which the mathematics items were assigned to a general higher order factor, which was labelled as Mathematics as well as to three specific correlated first-order factors of Arithmetic, Algebra and Geometry provided the best fitting model. In addition, in the confirmatory factor analyses undertaken, no evidence was found to reject the assumption of the existence of one general factor involved in the mathematics tests, in so far as in the nested model the Mathematics factor extracted more of the total variance than did the specific first-order factors taken together. Therefore, the mathematics test items in the FIMS and SIMS studies are considered to satisfy strongly the requirement of unidimensionality. The item cluster-based design procedure employed in the construction of the TIMS data sets would seem to preclude the use of confirmatory factor analysis to test the unidimensionality of the TIMS data set and confirmation of unidimensionality must be provided by the introductory steps in the Rasch analysis.

Developing a common mathematics scale

The calibration of the mathematics test items with concurrent equating permitted a scale to be constructed that extended across the three tests, namely FIMS, SIMS and TIMS test items on the mathematics scale. The fixed point of the scale was set at 500 with one logit, the natural metric of the scale, being set at 100 units. The fixed point of the scale, namely 500 was taken as the mean of the difficulty level of the calibrated items in the FIMS test administered in 1964.

Rasch Analysis

Three groups of students namely FIMS (4320), SIMS (5120) and TIMS (12,850) were employed in the calibration and scoring analyses. The necessary requirement for calibration in Rasch scaling is that the items and persons must fit the Rasch scale. In order to examine whether or not the items and persons fitted a unidimensional scale, it was important to evaluate both the item fit statistics and the person fit statistics. In FIMS two items (Items 13 and 29), in SIMS two items (Items 21 and 29) and in TIMS one item [(Item T1b No 148) with one item (Item 94), which was excluded from the international TIMSS analysis] were removed from the calibration analyses due to the misfitting of these items to the Rasch model with infit mean square values outside the range of 0.77 to 1.30 (Adams and Khoo, 1993). Moreover in this analysis no person was deleted, because the for all cases was less than 5 (Wright and Stone, 1979). However, students with a zero score or with a perfect score were automatically excluded from the calibration procedure, since they did not provide useful information for the purposes of scale calibration, although such students were included in the scoring of the data.

COMPARISON OF SEX DIFFERENCE IN FIMS, SIMS AND TIMS

In this section the differences in mathematics achievement between the two sexes are examined for the three occasions. These comparisons for Australia in 1964 and 1978, differ from those carried out in previous studies (Keeves, 1968; Moss, 1982) in that proper account can now be taken of the complex design of the samples employed in testing for statistical significance. However, in testing for significant differences, while multiple comparisons are involved no use is made of the Bonferroni adjustment (Finn, 1997), because the thrust of the comparisons is more toward the detection of no differences, than towards the detection of highly significant ones.

The t -test

In order to determine the level of statistical significance between the mean scores on FIMS, SIMS and TIMS in mathematics achievements a t statistic was calculated, which took into account errors from three scores: (a) sampling error, (b) errors of calibration, and (c) equating error. Since the samples all involved a cluster sample design with schools sampled with a probability proportional to size at the first stage and students sampled from within schools at the second stage, it was necessary to use the WesVarPC (Brick et al. 1997) computer program to test the data for statistical significance, taking into account both the stratification and cluster sample design employed in all three studies.

Effect size

In this paper both the standardized effect size and the magnitude of effect on the calibrated scales are used to examine the level of practical significance of the differences between FIMS, SIMS and TIMS in mathematics achievements over time.

In this study effect size values less than 0.20 are considered as trivial, while values between 0.20 and 0.50 are considered as small. Furthermore, effect size values between 0.50 and 0.80 are taken as moderate and values above 0.80 are treated as large (Cohen, 1992).

Effect of mathematics learning in one year

Tilahun (1998) has undertaken analyses to estimate the extent of learning achieved by Australian students in the TIMS study in one year. It is estimated that the growth in mathematics achievement between Year 8 and Year 9 is 0.32 logits. From this information it is possible to examine the practical significance of differences between the male and female students in terms of a year of learning of mathematics.

COMPARISONS OF SEX DIFFERENCES BETWEEN 1964 AUSTRALIAN STUDENTS

Table 1 and Figures 1 and 2 show the three comparisons which are considered in FIMS. The estimated mean score differences on the three comparisons are seemingly in favour of boys. This suggests that boys achieved at a higher level than girls. However, the effect size is trivial and t-values are non-significant. Thus, in all the comparisons the differences are neither practically nor statistically significant. It should be noted that Keeves (1968) using a very crude approximation for the design effect of the complex sample reported a significant difference. However, the procedures used currently in significance testing, make proper provision for the complex sample design using the jackknife routine in WesVarPC (Brick et al. 1997).

Comparisons of Sex Differences between 1978 Australian Students

In the SIMS data set four comparisons are undertaken and presented in Table 1 and Figures 1 and 2. The comparisons are between boys and girls in government schools, nongovernment schools, in all schools, and the restricted sample of schools.

The mean score differences between the two sexes in government schools (both Government and Restricted) indicated that girls achieved at a higher level than boys. However, the

Table 1

Descriptive statistics for mathematics achievement of all students for the three occasions by sex

	FIMS					
	FIMSA		FIMSB		Total	
	Male	Female	Male	Female	Male	Female
Mean	462.0	457.0	455.0	448.0	459.0	453.0
Standard Deviation	101.0	90.0	87.0	76.0	98.0	87.0
Jackknife Standard Error	9.7	7.3	10.1	7.5	9.3	6.5
Design Effect	14.9	9.6	17.3	11.2	20.5	11.4
Sample Size	1530	1386	1619	1462	2275	2044

	SIMS							
	Government		Non-government		Total		Restricted	
	Male	Female	Male	Female	Male	Female	Male	Female
Mean	442.0	443.0	488.0	464.0	453.0	448.0	440.0	443.0
Standard Deviation	105.0	100.0	117.0	97.0	110.0	99.0	105.0	99.0
Jackknife Standard Error	4.2	5.1	14.9	9.6	5.2	4.6	4.7	5.8
Design Effect	3.4	5.0	9.5	5.4	6.2	5.2	3.2	4.8
Sample Size	2095	1894	580	551	2675	2445	1614	1424

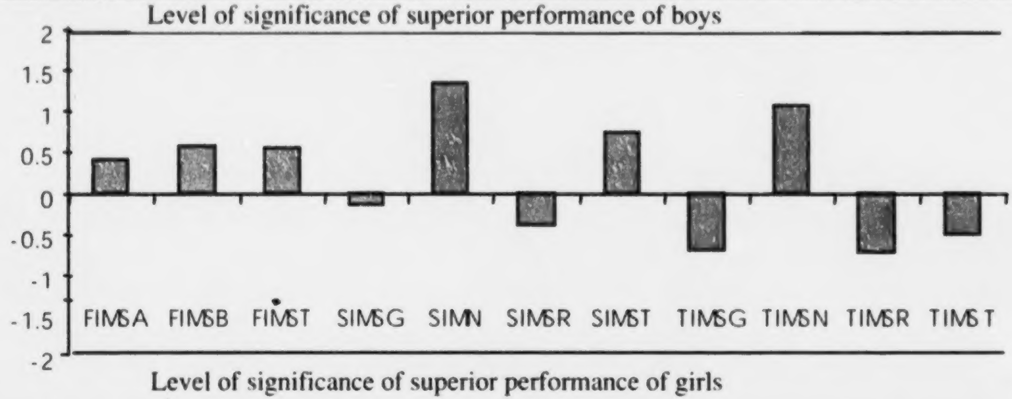
	TIMS							
	Government		Non-government		Total		Restricted	
	Male	Female	Male	Female	Male	Female	Male	Female
Mean	423.0	431.0	488.0	474.0	443.0	448.0	422.0	431.0
Standard Deviation	125.0	124.0	117.0	110.0	126.0	121.0	126.0	126.0
Jackknife Standard Error	7.7	8.4	10.2	8.4	7.0	6.4	8.4	9.2
Design Effect	9.4	9.9	8.5	9.6	11.1	10.7	8.9	9.4
Sample Size	2479	2168	1111	1633	3590	3801	2030	1755

	Mean difference	Effect size	t-value	Sign. level
FIMSA Male vs Female	5	0.05	0.41	Not Significant
FIMSB Male vs Female	7	0.09	0.56	Not Significant
FIMS Male vs Female	6	0.06	0.53	Not Significant
SIMS Government Male vs Female	-1	-0.01	-0.15	Not Significant
SIMS Nongovernment Male vs Female	24	0.22	1.35	Not Significant
SIMS Male vs Female	5	0.05	0.72	Not Significant
SIMS Restricted Male vs Female	-3	-0.03	-0.40	Not Significant
TIMS Government Male vs Female	-8	-0.06	-0.70	Not Significant
TIMS Nongovernment Male vs Female	14	0.12	1.06	Not Significant
TIMS Male vs Female	-5	-0.04	-0.52	Not Significant
TIMS Restricted Male vs Female	-9	-0.07	-0.72	Not Significant

Restricted = Those group of students in SIMS which are comparable with FIMSA, and students in TIMS which are comparable with FIMSB

Figure 1

t-values for differences between boys and girls in Mathematics test score for FIMS, SIMS and TIMS

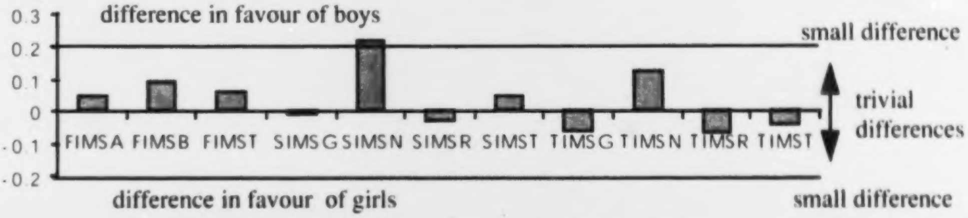


differences were neither practically nor statistically significant since the effect size and t-values were too small to be considered significant.

Furthermore, the estimated mean score difference between male and female nongovernment school students in SIMS is seemingly in favour of the boys, although the effect size (0.22) is small and the t-value (1.35) is non-significant. Thus, the difference is not statistically significant. Nevertheless, the achievement level of the boys is higher than that of the girls by approximately two-thirds of a year of mathematics learning in Australian schools as estimated in the mid 1990s, although this result may be influenced by the particular schools selected in the analysis and the result is not significant statistically.

The last comparison in SIMS is between boys and girls in all schools. The mean score difference is apparently in favour of boys. However, the effect size (0.05) and t-value (0.72) are very small. Hence, the difference between boys and girls in 1978 is not practically or statistically significant across Australian schools. Out of the four comparisons, only in government schools is the achievement of the girls slightly higher than that of the boys, in the remaining comparisons boys appear to achieve better than girls. These findings appear dissimilar to the findings in FIMS. In the 1964 data set all the differences are in favour of boys. This might indicate that over the 14-year period there was a small shift in achievement level differences between boys and girls. However, the differences are not statistically or practically significant.

Figure 2
Effect size values for differences between boys and girls in Mathematics test score for FIMS, SIMS and TIMS



Comparisons of Sex Differences between 1994 Australian Students

The estimated mean score differences between the two sexes in government schools (both Government and Restricted) are in favour of girls (see Table 1 and Figures 1 and 2). However, the effect size and t-values are very small and the mean differences are neither practically nor statistically significant.

The other comparison is between the two sexes in nongovernment schools. Unlike the government school students the difference is seemingly in favour of boys. However, the difference is not statistically or practically significant since the effect size is trivial (0.12) and the t-value is very small, because the design effects are large.

The last comparison in TIMS considers all students (government and nongovernment together). The mean score difference is apparently in favour of girls. However, the effect size (-0.04) and t-value (-0.52) are too small to be considered significant. Consequently the mean difference is neither practically nor statistically significant.

Three of the four different comparisons in TIMS between boys and girls in mathematics achievement reveal that the mathematics achievement levels of the girls is slightly greater than that of the boys. When the findings in TIMS are compared with the findings in FIMS and SIMS, the direction of the difference between the sexes has apparently changed, but the changes are not significant. In Figures 1 and 2 these results for the differences in effect size and in the t-values employed for significance testing are presented graphically in order to summarize the results.

DISCUSSION AND CONCLUSION

In 1964 the differences are in favour of the boys, while in 1978 the difference is in favour of the girls only in government schools. However, in 1994 the differences, except in nongovernment schools, are in favour of the girls. Nevertheless none of the differences are found to be statistically significant. These nonsignificant differences might seem to suggest that a difference between the sexes on mathematics achievement is starting to emerge in the 1990s in favour of girls and in contrast to the findings of Keeves (1972), Carss (1980), Moss (1982), Leder and Forgasz (1992). These researchers would appear to argue that a difference between the sexes in mathematics achievement in Australian schools starts to emerge at the junior secondary school stage in favour of boys. The apparent change in direction in the 1990s could then be argued to have been a result of the implementation of different government policies to increase the participation and the mathematics achievement level of girls by the State and Federal Governments. Alternatively, it could be argued that the level of performance of the boys has declined more than that of the girls over time, as can be seen from Table 1 there is a general decline in achievement of all Australian students over the 30-year period.The possibility of a significant decline in the level of achievement in mathematics of boys, rather than a noticeable gain in the achievement of girls must be of some concern for Australian education.

Until, there is clear evidence of statistically significant differences between the sexes in mathematics achievement and of sufficient magnitude to warrant resources being diverted to addressing the problem, there would appear to be more pressing problems to be considered in Australian education than the nature and origin of differences between the sexes in mathematics achievement at the lower secondary school stage. Moreover, such programs as have been undertaken in Victoria of establishing separate classes for boys and girls in mathematics at the lower secondary school stage would seem to be a misdirection of effort and resources until it can be shown that a substantial problem exists.

This comment should not be taken to divert attention from the problem of the lower participation of girls in advanced mathematics classes at senior secondary school level.

The failure to detect significant differences between the sexes in mathematics achievement in this study which uses the largest and most carefully designed samples available in Australia over a period of more than 30 years has several important implications for educational research in Australia. There is a need to :

employ large well designed random samples and to attain high response rates in order to

- test relationship that are widely assumed to exist;
- use sound statistical procedures to test for the statistical significance of differences;
- report and discuss the magnitudes of effect sizes and the pattern of results, as well as the use of procedures of inferential statistics;
- monitor in a systematic way changes in mathematics achievement over time, as well as changes in the performance of clearly identified subgroups of students; and
- maintain the view that all students can learn mathematics successfully irrespective of sex, race, ethnic background and social class, at all stages of schooling and in particular at the lower and middle secondary school levels.

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Acknowledgment

The first author was sponsored by The Flinders University of South Australia Overseas Postgraduate Research Scholarship and the Flinders University Research Scholarship.

THE CROSS-CULTURAL PERSPECTIVE OF TEACHERS' BELIEFS AND THEIR INFLUENCE ON TEACHING PRACTICES: A CASE STUDY OF TWO TEACHERS TEACHING SECONDARY MATHEMATICS IN AUSTRALIA AND INDONESIA[1]

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Research indicates that a teacher's beliefs have a powerful impact on the practice of teaching, since a teacher's beliefs constitute the reasons which account for the differences of individual teachers in teaching mathematics, without ignoring the influence of his or her content knowledge. This paper describes the cross-cultural aspects of a teacher's beliefs and its effect on his or her instructional practices under the constraints and opportunities provided by the social context of teaching.

This article documents the beliefs of an Australian and an Indonesian senior mathematics teacher and examines how these beliefs influence the teachers' practices. Knowledge of the beliefs of teachers from different countries given their cultural differences and equities will support a more explicit understanding of one's own implicit theories on teaching and learning. Interviews with the teachers and observations of them carrying out their teaching practices provided a window through which to see their beliefs and their influence on the underlying frameworks of the teachers' practices.

THEORETICAL FRAMEWORK

Research on Exemplary Practitioners

It has been proved that research conducted on exemplary teachers provides much to learn for teachers and curriculum developers (Driscoll, 1985; Kennedy, 1986; Lloyd, 1998; Tobin & Fraser, 1987; and Berliner, 1986). Additionally, there have been suggestions that case studies of expert teachers should form a part of teacher education programs (Berliner, 1986; and Shulman, 1986). Studying advanced practices could motivate and guide preservice and inservice teachers in their practice.

The Links of Teachers' Beliefs to Instructional Practices

Green (1971) and Rokeach (1960), as quoted by Thompson (1992) introduced the concept of a beliefs system and use this notion as a metaphor for examining and describing how an individual's beliefs are organised. However, Ernest (1989) suggests the definition of *beliefs* as views regarding the nature of teaching that topic (the purpose, the self-perceived role of the teacher and the relationship between students and teachers), the process of students' learning of that topic, and the nature of that topic. The categorisations of teachers' beliefs in this study are based on Ernest's views.

In devising approaches to the subject matter content and in adopting certain models of learning, teachers' beliefs do have an important role since beliefs act as a filter and a lens in seeing and understanding new information. Brophy & Good (1974) indicated that teachers' beliefs system are especially important in guiding their instructional decisions. Pajares (1992) suggests that beliefs about teaching are well established by the time students go to college, and as a result, such beliefs act as a filter through which new information on teaching is sifted. Prawat (1989) asserts that teachers with the same level of conceptual understanding may teach differently depending upon their educational beliefs (ie. their beliefs about teaching and learning). Further, Putnam, Heaton, Prawat, & Remillard (1992) found that teachers' beliefs and knowledge affect how they perceive and act upon various messages about changing the way they teach mathematics. Teachers' conceptions of what mathematics is, and how it is learned, impacted upon their decisions about what and how to teach. However, research done by Kesler (1985), Parmelee (1992), and Van Zoest, Jones, & Thornton (1994) found that actions are not always consistent with stated beliefs.

Focus of the Study

The main focus of this study was on the teacher's beliefs and their impact on their instructional practices within the constraints and opportunities provided by the social context of teaching. The following questions helped to guide the collection of data:

(i) What kind of beliefs regarding the nature of mathematics, and the teaching and learning of mathematics, does the teacher have?

(ii) How do these beliefs influence his or her teaching practices on teaching mathematics?

METHODOLOGY

A qualitative case study approach was adopted (Merriam, 1988), using an ethnographic framework (Wolcott, 1988; Zevenbergen, 1998). The interpretive research methodology (Erickson, 1986) utilised both quantitative and qualitative techniques. Mr Ton, along with his 48 Year 11 Indonesian students, and Ms Skid, along with her 20 Year 12 Australian students were willing to participate in this study. The study was conducted in an East Javanese and a Western Australian senior high school. The schools were located in the metropolitan areas of Surabaya, the capital city of the province of East Java, and of Perth, the capital city of the state of Western Australia. The author proposed to describe the beliefs or the views of the two teacher—on the nature of mathematics, the nature of mathematics teaching, and on the learning process—and their impact on teaching practices.

Data Collection

Beside the researcher herself as the research instrument (Merriam, 1988), the study utilised an interview guide and a sequence of observations. The interview guide developed by the researcher consisted of questions devised to ascertain the interviewee's views of the nature of mathematics, and of mathematics teaching and learning. Information about Mr Ton's and Ms Skid's subject matter knowledge and beliefs obtained by the interview guide were complemented by a series of observations of their practices in the natural setting of the classroom. Interviews with them were conducted and audiotaped in between the observations and the summaries of the interviews were shown to them. Observation focused on the pattern of Mr Ton's and Ms Skid's teaching and their ways of representing and formulating the topic to the students; in order to establish the impact of their and beliefs on their teaching practices. The 6 lessons observed, 45 minutes each, on the teaching of Graph Theory in Indonesia lasted over a two-week period and the 11 lessons of 60 minutes each, on the teaching of Networks in Australia occurred over a three-week period. The smaller number of lessons in Indonesia was the result of shortening the original plan of teaching 14 lessons on the Graph Theory topic, due to the country's political situation. The teaching of the Networks topic in Australia was broken into two parts, before and after the term exams.

Mr Ton's and Ms Skid's lessons were also videorecorded and the resulting analysis and interpretation of their teaching was shown to them. In addition, the researcher discussed various aspects of their teaching practices with the two teachers. This was to sanctify that the research was conducted in a manner that "respects the rights of the individual whose privacy is not invaded and who is not harmed, deceived, betrayed, or exploited" (Burgess, 1989, p. 60). Interviews were also conducted with three top students, three middle ability students, and three low ability students from each setting. This was done in order to obtain information from three groups of students of differing ability. The interviews sought students' opinions on Mr Ton's and Ms Skid's teaching, on the teaching materials, and on their learning.

Analysis

Initial analysis and data collection occurred simultaneously and evolved as the interpretation of each phase of data collection influenced the subsequent phases. Each interview conducted was based on the themes of previous observations and interviews, while taking consideration of the interview guidelines. The processes of data collection and analysis were recursive and dynamic (Merriam, 1988, p. 123). All the data for the study were organised into a case study database (Yin, 1994) which was constructed by listening to tapes, watching the videos, reading the transcripts, taking fieldnotes, and examining classroom artefacts. The videos helped to create a chronological account of lessons and a base for interview themes. Retracing the phases of data collection helped in identifying the emerging patterns and providing a basis for coding them thematically. The final stage of analysis categorised and synthesised the emergent themes within and across the data sources to lend coherence and richness to the written case study report.

THE SOCIAL CONTEXT

The study was conducted during a period of social, economical, and political crisis in Indonesia. The mass demonstration in and surrounding the Parliament Building not far from the school blocked the traffic and thus affecting the school's teaching and learning. Parents and teachers feared for students safety and the school was closed for two weeks at the height of the unrest.

The Indonesian school syllabi are determined at the central level by the Ministry of Education and Culture and implemented using a top-down approach. A single mathematics syllabi for K-11 and three strands of mathematics for Year 12. Every school in Western Australia designs their own syllabi, except those for Year 11 and 12. An external body, the Curriculum Council of Western Australia, develops these.

Indonesian Teaching Philosophy

Ki Hajar Dewantara, an Indonesian educational expert, inspired his countrymen with his famous teaching philosophy: *Ing Ngarso Sung Tulodo, Ing Madya Mangun Karso, Tut Wuri Handayani* (Tirta-Seputro, 1998, p. 9). This meant that being in front, the teacher is the model of the class; being in the middle, a teacher should generate their spirits to learn; being at the back, a teacher should facilitate and encourage their learning. In a loose way, a teacher's roles in these three different positions in the classroom can be translated as the roles of a model, a friend, and a parent. *Guru* or 'teacher' in Javanese, the dialect spoken by people who live in Central and Eastern Java Island, meant 'digugu'—to be listened and 'ditiru' to be modelled and followed.

THE TEACHERS

Participants and their Background

Both Mr Ton and Ms Skid were full time teachers at their schools. Mr Ton had been teaching in his present school for nine years and Ms Skid in hers for twelve years. Both teachers were regarded as caring teachers with a high standard of moral integrity and honesty. Mr Ton was the Head of Mathematics Department in his school and Ms Skid was the Head of Year 12, which was a pastoral care role.

Mr Ton's and Ms Skid's Schools

Being established in 1967, Mr Ton's school had a total number of 1200 students of Year 10 - 12 and 40 teachers at the time of study. Schooling were conducted in two shifts: morning classes from 6.45 a.m. to 12.20 p.m. and afternoon ones from 12.40 p.m. to 6.15 p.m. Ms Skid's school had been one of the most prestigious girls schools in Perth and was established in 1908. At the time of study the school had a population of 1000 students from K-12 and 75 teachers. Both schools were independent private establishments and recognised for the good ethics and sound teaching methods practised in the schools. In Mr Ton's classroom the students sat by pairs in unmovable desks, four lines of six rows, facing the teacher. The students in Ms Skid's classroom sat in-groups of three or four and sometimes two as they studied their single mathematics course.

Mr Ton's Beliefs

At the time of study Mr Ton was exposed to teaching a new topic in the Indonesian school mathematics and this resulted in his amazement at the uses to which mathematics could be put to solve real life problems in many different fields. He believed that this new mathematics topic offered much to students in seeing mathematics in a different way than he experienced it himself. He felt that the real world applications of mathematics would be especially attractive to students and would motivate them to learn more mathematics. Reflecting his view of the nature of mathematics as a mixture of a static unified body of knowledge and an accumulation of facts, rules and skills to be used in pursuance of some external end, Mr Ton often said, "Mathematics has absolute truth". But, his experience in studying the new topic led him to develop a view of mathematics as a dynamic, continuing, expanding field, "Mathematics develops according to world's progress. It was demanded to contribute its values there".

Mr Ton believed that he had to be firm in teaching, in deciding what was right and wrong when students asked about the solution of problems or definition of concepts, and in controlling the class, so that each student had an equal opportunity to learn. He admitted, "Each student had to learn in the classroom. The classroom must be quiet and be in order to develop some degree of tolerance among the class' members". Mr Ton had the view that it was important to cover all the material listed in the syllabus in the space of time provided, especially in preparing the students for sitting exams. Consequently, there were times that he felt the need to go ahead with a lesson regardless of the less-able students' questions and confusion with details of current or previous lessons. Mr Ton believed that this pressure would somehow push the less-able students to do more study in mathematics by themselves. The risk of being ineligible to move on to the higher grade was of deep concern to most students.

Mr Ton viewed the learning process as an active study of unknown objects until one knew them well. For him passive learning meant an absence of students' effort to learn: Students had to realise that they should not depend on the teacher for their learning, and that the effort to learn should come from themselves. Additionally, he believed that a teacher could not change the students, they should try seriously to change themselves. However, Mr Ton's effort to facilitate students' active construction of knowledge was limited by time and his teaching duties. "There is too much material to teach. …". "The biggest problem is time". Although Mr Ton perceived his role in teaching as a learning facilitator and favoured active learning, the situation he faced in the classroom enhanced his acceptance of a view of learners as submissive and compliant.

Ms Skid's Beliefs

Ms Skid thought that mathematics was a growing active discipline, and that its connections were discovered through different tools, then expanded. She believed that the basic skills in mathematics were important to students and that teachers were only teaching a fraction of mathematics as basic skills and basic applications, but some of the students were very excited about mathematics and wanted to learn more. Ms Skid had the view that the skills of mathematics are heading more to the application of mathematics and problem solving. She realised that the problem solving helps more students to enjoy mathematics a lot more than in her time when mathematics was a lot more rigid and only students who had a good analytical base, really experienced a positive sense of achievement.

Ms Skid embraced the view that the trend these days was towards teaching mathematics implemented with, rather than in isolation from, other subjects. She had the view that it was important for the students to be able to analyse and use the equipment and technology which were now available. She believed in attending regular inservice activities to catch up with new developments in teaching and learning. Ms Skid had the view that well-prepared lessons resulted in good teaching. She had the notes, an idea of what she wanted to achieve and how she wanted to achieve it for each lesson, "I need to know what the students need to do as homework. I also need to analyse the effectiveness of the lesson before I prepare the next one". She believed that students could be affected dramatically by teachers. As a result, she embraced the view that she had the power of changing students, to change their attitude to learning if it was negative, but not their fundamental personalities. At the same time she realised that she could have the reverse effect, the students could hate mathematics because of her. Additionally she had the opinion that it was an enormous responsibility that teachers can affect people's lives forever, hopefully in a positive sense.

Ms Skid perceived her main role in teaching as a learning facilitator and sometimes as the resource person of knowledge. "The students need to discover the concepts", she viewed herself as working as a team with her students: she directed the activities and they discovered the mathematics. Students should not just know the concepts they need to think them through to make sense. If she noticed students were regularly not doing their homework she would ask them why this was so. She tried to work with them because her aim was to prepare students for university where they would not have a teacher telling them what to do all the time. Ms Skid believed that students would have to be disciplined—self disciplined. She thought that one of the biggest problems for Australian students was to find that self-discipline, when they were in the university. Further, she believed that in different situations, a teacher might have a different role.

Ms Skid stressed the importance of success in learning to ensure students' enjoyment in studying the subject. She admitted, "I have to know that the students can work toward their final results". "I need to ensure that there must be success and the success must be according to the person's ability". Ms Skid believed that her main goal in teaching mathematics was that students graduated from school with skills developed to their best ability and with a positive attitude to their experiences. Ms Skid strongly believed that meaningful learning occurred only when students found enjoyment in their study. She embraced the view that students need to experience different approaches to learning. "I need to know that I am using the sort of learning that they can relate to". Ms Skid admitted that this activity might not be used with other classes, different classes have different styles of learning, and she responded according to the type of students she had. In promoting students' learning, Ms Skid believed in the freedom of the students to express their opinions and understanding in class. She liked them to focus on what they were doing not only on what they wanted to talk about. Ms Skid embraced the view that her responsibility to her students was to provide an atmosphere which was safe for them to learn, to ensure that the curriculum was covered and to ensure that she could find ways to extend their learning. The students did have to feel safe, free of sarcasm and free of criticism in a negative sense. They should get support and praise and learn. She believed in building on knowledge, and that was her responsibility.

The General Pattern of Mr Ton's Lessons

Certain patterns became visible during the observations. Mr Ton started his lessons by reviewing the materials taught in the previous lesson and raised several review questions. He then moved on to discuss the assigned homework, explained the difficult problems or asked one or two students to share their work with the rest of the class by working the problems on the board if the time permitted. Following this, he introduced the topic he intended for that day. He usually showed, read and explained the definitions of the concepts and provided some examples. Then he showed some example problems and tried to involve his students in working them out. He finished the lesson by giving the day's homework. It was usual for each lesson segment to flow smoothly to the next. The whole class approach was utilised in Mr Ton's class.

The General Pattern of Ms Skid's Lessons

Ms Skid started the lesson in a variety of ways. When introducing new concepts she always started from her students' point of view, and if necessary added her own illustrations of the concepts. She utilised many different techniques in generating the students' understanding of the concepts, always involved the students either individually or as a group in a whole class interaction. She was directing the students in their learning activities and at the same time the students were hoping to discover the concepts and build them into their own knowledge. She encouraged self-discovery by students and strengthened their understanding by giving them problems to work on, on handouts or in the textbook used. Meanwhile she raised questions to remind the students of any new terms they had met. While walking around the class visiting and talking to individuals or groups of students she assessed their understanding. When some students finished working on the problem she would ask them individually or in a group to share their work on the boards. Then she invited the participation of the rest of the class by asking them to make judgement of it. Ms Skid might even ask individual students to give their opinions of the work on the board.

DISCUSSION AND CONCLUSION

Mr Ton's dominant existing view of mathematics was Platonist and Instrumentalist. He had started to develop a Problem Solving view of the subject, but its implementation depended much on the opportunities and constraints of the social context of teaching in his setting. As a result, his actions reflected his dominant role as a model, which tended to maintain the one way of communication, in contrast to his perception of his role as a learning facilitator. This showed his adoption of Dewantara's teaching philosophy with more emphasis on the role model of a teacher. Consequently, Mr Ton's instructional practices tended to be autocratic. Ms Skid's view of mathematics, which was a combination of Platonist, Instrumentalist and Problem Solving, was supported by the class size, the facilities and resources available in her school, and the curriculum and syllabus utilised. Her practices tended to be more democratic. She managed to implement her role of a learning facilitator's view into action and viewed herself as standing on an equal base with her students.

In spite of his view of active learning, Mr Ton conducted a teaching that required submissive and compliant behaviour of his students in learning. Due to his embraced view that the class should be quiet and in order for learning to occur, Mr Ton tended to perceive active learning to be mainly 'active in mind', not necessary 'verbally active'. This perception was developed due to the normal large class size and other contextual factors of teaching in Indonesia. The course of learning of the students in Ms Skid's class reflected her view of active learning. Her students challenged her by raising questions and sometimes interrupted her explanations.

The findings of this study support the strong influence of a teacher's beliefs on his or her instructional practices, lend strength to the previous research which found that actions were not always consistent with stated beliefs. These also in line with research that revealed some degree of cultural differences in the perception of teachers' and students' roles, of the nature of teaching and learning, and of the nature of mathematics itself (Stigler & Perry, 1988). One of the causes of the inconsistencies was that personal perceptions were based on values and norms accepted by the person.

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Endnote

[1] The study is part of a larger case study of senior mathematics teachers in Australia and Indonesia teaching the topic of Networks or Graph Theory.

GRAPHICS CALCULATORS IN VICTORIAN SECONDARY SCHOOLS: TEACHER PERCEPTIONS OF USE

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Since Victoria permitted the use of graphics calculators in final external examinations, their use has become quite widespread. A survey of secondary schools, undertaken to gauge the response of teachers to these tools, provides information on how teachers view graphics calculator use in secondary mathematics courses.

INTRODUCTION

As a follow-up to the decision in Victoria (Board of Studies, 1995) to remove the ban on graphics calculator use in Victorian secondary schools from 1997 in all external examinations, a statewide survey of schools was conducted in late 1997 to assess the impact of the decision on schools. The survey had two aims - the first to determine from school mathematics coordinators the level of ownership or access of calculators by students and the second to determine teacher attitudes and use of the calculators in various subjects or topics (Routitsky and Tobin, 1998; Routitsky, Tobin and Stephens, 1998). Information on effective use of graphics calculators in mathematics classes at all levels has generally been based on small group or case work previously and this was an opportunity to examine a larger group. A review of previous research on all aspects of teaching and learning with graphics calculators is provided by (Penglase and Arnold, 1996).

THE SURVEY

The survey was initially sent in September 1997 with two follow ups to non-responding schools, some by mail and others by telephone, in March 1998. The information discussed in this paper arose from the full survey data set. Some preliminary results on the earlier part of the survey have been presented elsewhere (Tobin, Routitsky and Jones, 1998).

The main target group of the survey was teachers of mathematics subjects in the Victorian Certificate of Education (VCE). This spans years 11 and 12 and includes the five subjects, Mathematical Methods 1 & 2, General Mathematics Units 1 & 2, Mathematical Methods Units 3 & 4, Further Mathematics Units 3 & 4 and Specialist Mathematics Units 3 & 4. It is only the unit 3 & 4 subjects which have external examinations affected by the new policy - these are ninety minute papers involving multiple choice, short answer and analysis tasks called Common Assessment Tasks or CATs.

The survey was sent to approximately 480 educational providers screened on the basis of whether they taught Mathematical Methods. The number of responses from school coordinators was high (about 73%). The brief Coordinator Questionnaire included information on school size (indirectly) calculator models, booklisting policies, class sets, student access and estimations of student calculator ownership. The focus of this questionnaire was depth of penetration of the graphics calculators in schools.

The more detailed teacher surveys were completed by over 1000 teachers, although not all teachers completed all sections. The focus of the teacher survey was on attitudes to the Board's calculator policy and attitudes to the graphics calculators themselves. This Teacher Survey enabled us to assess any variations in responses between school regions, types or sectors as information on these was gathered also. In Victoria there are seven educational regions classified by the Ministry: Eastern Metropolitan, Western Metropolitan, Barwon South Western, Central Highland, Gippsland, Goulburn-North Eastern and Loddon-Campaspe-Mallee.

The four sectors were Government Secondary, Catholic, Independent and TAFE. The three types of educational providers classified were boys, girls and co-educational. Most respondents in the first group were schools - only two TAFE colleges out of a possible ten TAFE VCE providers gave any feedback.

RESULTS

The methodology of data collection and results of the survey relevant to market penetration of the graphics calculators and teacher perceptions of equity issues have been discussed in a previous paper (Routitsky and Tobin, 1998). In that paper, regional, sector and school type variations were examined. The paper raised the issue of teacher support for the decision to use graphics calculators and found that there was broad agreement for the policy across all sectors, regions and school types. This support level ranged from 64% to 70%, depending on the VCE subject, and this occurred, despite there being a similarly common perception (about 73% of respondents) that the use of calculators raised serious equity issues in terms of student access.

Taking up this theme, Routitsky, Tobin and Stephens (1998) analysed further the data on teachers who disagreed with the Board's policy. The purpose of this investigation was to determine if this were linked to their personal level of access to graphics calculators, or whether it related to the level of access which their students had, either through ownership or school access. The results of that analysis demonstrated, perhaps unsurprisingly, that teachers who disagreed with the Board policy tended to come from schools where they and/or their students had limited access to a graphics calculator. This is consistent with a previous study on teacher attitudes to use of graphics calculators in a college algebra course in the USA, where it was found that the only significant variable on level of teacher support was degree of familiarity of the user (Chamblee, 1995). Reduced familiarity is an immediate consequence of limited access.

The issue of teacher perceptions of the usefulness of the calculator in teaching mathematics remains to be considered and in particular its relationship to calculator access.

Because information was gathered on both teacher and student access to graphics calculators, for the purpose of this study it was found useful to create a single three level access variable which classifies access as poor, basic or good. This required recoding of the data as shown in the appendix and was justified on the basis of the strong correlation between the teacher access and their students access (see Table 1).

Table 1
Cross Tabulation of Student and Teacher Access to Graphics Calculators

	Students can use graphics calculators in class			
Teacher can use graphics calculators in class	Never / rarely	Sometimes	Often / always	Total
Never / rarely	47	8	0	55
Sometimes	15	42	8	65
Often / always	139	75	591	805
Total	201	125	599	925

The data reported in the following tables looks at teachers responses to several statements related to the graphics calculator and teaching mathematics.

The Graphics Calculator is Useful in your Teaching

The first aspect examined was on usefulness of graphics calculators in teaching. The exact statement is given following and teachers rated a level of agreement with the statement. Results are shown in Table 2.

Table 2
Perception on Whether the Graphics Calculator is Useful in the Classroom

	Graphics Calculator Access in the Classroom % (count)			
	Poor	Basic	Good	Total
Never/rarely	22.49 (74)	10.86 (24)	6.79 (25)	13.40 (123)
Sometimes	38.91 (128)	35.75 (79)	26.90 (99)	33.33 (306)
Often/always	38.60 (127)	53.39 (118)	66.30 (244)	53.27 (489)
Total	100 (329)	100 (221)	100 (368)	100 (918)

The table shows a statistically significant relationship between perception of usefulness and access; the greater the access level, the greater the perception of usefulness.

The next statement examined the time factors associated with calculator use and the statement is given following. Results are given in Table 3.

Teaching with graphics calculators reduces the time needed for explanation

Again a statistically significant relationship was found between perception of time needed for explanation and access with time saving being seen as more common as the access level increased.

Table 3
Perception on Whether the Calculator Reduces the Time Needed for Explanation

	Graphics Calculator Access in the Classroom % (count)			
	Poor	Basic	Good	Total
Never/rarely	57.49 (188)	49.54 (108)	44.81 (164)	50.49 (460)
Sometimes	24.77 (81)	27.52 (60)	27.60 (101)	26.56 (242)
Often/always	17.73 (58)	22.94 (50)	27.60 (101)	22.94 (209)
Total	100 (327)	100 (218)	100 (366)	100 (911)

Teaching with graphics calculators makes learning mathematics easier for students

In the next statement we look at whether the calculators ease the learning process. A previously the actual statement given to teachers is supplied here. This is based on teacher perception not student perception and results are in Table 4.

Table 4
Perception on Whether the Calculator Makes Learning Easier

	Graphics Calculator Access in the Classroom % (count)			
	Poor	Basic	Good	Total
Never/rarely	34.76 (114)	22.37 (49)	16.89 (62)	24.62 (225)
Sometimes	35.98 (118)	36.99 (81)	34.60 (127)	35.67 (326)
Often/always	29.27 (96)	40.64 (89)	48.50 (178)	39.72 (363)
Total	100 (328)	100 (219)	100 (367)	100 (914)

The table shows a statistically significant relationship between perception of easing learning and access. For teachers with good or basic access, the belief that graphics calculators makes learning easier is much stronger. For those with poor access opinion is more divided.

Graphics Calculators Improve Students' Understanding of Mathematics

One aspect of interest is the quality of the learning. Teachers were asked to rate the graphics calculator's ability to improve students understanding. The actual statement is supplied.

Results are shown in Table 5. There is a statistically significant relationship between a belief that the graphics calculators aid understanding and access. The results show that teachers are very divided on the issue of whether graphics calculators improve understanding. This division occurs particularly in the lower access groups with the poor access group having the most negative perception. The 'good' access group shows a strong degree of support for calculators aiding in understanding.

Table 5
Perception on Whether the Calculator Improves Understanding

	Graphics Calculator Access in the Classroom % (count)			
	Poor	Basic	Good	Total
Never/rarely	39.51 (130)	26.82 (59)	19.89 (73)	28.60 (262)
Sometimes	32.22 (106)	39.09 (86)	37.60 (138)	36.03 (330)
Often/always	28.27 (93)	34.09 (75)	42.51 (156)	35.37 (324)
Total	100 (329)	100 (220)	100 (367)	100 (916)

Usefulness of Graphics Calculators in Your Lessons Depends on Your Teaching Style

The final aspect of teaching and learning was a personal one - the interaction between teaching style and the use of calculators. This was not a result which appeared to be dependent on access. Results are in Table 6.

Table 6
Perception of Dependence of Use on Style

	% (count)
Never/rarely	15.97 (165)
Sometimes	29.82 (308)
Often/always	54.21 (560)
Total	100 (1033)

Over all groups, the teachers showed a similar belief in the usefulness of calculators depending on teacher style. The statement implies that usefulness is not intrinsic and suggests a recognition of the possibility that more could be gained from the calculators by some teaching approaches.

CONCLUSION

The results of the analysis shows that teacher perceptions of usefulness of the graphics calculators generally depends on the level of access to the calculators which occurs in the classrooms. The 'good' group represent the presumed optimal classroom situation with everyday student/teacher access to a graphics calculators and, in this 'expert' group, the perception that the graphics calculators improves understanding and makes the work easier is quite striking.

However, the degree to which access affects perceptions varies with the aspect of teaching under discussion however and even the regular users (the good access group) found that the calculators do not seem to save time in explanations. This may be partly due to the grafting of the graphics calculator use onto an existing syllabus instead of creating a syllabus which anticipates their use. For example, it is easy to imagine that using a calculator to perform log/antilog calculations might still have been time consuming although it was unnecessary.

Finally, theresults reflect views of student learning filtered through teachers perceptions as the survey only went to teachers themselves and teachers may have different views from the students. For comparison, student views on learning wereanalysed directly in a small survey of fifty students selected at random in a UK undergraduate course (Zand and Crowe, 1997) conducted by distance education. In this survey, 45% of the group claimed graphics calculators contributed to understanding to a 'large extent' and 27% to a 'reasonable extent'. These results compare well with the views of the teachers in our own 'good' access group on the issue of understanding where 42.5% said this always or often occurred with their students.

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APPENDIX

Data Recoding

in cases where both students and teachers can always use graphics calculators in the classroom the access level is defined as **good**. In the survey 370 of 925 cases (40%) were in this group. Where students often have access to classroom use and teachers often or always do too or students always have access and teachers often or sometimes do the access level is rated as **basic**. This group included 222 respondents (24%). All other cases are considered **poor** in this discussion.

In some survey forms some lack of completeness means that there are minor fluctuations in the actual case numbers. In addition a large number of respondents did not complete most or any of this section.

CABRI GEOMETRY: A CATALYST FOR GROWTH IN GEOMETRIC UNDERSTANDING

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In a recent study (Vincent, 1998), van Hiele levels were used to monitor students' progress in geometric understanding when learning with Cabri geometry. The following report focuses on the experiences of two of the twelve participants in the study: Student D who was initially at van Hiele Level 0/1 and Student L who was at Level 2 for most concepts.

INTRODUCTION

Cabri geometry was developed in 1986 by Jean-Marie Laborde, Yves Baulac and Franck Bellemain, at the Université Joseph Fourier in Grenoble, France. Cabri primitives may be classified as pure drawing primitives, where the user can draw a point, line or circle as in any computer drawing tool, or geometric primitives such as perpendicular bisectors or midpoints.

Visual Versus Geometrical Strategies with Cabri

While it might be expected that dynamic geometry software such as Cabri would enhance students' learning of geometry, research has shown that students experience considerable difficulty constructing particular drag-resistant shapes such as rectangles or equilateral triangles since they base their constructions on visual appearance. They expect to be able to apply the same by-eye methods as with pencil and paper.

Laborde (1993) suggests that the van Hiele levels can be recognised in students' use of the drag mode:

> One can recognize the van Hiele levels in the use of the variations of the drawing. At a low level the figure is viewed as an entity but not analysed into parts or elements: all parts of the drawing must move together under the drag mode. At an intermediate level the figure is viewed as a shape which can be distinguished from other shapes, the drawings are instances of the shape but not yet analysed. At a higher level the figure is made of elements linked by relations which remain invariant when dragging the drawing. (p. 66)

In her experimental observations, Laborde (1995) notes the occurrence of three strategies: purely visual strategies, combinatorial use of geometrical primitives without definite intention and geometrical strategies aimed at a definite result. Students did not necessarily follow these pathways hierarchically - they would sometimes revert to "messing-up" constructions even after successfully completing quite difficult drag-resistant constructions. When faced with a new problem, students were observed to revert initially to by-eye strategies, even though they had previously found these visual strategies to be unsuccessful. Healy, Hoelzl, Hoyles and Noss (1994) made similar observations when working with 14-year old pupils in a London comprehensive school. Despite the pupils' previous experience of "messing-up" they usually started with a by-eye drawing and "were not always convinced that a shape which looked correct might still be wrong" (p. 16). Noss and Hoyles noted that the pupils' by-eye approach provided them with a pathway which showed them that their initial solution was inadequate as well as leading them to a resolution. Cabri "provided a route which could be progressively abandoned as an alternative strategy presented itself" (p. 128).

THE STUDY

The study investigated students' progress in geometric understanding when using the dynamic geometry software, Cabri geometry™, MS-DOS version 1.7. The participants in the study were twelve 11-12 year old girls in a private girls' school in Melbourne. The girls had their own notebook computers, but had not used Cabri prior to this study.

Methodology

The study was essentially a case study: both quantitative, involving measurement of changes in van Hiele levels associated with the use of Cabri, and qualitative, documenting and analysing students' geometric language and methods of construction of geometric figures with Cabri. Data collection, in the form of students' test responses, saved Cabri files and taped conversations, took place over a period of four weeks. The methodology involved the following stages:

1. PowerPoint presentation of triangles and quadrilaterals by each student.
2. Pre-tests:
   Geometric Terms test
   Shapes Recognition test
   CDASSG Project multiple choice van Hiele test
   Mayberry/Lawrie van Hiele test

At this stage, on the basis of the van Hiele pre-tests, the twelve students were divided into two groups: Group I comprising the students who were at Level 0 or 1 on most concepts and Group II comprising those who were at Level 2 or 3 on most concepts.

3. Cabri worksheet lessons (each lesson was of 45 minutes duration):
   Lesson 1: Cabri introduction where students played with the software.
   Lesson 2: Exploring the difference between the various Cabri points.
   Lessons 3-8: Learning with Cabri using structured worksheets.
4. Cabri constructions for Group II students.
5. Shapes Recognition post-test
6. Van Hiele post-test (Mayberry/Lawrie van Hiele test).
7. Further Cabri constructions for selected students (letter A and House shapes)

The Cabri Lessons

During lessons 3-8 the students recorded information from their Cabri activities and answered questions on their worksheets. It was anticipated that these activities would develop Level 2 thinking by encouraging the students to think about properties of the figures, but no explicit reference was made to relationships between properties.

The Cabri Constructions

Students were required to produce drag-resistant geometric figures: parallelogram, rectangle, right-angled triangle and isosceles triangle. The *History* option of Cabri enabled the students' construction steps to be retraced in the saved files and assisted in analysing the levels of thinking displayed by the students in their constructions.

RESULTS AND DISCUSSION

PowerPoint Presentation

The purpose of this activity was to ascertain the students' prior understanding of triangles and quadrilaterals using the drawing tool shapes which could be freely rotated. In her slide for Squares Student D placed her three squares in standard positions while Student L rotated two of her squares and classified squares as rectangles. In her Rectangles slide Student D described rectangles as having "two equal sides which are parallel", indicating that she was unable to correctly verbalise her understanding.

Geometric Terms Test

The Geometric Terms test involved matching a geometric term with a diagram. The correct (✓) and incorrect (✗) responses of Students D and L are shown in Table 1.

Table 1.
Responses in the Geometric terms test

Student	Intersection	Parallel	Scalene triangle	Rectangle	Perpendicular	Isosceles triangle	Midpoint	Perpendicular bisector	Quadrilateral	Parallel-ogram
D	✓	✓	✗	✓	✗	✓	✓	✗	✓	✓
L	✓	✓	✓	✓	✗	✓	✓	✗	✓	✓

Neither student, in common with most of the other students in the study, knew the meaning of *perpendicular* or *perpendicular bisector*. Student D confused isosceles and scalene triangles, although there were inconsistencies in her selection of isosceles triangles in the Shapes Recognition test, as shown in Table 2.

Shapes Recognition Test

The Shapes Recognition test involved selecting rectangles and parallelograms from a page of quadrilaterals and selecting isosceles triangles from a page of triangles.

Table 2
Numbers of Shapes Correctly Identified in the Shapes Recognition Test

	Isosceles triangles	Rectangles	Parallelograms
Possible number of correct responses	10	6 (including 3 squares)	10 (including 2 rhombuses and 6 rectangles)
Student D	5 + 1 incorrect	3 rectangles (no squares)	2 (included 1 rotated rectangle + 3 incorrect trapezia)
Student L	8	6	4 (included rhombuses but not rectangles)

Student D made several incorrect selections and did not recognise class inclusion. By contrast, Student L classified rhombuses as parallelograms and squares as rectangles, although she did not include rectangles as parallelograms.

CDASSG van Hiele Test

This test consisted of the first 15 items, representing van Hiele Levels 1-3, of the 25 item test developed in the CDASSG Project (Usiskin, 1982). Table 3 shows the numbers of correct responses and the assigned van Hiele levels. Student D was classified as Level 1 using the CDASSG 3 of 5 criterion but as Level 0 using the CDASSG 4 of 5 criterion. Student L could not be assigned a level as she scored 5/5, 2/5 and 4/5 respectively for the Level 1, 2 and 3 questions, perhaps indicating that she was in transition from Level 2 to Level 3.

Table 3
Assigning van Hiele Levels from the CDASSG Test (Items 1-15)

Student	Number of correct responses			Level	
	Questions 1-5	Questions 6-10	Questions 11-15	3 of 5 criterion	4 of 5 criterion
D	3	0	0	1	0
L	5	2	4	NC	NC

NC Not classified

Mayberry/Lawrie van Hiele Test

This written test used the 25 items concerning the concepts squares, right-angled triangles, parallel lines and isosceles triangles from the 58-item Mayberry/Lawrie test (Lawrie, 1993). Table 4 shows the levels assigned by the test for the four concepts for Students D and L. Student D, who performed poorly for isosceles triangles on the Shapes Recognition test and Geometric Terms test, was at Level 0 for isosceles triangles. These observations suggest that she was probably in transition from Level 0 to Level 1. Although Student L could not be assigned a level on the CDASSG test, she was clearly at Level 2 for three of the four concepts on the Mayberry/Lawrie test.

Table 4
Assigning van Hiele Levels from the Mayberry/Lawrie Test

Student	Squares	Right-angled triangles	Isosceles triangles	Parallel lines
D	1	1	0	1
L	2	1	2	2

Following completion of the Cabri worksheet lessons, the students were re-tested with the Mayberry/Lawrie van Hiele test and the Shapes Recognition test. Table 5 compares the pre-test and post-test responses for the Shapes Recognition test. Student D showed very little change except that she made fewer incorrect selections on the post-test. Student L, on the other hand, now clearly identified rectangles as parallelograms, although she incorrectly classified two parallelograms as rectangles.

Table 5
Comparison of Responses in the Shapes Recognition Pre-test and Post-test

Student	Isosceles triangles (10)		Rectangles (6)		Parallelograms (10)	
	Pre-test	Post-test	Pre-test	Post-test	Pre-test	Post-test
D	5 + 1 incorrect	7	3	3	2 (included 1 rotated rectangle + 3 incorrect trapezia)	2
L	8	9	6	6 + 2 incorrect parallelograms	4 (included rhombuses but not rectangles)	9 (missed 1 rhombus)

Table 6 compares the pre-test and post-test van Hiele levels assigned from the Lawrie/ Mayberry van Hiele test. Both students progressed in three of the four concepts, with Student L increasing from Level 1 to Level 3 for right-angled triangles. Shaded cells indicate where an increase in van Hiele level occurred.

Table 6
Pre-test and Post-test van Hiele Levels for Four Concepts.

Student	Squares		Right-angled triangles		Isosceles triangles		Parallel lines	
	Pre-test	Post-test	Pre-test	Post-test	Pre-test	Post-test	Pre-test	Post-test
D	1	2	1	2	0	1	1	1
L	2	3	1	3	2	2	2	3

Based on the van Hiele levels assigned by the tests, Student D was placed in Group I and Student L in Group II.

Cabri Constructions

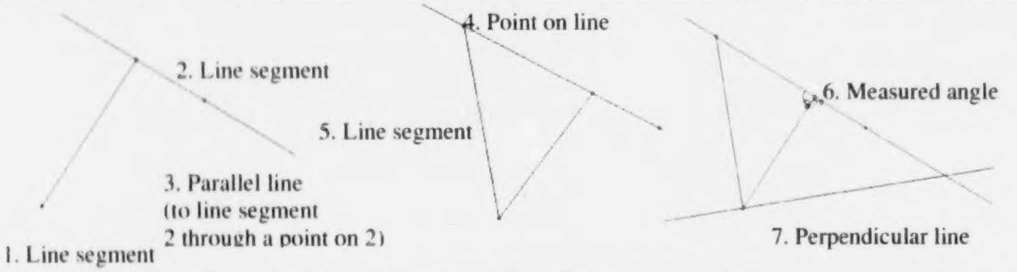
Student D

Since Group I students were still consolidating their Level 2 thinking with relationships between properties not yet being recognised, the construction tasks were generally not given to these students. However, Student D, when asked to draw an isosceles triangle, used the triangle tool, dragging it until it appeared to be isosceles. When she realised that it did not remain isosceles when dragged, she had no idea how to proceed and became distressed: she could not see beyond a by-eye construction. Student D differed in this respect from the Group II students, who were not worried by the "messing up" of their by-eye constructions, but used it to help them with a geometric construction.

Student L: Right-angled Triangle

Student L at first used a by-eye method (Figure 1), starting with two *line segments* (1 and 2). She confused *parallel* and *perpendicular* lines and, instead of constructing a line *perpendicular* to line segment 2, she constructed a *parallel line* (3) to this line segment. She then constructed a *point on line* (4) and a *line segment* (5) to form a triangle. After measuring the angle (6), Student L abandoned this angle as her right angle and constructed a *perpendicular line* (7) but then decided not to complete this triangle.

Figure 1
Student L: Right-angled Triangle 1


2. Line segment
3. Parallel line (to line segment 2 through a point on 2)
1. Line segment
4. Point on line
5. Line segment
6. Measured angle
7. Perpendicular line

In her second attempt (Figure 2), Student L again initially confused parallel and perpendicular and, although she did construct a right-angled triangle, it could not be rotated since the triangle depended on a basic line which cannot be rotated once placed. Student L's third attempt (Figure 3) satisfied the requirements of the task.

Figure 2
Student L: Triangle 2

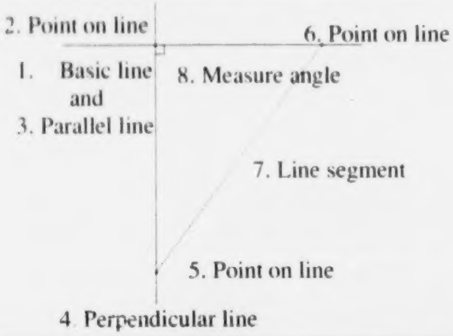
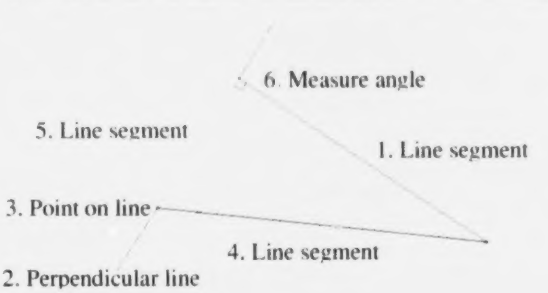

2. Point on line
6. Point on line
1. Basic line and
3. Parallel line
8. Measure angle
7. Line segment
5. Point on line
4. Perpendicular line

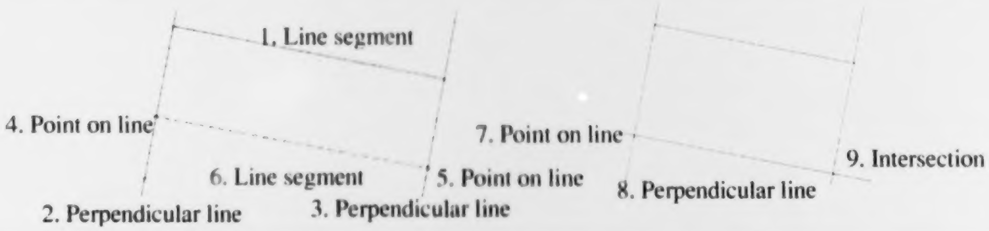
Figure 3
Student L: Triangle 3


6. Measure angle
5. Line segment
1. Line segment
3. Point on line
4. Line segment
2. Perpendicular line

Student L: Rectangle

Student L completed her rectangle (Figure 4) with little difficulty, having already discovered the problems of by-eye methods with her right-angled triangle and now understanding the meaning of *perpendicular*. She quickly realised that *line segment* (6), which was based on by-eye placing of *points* (4) and (5), would not stay parallel to *line segment* (1), so she deleted *points* (4) and (5), constructing another *point* (7), a *perpendicular line* (8) and an *intersection* (9).

Figure 4
Student L: Rectangle


1. Line segment
4. Point on line
6. Line segment
5. Point on line
2. Perpendicular line
3. Perpendicular line
7. Point on line
9. Intersection
8. Perpendicular line

Student L: Letter *A*

Despite her previous experiences, Student L's first attempt at constructing the letter *A* was a by-eye placing of three *lines by 2 points* (1, 2 and 3) - see Figure 5. In her second attempt (Figure 6), she constructed a pair of perpendicular lines using *line by 2 points (1)* and *perpendicular line (2)* to form the "legs" of her Letter *A* but then placed the cross-line by eye: *points on lines* (3 and 4) were connected with a *line by 2 points* (5). Student L then asked "This is not one of those impossible puzzles is it? "

Figure 5
Student L: Letter A (1)

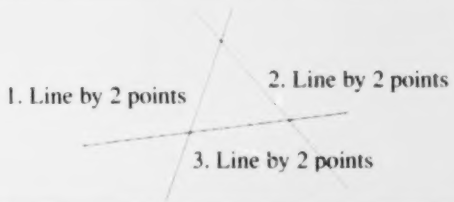
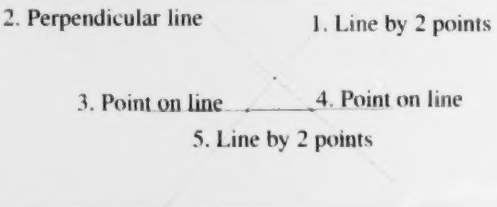

1. Line by 2 points
2. Line by 2 points
3. Line by 2 points

Figure 6
Student L: Letter A: (2)


2. Perpendicular line
1. Line by 2 points
3. Point on line
4. Point on line
5. Line by 2 points

After being assured that the construction was possible, Student L abandoned the idea of perpendicular lines and tried using a *triangle* (Figure 7), dragging it until it appeared to be an equilateral triangle. This triangle became the framework for constructing three *lines by 2 points* to make the Letter *A* shape. Finding that this distorted when dragged, Student L placed a *point* (2) visually above the centre of a *line by 2 points* (1) (Figure 8) but seemed unsure how to continue. She had realised that visual constructions would not work but her progress was perhaps hindered by her lack of understanding of *perpendicular bisector*. Although she had explored the Cabri menu options, unlike other Group II students, she made no attempt to use this construction tool.

Figure 7
Student L: Letter A (3)

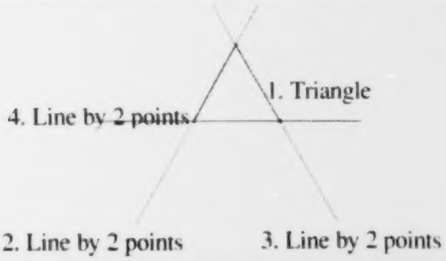
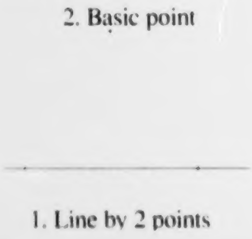
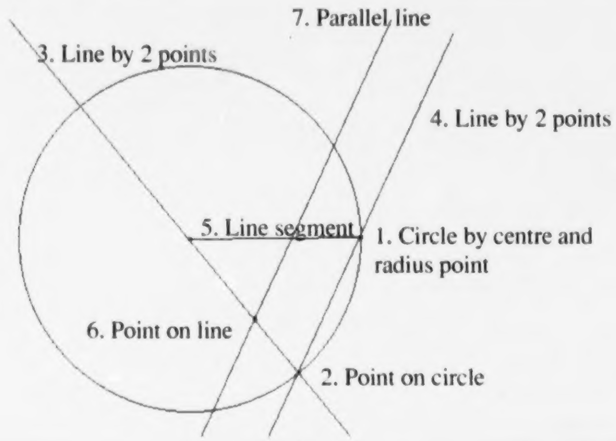

1. Triangle
4. Line by 2 points
2. Line by 2 points
3. Line by 2 points

Figure 8
Student L: Letter A (4)


2. Basic point
1. Line by 2 points

Having apparently been trying to utilise the symmetry property of the letter *A* in her first four attempts, Student L instead focused on the equal sides of an isosceles triangle in her next attempt, using two radii of a circle to construct two equal sides of an isosceles triangle (Figure 9). By constructing a line (7) parallel to her secant line (4) she was able to successfully complete the letter *A*.

Figure 9
Student L: Letter A (5)



CONCLUSIONS

Student D, who initially had very limited understanding of geometric figures or properties, was now able to correctly identify the relevant geometric shapes and was aware of at least one property of each. Unlike Student L, however, she was not yet aware of relationships between properties and was not ready for using Cabri to construct drag-resistant geometric figures. After her initial by-eye drawings, Student L realised that appropriate geometric constructions were required and was able to analyse a geometric figure and make links between its properties and the properties of another figure. Healy, Hoelzl, Hoyles and Noss (1994) suggest that "finding a 'solution' by-eye so that they know where they are going, and thereby have something on the screen on which to reflect, seems to be crucial scaffolding for many children" (p. 16).

Student L's ultimate successful construction of the letter *A* involved the linking of several properties: radii of a circle are the same length, the equal sides of an isosceles triangle can be constructed using the radii of a circle and the required "horizontal line" of the Letter *A* can be achieved by constructing a line parallel to the base of the triangle. In her search for a method by which Cabri would allow her to construct equal sides for her isosceles triangle, Student L used a property independent of the isosceles triangle, thereby linking properties of different geometric figures. Student L's constructions support observations by Laborde (1993), who noted that the use of visualisation by students, both in exploratory tasks and in validating constructions, is related to their conceptual understanding of the figure they have constructed.

Students L's preliminary visual attempts seemed to focus her thinking on which aspects of the letter *A* must remain invariant and, because she was unfamiliar with the term *perpendicular bisector*, she did not recognise the role this Cabri tool could play in the construction. Van Hiele-Geldof in Fuys, Geddes and Tischler (1984) notes that "concept and language can be distinguished, but cannot be separated. The thinking operation itself first has to be made conscious through language symbols and the language symbols are a consequence of the thinking operation" (p. 232). Observations of another pair of students in the study, who worked together on the letter *A* and House constructions, suggest that Cabri has greater potential to enhance the understanding and use of appropriate geometric language when students are actively discussing their constructions.

The study suggests that Cabri can result in significant progress in understanding of geometric properties and relationships even after relatively few lessons. Further studies are essential to explore the potential of dynamic geometry software to develop higher levels of geometric understanding.

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THE RELATIONSHIP BETWEEN THE PURPORTED USE OF ASSESSMENT TECHNIQUES AND BELIEFS ABOUT THE USES OF ASSESSMENT

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The purpose of this study was to explore teacher's beliefs about the uses of assessment and their actual assessment practices. The sample consisted of 387 primary teachers. The results indicate that these teachers are using a wide variety of assessment methods, but assessment is predominately used to provide feedback for the teacher rather than the learner and parents. The types of assessment methods chosen appear to support these beliefs about the uses of assessment.

INTRODUCTION

The growing awareness of the links between assessment and learning has resulted in several critiques of existing practice and proposals for reforms at the international and national level. In the USA the almost exclusive use of multiple-choice pen-and-paper tests has been challenged (Murname & Raizen, 1988; Marzano, Pickering & McTighe, 1994). The need for more varied types of assessment procedures has been called for. In the UK, national assessment involved the use of broadly based tests employing a wide range of modes of presentation and responses (DES, 1989), and in Australia there has been a call for developing and using additional modes of assessment besides the traditional pen-and-paper tests as it is believed that such testing cannot address all areas of the curriculum (*A National Statement on Mathematics in Australian Schools*, 1991). Changes in mathematical instruction seems to be influenced by teachers' beliefs about mathematics (Merseth, 1993). Missing from the literature is information regarding the relationship between teachers' beliefs about assessment and assessment practices.

Types of Assessment

The debate of when, how and what to assess is still prevalent throughout educational literature. For example, NCTM Assessment Standards (1995) states that student assessment should be an integral part of instruction and multiple methods need to be employed. This is supported by Webb (1993) who believes that any single form of assessment fails to describe fully a student's knowledge about mathematics. Limiting assessment to pen-and-paper examinations results in assessment that is distorted as it fails to sample tasks where there are multiple answers, where there are many problem solving strategies, or where the skills cannot be demonstrated easily in pen-and-paper format (Haines & Izard, 1994). The implications for the classroom are that students need to write and talk about mathematics. The teacher must not only become proficient at utilising a variety of assessment practices such as checklists, but also regularly use these methods in order to gauge students' progress and improve classroom instruction (Grouws & Sherry, 1992).

A National Statement on Mathematics for Australian Schools (1991) identified a range of assessment practices, which can be used to gather information. They include: teacher observation and questioning; structured interviews with students; paper-and-pencil tests; oral tests; practical skills tests; work- or project-based assessment; collected samples of students' independent work; individual homework assignments; group reports; anecdotal records; self assessment and peer assessment. The assessment initiatives in the National Statement have been supported by a variety of publications such as the Board of Studies (1996) and the Department of Education, Queensland (1994). These publications illustrate how to incorporate different approaches to assessment in classroom teaching at all levels. Thus assessment should be varied, provide starting points for further learning and be reported in a clear and precise manner.

In spite of the wealth of literature delineating differing assessment techniques and the range of professional development opportunities provided for teachers, the literature reports many instances where the assessment procedures utilised tend to be narrow. For example, Haines and Izard (1994) claim that there is a disturbing tendency to limit assessment to what is easy to assess rather than to what is important to assess. If one accepts the claim by Merseth (1993) about the influences of beliefs on instruction, then it is more than likely that the wide range of assessment practices existing in schools is related to a similar range of beliefs about assessment and the uses of assessment.

Beliefs About the Uses of Assessment

Assessment appears to consist of three crucial components, one relating to the provision of valid evidence of learning achievement and another relating to the need for accountability. Assessment also provides feedback to teachers. These factors appear to influence the type of assessment implemented in the classroom. It is generally believed that teachers view test results as an indication of the effectiveness of their teaching (Grouws & Sherry, 1992). Thus assessment, teaching and learning are closely linked, with assessment being the more dominant component and in some instances acting as a catalyst for change. Clarke, Clarke & Lovitt (1990) claim that the major uses of assessment focus on three areas, the teacher, the student, and the parent. First, the teacher uses assessment to improve instruction by using students' responses to help identify (i) instructional strategies that are most successful; and, (ii) student learning behaviours that need to be "encouraged and developed or discouraged and replaced". Second, assessment informs students of their identified strengths and weaknesses and informs subsequent teachers of students' competencies. Finally, parents are informed of their child's behaviour so that they can give more effective support. The NCTM Assessment Standards (1995) delineates four purposes for assessment. These are promoting student growth, improving instruction, recognising accomplishments, and modifying programs. These purposes are teacher-learner focussed. The role of parents in the learning paradigm is not acknowledged in this document, yet it is believed that positive parent-teacher communication is a vital key in determining learning outcomes of children at school (Arthur, 1996; Greenberg, 1998). Philipou and Christou (1996) report that although teachers are aware of and accept contemporary ideas about assessment there is a gap between their knowledge and intentions on the one hand, and their actual practice. This probably indicates that assessment practices depend not only on knowledge and intentions, but also deeply held beliefs about assessment. This research reports on the relationship between primary school teachers' beliefs about the uses of assessment and current assessment practices.

METHODOLOGY

In part of a survey, primary teachers were asked to indicate their uses of assessment, the frequency with which they used various assessment techniques and to indicate whether or not the introduction of the Year 2 Net (a diagnostic instrument in early primary mathematics) had influenced their teaching and assessment of mathematics. The results of the Year 2 Net component are reported in Nisbet and Warren, 1999.

The questions pertaining to the uses of assessment focussed on three main areas, namely, the teacher, the pupils and the wider community including parents. There were also four general questions relating to purported assessment practice. One related to informal assessment versus formal assessment. Informal assessment is seen as the collection of assessment information coincident with instruction. By contrast, formal assessment involves an "assessment event" (Clarke, Clarke, & Lovitt, 1990). The other three questions related to the types of assessment data teachers tended to collect, for example, observational data, and information about rules and facts. These four questions were embedded within the questionnaire itself and were used to confirm and illuminate information given within the assessment component. The assessment techniques chosen for the questionnaire are those recommended in the National Statement for Mathematics (1991), namely, observations, oral tests, practical work, interviews, timed tests, projects, assigned homework and assignments. Journals and investigations were also included as both have received considerable attention in the literature as useful assessment alternatives (e.g. Clarke, Clarke, & Lovitt, 1990; Waywood, 1994).

Fifteen hundred surveys were sent to a random selection of primary schools representing different school systems (government & catholic), socio-economic areas (high & low) and geographic locations (metropolitan, provincial & rural). The return rate was 26% (n = 387) and the resulting sample was representative of the different systems, socio-economic areas and geographic locations. The sample was also representative of school year levels.

RESULTS

The questionnaire contained items relating to three overall purposes of assessment, namely, to inform the teacher (6 questions), to inform the learner (4 questions) and to report to others (3 questions). Teachers were asked to indicate how often they used assessment for these purposes. To test the reliability of these components Cronbach as were calculated. These results indicated that the items were reliable for measuring informing the teacher (a = .78) and for informing the learner (a = .72). With regard to informing others a was .322. This could simply reflect the fact that only 3 items represented this component. Paired t tests were performed to compare the frequency of use of assessment (i) for informing the teacher, (ii) for informing the learner, and (iii) to report to others. The overall significance level was set at 0.05, and the Bonferroni Inequality (Stevens, 1992) was employed because three different analysis were conducted. The conservative application of this inequality required the planned Type 1 error for each analysis to be set at the family-wise level divided by the number of analyses (i.e. .05 ¸ 3 » 0.016). All three paired t tests were significant at this level. A summary of the results of these paired t-tests is presented in Table 1.

Table1
Paired t tests Between the Purposes of Assessment

	Inform the learner	Inform others
Inform the teacher	13.91**	28.56**
Inform the learner		14.91**

The results indicate that primary teachers seemed to use assessment more often to inform the teacher with regard to teaching than to inform the learner with regard to learning and that using assessment for reporting to others was not as important as informing teaching and learning.

Tables 2, 3 and 4 present the statements from the questionnaire relating to these three purposes and the frequency of use of assessment to fulfill the respective aspects of assessment.

The above results indicate that teachers believed the predominant uses of assessment were to identify individual students with problems and to evaluate the effectiveness of their teaching. They also used assessment to inform them about the ability levels of students and the progression of the class. Of less importance was the use of assessment to inform them about which students are working and planning for the next phase of teaching. The next cluster of statements related to the use of assessment to inform the learner. The statements and percentage frequency of agreement with each are presented in Table 3.

Table 2
Percentage Frequency of Responses for Questions Relating to the Use of Assessment to Inform the Teacher (n=387)

I use assessment to	Almost never	Seldom	Sometimes	Often	Very Often
Help me identify individual students with problems	0.5	0.5	7.5	44.7	46.7
Help me evaluate how effective my teaching has become	0.8	3.0	17.3	46.1	32.6
Inform me about the ability levels of the students	1.5	4.1	19.6	49.9	24.4
Judge how well the class is progressing	1.3	2.8	19.7	45.1	30.4
Inform me about which students are working	7.4	20.0	29.5	31.3	11.8
Help me plan the next phase in teaching	7.4	19.6	29.3	31.0	11.7

For the learner, primary teachers saw assessment as a positive experience, giving feedback on student's strengths and encouraging them to learn their work. Of less importance was the use of assessment for reward purposes or to inform students of their weaknesses.

Table 3
Percentage Frequency of Responses for Questions Relating to the Use of Assessment to Inform the Learner (n=387)

I use assessment to	Almost never	Seldom	Sometimes	Often	Very Often
Give students feedback on their strengths and abilities	1.3	2.3	20.8	55.4	20.0
Encourage students to learn their work	4.3	7.8	27.0	44.9	14.9
Provide rewards for successful students	10.4	18.9	30.1	27.5	12.4
inform the students about what they do not know	10.1	11.6	39.4	30.8	7.3

The statements relating to the use of assessment for reporting purposes were distributed throughout the questionnaire. Teachers were asked to indicate if they strongly disagreed, disagreed, unsure, agreed, or strongly agreed with these statements. The statements and percentage frequency of agreement with these statements are presented in Table 4.

Table 4
Percentage Frequency of Responses for Questions Relating to the Use of Assessment for Reporting Purposes (n=387)

Statement	Strongly Disagree	Disagree	Unsure	Agree	Strongly Agree
Assessment is really important in terms of public accountability	3.8	18.2	7.3	57.6	12.4
The main reason why I assess is to meet the school's requirements	12.4	62.7	5.8	17.3	1.5
The main reason why I assess student performance is to inform parents	8.3	70.5	5.8	14.4	0.5

With regard to the wider community, it seems that the teachers' focus was on public accountability rather than meeting school requirements or informing parents. In fact there was a strong disagreement with both of these statements, with only 15% of the sample seeing informing parents as important. This could reflect the wording of the question and the inclusion of 'main'.

In the next section of the survey teachers were asked to report on how often they used a variety of assessment techniques. The results are presented in Table 5

Table 5
Percentage Frequency of Use of various Assessment Methods (n=387)

How often do you use each of the following methods of assessment	Almost never	Seldom	Sometimes	Often	Very Often
Observation	1.0	1.8	15.5	37.6	43.1
Practical work	2.3	3.0	27.0	41.9	24.7
Oral testing	5.8	9.1	34.8	37.1	12.4
Investigations	2.5	9.6	40.2	34.6	11.9
Assigned homework	18.9	13.6	23.5	28.3	14.6
Informal interviews	6.3	18.0	38.3	27.2	8.6
Timed test	22.2	19.9	35.1	18.4	3.0
Projects	17.9	31.1	32.6	13.6	2.8
Journals	27.7	23.1	29.4	12.7	4.8
Assignments	26.6	26.8	31.1	12.2	2.0

The most frequently used method for assessing students in the primary schools seemed to be observations, with 80% of the sample indicating that they used this form of assessment often or very often. The next most commonly used form of assessment was practical work (66.6%) followed by oral testing (49.5%), investigations (46.5%), and assigned homework (42.9%). Of less importance was the use of timed tests, projects, assignments and journals with up to 50% of the sample using these forms of assessment either almost never or seldom.

Four general statements relating to assessment practice were also embedded within the questionnaire. Teachers were asked to indicate their agreement with these statements. Table 6 presents the percentage of agreement with these statements.

Table 6
Percentage of Agreement with Statements Relating to Assessment (n=387)

Statement	Strongly Disagree	Disagree	Unsure	Agree	Strongly Agree
Informal assessment is as important as formal assessment	0.3	2.0	1.5	62.9	33.3
The main tools I use for gathering data are timed tests rather than projects and investigations	15.6	57.9	3.8	21.2	0.5
It is worth spending time on collecting observational data on students	1.8	1.8	4.5	42.8	48.6
I tend more to test rules and facts than have students solve problems	7.9	32.5	44.8	12.5	1.5

The majority of primary teachers agreed that informal assessment was important. Their responses to the second statement confirmed that for them timed tests were not the principal tool used for assessing students. Teachers felt that it was worthwhile spending time collecting observational data and this was the most frequently used method of assessment (See Table 5). Nearly half of the sample seemed unsure just what type of mathematics they tended to assess more frequently. Was it rules and facts or problem solving? This is another source of concern that many teachers are not aware of the emphasis of their own assessment items.

In summary, the primary teachers predominantly used assessment to inform them as teachers about the status of their students (what can they do, what can't they do) and the effectiveness of their teaching. Of less importance was using assessment to help them with their ongoing planning. The most frequently used assessment methods were observations, practical work oral testing, investigations, and assigned homework. With regard to the learner, teachers feel that assessment is used mainly to inform the learner about what they can do rather than what they cannot do. This is in contrast with how they themselves use assessment. It could be conjectured that this represents a model of teaching where the teacher is in control of student learning. This is supported by the infrequent use of projects, journals, assignments, and tasks for which the student has the main control.

DISCUSSION

From the results of the survey, it seems that these primary teachers have adopted many of the suggestions presented in the National Statement on Mathematics. They are no longer relying heavily on pen-and-pencil tests to gather information (Haines & Izard, 1994). They seem to be proficient at using a number of assessment practices and are using these practices regularly to improve classroom instruction (Grouws & Sherry, 1992). These teachers also believed that the collection of assessment incident to instruction (informal assessment) was as important as the assessment event (informal instruction). For these teachers this does represent a change in practice and it seems that the introduction of the Year 2 Diagnostic Net has been a catalyst for this change (Nisbet & Warren, 1999).

Clarke, Clarke, & Lovitt (1990) claim that assessment is used to provide feedback to the teacher, the learner and parents. These results indicate that teachers feel that the feedback to them is more important than feedback to the learner with feedback to the parents playing an insignificant role in the uses of assessment. This study confirms that teachers do use assessment to inform their instruction by identifying successful instructional strategies and student learning behaviours. With regard to the learner, teachers seemed more concerned about informing them of their strengths and abilities (75.4%) rather than of their weaknesses (38.1%). This imbalance needs to be addressed. In order for students to apply the most effective strategies in mathematics they need to know where their mistakes are and why they are occurring. Most of the teachers believed that assessment is very important in terms of public accountability. Again the public accountability emphasis could reflect recent State initiatives such as the introduction of the Year 2 Diagnostic Net and the Year 6 Test. The role that teachers believed parents play in students' learning needs to be explored further as these primary teachers did not appear to believe that using assessment for reporting to parents was a prime concern. Yet, many educators believe that parents play an important role in providing effective support for student learning (Arthur, 1996; Clarke, Clarke & Lovitt, 1990; Greenberg, 1989). Of concern is teachers' inability to discriminate between facts/ rules and problem solving.

This study indicates that a relationship between beliefs about the uses of assessment and the frequency of use of various assessment techniques appears to exist. The reasons why teachers are failing to use methods such as project, journals, and assignments more frequently needs further exploration. Does this reflect a belief that assessment is mainly used to inform the teacher rather than to inform the learner and parents? All of these assessment methods can have significant learner and/or parent participation. Or do teachers simply see these forms of assessment as being unmanageable.

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LONGITUDINAL UNDERSTANDING OF CONDITIONAL PROBABILITY BY SCHOOL STUDENTS

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Two survey items asking for estimates of probability or frequency of conditional events, (A|B) and (B|A), were completed by 2719 school students in grades 5 to 11. Cross-sectional and longitudinal analyses revealed improvement with grade in expressing probability numerically and in distinguishing conditional events. Conditional events were better distinguished for the frequency item than the probability item. Comparisons with responses to other probability items indicated understanding of conditional probability was related to development of basic chance measurement.

Historically understanding of conditional probability was explored with tertiary students. Not only were tertiary students readily available as participants in the researchers' classes but also there was a lack of emphasis on probability in the school curriculum generating little interest by mathematics education researchers. Over the last decade, however, mathematics curricula of most western countries have recognised the importance of chance and probability generally and included specific reference to conditional probability. In *Mathematics - A Curriculum Profile for Australian Schools* (Australian Education Council, 1994) secondary students are expected to "assign conditional probabilities based on data in two-way tables" (level 7.23, p. 124). In the *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989) in the United States, recommendations for grades 9 to 12 include addressing probabilistic intuitions (Fischbein, 1975; Kahneman & Tversky, 1972).

> Concepts of probability, such as independent and dependent events, and their relationship to compound events and conditional probability should be taught intuitively. Formal definitions and properties should be developed only after a firm conceptual base is established... (National Council of Teachers of Mathematics, 1989, p. 171)

What it means to teach or learn intuitively however may still be open to debate following the research of Fischbein and others in probability generally (Fischbein, 1975; Fischbein & Gazit, 1984; Fischbein & Schnarch, 1997) and more specifically in relation to conditional probability (Bar-Hillel & Falk, 1982; Falk, 1986). Findings show mixed results in terms of the development of intuitive understanding; for example, Fischbein and Schnarch (1997) found diminished performance in relation to some concepts with increasing age. In discussing misconceptions associated with significance testing, Falk (1986) suggested that much of the difficulty is related to confusing conditional probabilities and equating the probability of rejecting a null hypothesis when it is true, $P(R|H_0)$, with the probability of it being true when it is rejected, $P(H_0|R)$.

The contexts in which conditional problems have been set have varied widely. In questioning middle school students' understanding of replacement and non-replacement aspects of conditional probability, Fischbein and Gazit (1984) and Tarr and Jones (1997) employed events of drawing objects from bags or choosing among alternatives in games. In questioning tertiary students who had not undertaken a statistics course, Pollatsek, Well, Konold, Hardiman, and Cobb (1987) employed events from everyday social settings, such as having green eyes, for which some degree of topic knowledge was assumed. No studies were found that used everyday definitions of events in terms of social characteristics with school students.

The format of response for conditional problems has also varied. Pollatsek et al. (1987) found slightly better performance for items requiring probability estimates than for forced choice items concerning the equality or inequality of P(A|B) and P(B|A). They concluded that asking for "numerical estimates may have resulted in a more careful analysis of the problem" (p. 264). Gigerenzer and Hoffrage (1995), using more complex problems with tertiary students, found that frequency items were easier than probability items. Results from other studies of development of chance measurement for students of grades 3 to 11 (Watson, Collis & Moritz, 1997; Watson & Moritz, 1998) have indicated that younger students often use words or non-normative numerical expressions for measuring chance, and that sex differences favouring males are evident at some secondary grade levels. Thus the expression of response to open-ended tasks and sex differences may be relevant factors in the development of conditional probability reasoning for school students.

The current study of school students' responses to conditional problems investigated not only students' conditional estimates of probabilities or frequencies, but also the ways that students express likelihood estimates. As the items on conditional probability in this study were part of a larger survey, it was not possible to include as many formats and settings for questions as covered by earlier researchers. Based on the experience of Pollatsek et al. (1987), it was decided to use the open-ended format allowing students to produce their own estimates of likelihood. Rather than using in-school contexts such as drawing objects from a bag with and without replacement, out-of-school contexts similar to those of Pollatsek et al. were used in an effort to explore student reasoning in everyday contexts and to reflect the application goals of the school curriculum. Three research questions were of interest. (1) In everyday contexts, do school students interpret conditional events in an appropriate fashion? What alternative interpretations arise and in what formats are responses offered? (2) Does student performance on these tasks improve with age, or differ between cohorts or sexes? (3) Is there an association between performance on conditional probability items and performance on other chance measurement items?

METHOD

Participants

Survey responses were gathered from 2719 students at 20 government primary schools, secondary schools, and matriculation colleges distributed throughout Tasmania. Responses were collected in 1993, 1995, and 1997, and totalled 3730 responses, including 785 responses from the same students surveyed again after a two-year interval, and a further 113 responses from students surveyed three times, in 1993, 1995, and 1997. The numbers of students surveyed from different schools varied across years and grades (see Table 1) due to availability of students. Approximately equal numbers of males and females were surveyed in each year at each grade level. More details of the cross-sectional and longitudinal aspects of the larger study are found in Watson and Moritz (1998).

Items and Procedure

Two short answer items, shown in Figure 1, were based on questions of Pollatsek et al. (1987; Q5, p. 269, and Q3, p. 260). Both items required subjective likelihood estimates of everyday events, Item 1 in frequency form and Item 2 in probability form. For each item, it was expected that part (b) would be estimated as approximately half of the total, whereas part (a) would be estimated as less than half of the total. These items were questions 14 and 16 of a 20-item chance and data written survey (Watson, 1994). The survey was administered to whole class groups during 45 minutes of class time. Some students who did not respond to these or later items, due to time or inclination, were excluded from this analysis.

Figure 1
Two Items Involving Conditional Events

1. Please estimate:
   (a) Out of 100 men, how many are left-handed.
   (b) Out of 100 left-handed adults, how many are men.
2. Please estimate:
   (a) The probability that a woman is a school teacher.
   (b) The probability that a school teacher is a woman.

Coding and Analysis of Responses

Responses were entered into a spreadsheet in the form students wrote them. Responses to each part were then (1) assigned a numerical value for the probability or frequency, and (2) coded according the type of expression used. Chance words were assigned numerical values where possible: "unlikely" and "low" were assigned 0.25, "maybe", "medium" and "average" were assigned 0.50, and "likely" and "high" were assigned 0.75, with modifiers such as "very" being assigned more extreme values. Chance words were not assigned values if responses to different parts could not be differentiated to determine which was intended to have a higher probability value. Responses to each of the two items were then coded according to the numerical relation between responses to parts (a) and (b), as well as the type of expression. Numerical Relations are represented symbolically by $b>a$ if P(part(b))>P(part(a)), $b=a$ if the numerical values of the two parts were the same, $b<a$ if P(part(b))<P(part(a)), and $b=a/2$ if P(part(b))=0.5P(part(a)). The last of these numerical relations appeared to be associated with a failure to distinguish a new conditional statement in part (b), and thus, for example, "a=30, b=15" was interpreted as, "a=30 [males are left-handed], b=15 [of these 30 left-handed people are male]." An *undefined* code was entered for responses where a numerical value was not given or assigned for both parts to the item, for example, "not enough info" or "yes Miss Alan is [a woman]" for Item 2. In coding Expressions, whole number answers between 0 and 100 were coded as *frequency* expressions. The *percentage* category refers to responses with "%" attached. The *fraction* category included common or decimal fractions or odds. Using a *word* such as "likely" or answering *yes/no* was common for Item 2 but not Item 1. The few such responses for Item 1 were included in the *mix/other* category. Examples of this last category included: "a=.5 [men are left-handed], b=80% [of left-handed adults are male]" and "a=likely, b=50%."

Students' responses were analysed in three different ways. *(1) Cohort and cross-sectional analyses* using χ^2 tests involved the independent factors of cohort (1993, 1995, 1997), sex, and grade level. Responses from comparable grades collected from different cohorts were compared to investigate whether recent curriculum reform and implementation had affected students' estimates of conditional probability and their expressions of chance. Responses of students from a cross-section of grades (and both sexes) were compared as one method of investigating conceptual development of students. *(2) Longitudinal analysis* was a second method for exploring conceptual development, in this case analysing differences in responses of 113 individual students gathered longitudinally over two 2-year intervals. *(3) Cross-item analyses* involved comparing students' responses to the probability (Item 2) and frequency (Item 1) forms. Responses were also compared to those of chance measurement tasks reported in previous studies (Watson, et al., 1997; Watson & Moritz, 1998) to explore understanding of other probability concepts that may impact on estimating conditional probabilities.

RESULTS

Cohort and Cross-Sectional Analyses

Tables 1 and 2 show the numerical relations and expressions used in responses to Items 1 and 2 respectively, for all 3730 responses grouped by grade and year of survey. The results reported include both those who were completing it for the first time as well as those who were repeating the items in each year. There were significant differences between non-repeating and repeating students (1) in grade 7 for the expressions used for Item 2, (2) in grade 10 for the expressions used for Item 1, and (3) in grade 11 for the numerical relations and expressions used for Item 1; in each case repeating students performed better. It is difficult to determine if these differences indicate repeat effects or if they are due to non-repeating students being drawn from different feeder schools, particularly for those in grades 7 and 11. Further analysis assumed repeating and non-repeating students could reasonably be grouped together.

For Item 1, χ^2 tests for equivalent grades in the different cohorts (grades 6 and 9 in all three years and grades 5, 8, and 11 in 1995 and 1997) indicated no significant differences across the cohorts. There was improvement in performance with grade from grade 5 but this tended to level off from grade 8. Percentages of responses in the undefined category varied from 2% to 10%. At all grades, most students responded with a frequency. Over all students, 17% responded "a=50, b=50" (using various expressions); a further 2% of students gave responses to parts (a) and (b) which summed to the total frequency but were not equal to 0.50.

Table 1
Percentage Responses to Item 1 Coded by Numerical Relation and Expression

Response	1993 Grade		1995 Grade					1997 Grade						
category	6	9	5	6	8	9	11	5	6	7	8	9	10	11
Numerical Relation														
b>a	16	29	8	10	25	27	39	11	15	15	23	20	32	41
b=a	44	52	38	46	57	45	50	41	46	51	56	63	56	49
b<a	4	2	6	6	4	1	4	4	6	4	3	1	2	0
b=a/2	0	0	0	1	1	0	1	0	0	0	0	1	0	2
Undefined	35	16	48	38	13	27	7	44	33	30	18	15	9	8
Expression														
Frequency	3	3	5	5	4	1	1	8	6	4	2	4	3	2
Percentage	30	42	23	34	54	36	54	17	31	38	50	45	51	45
Fraction	14	28	10	12	17	23	30	16	13	11	17	11	26	43
Word	15	10	15	12	10	11	4	14	16	15	11	20	9	0
Yes/No	18	4	27	18	4	12	1	27	19	17	5	4	3	0
Mix/Other	20	14	20	19	11	17	9	18	15	16	15	16	8	10
N	307	377	435	321	361	345	215	216	215	305	186	104	292	51

For Item 2, there were significant differences between grade 9 cohorts for both numerical relation and expressions; there were more *Yes/No* expressions in 1995, and more *b=a* numerical relations in 1997, as can be seen in Table 2. The level of correct performance increased with grade from 8% for one grade 5 group to 41% for one grade 11 group. Responses for all grades were more likely to be in the *b=a* category than the *b>a* category. Percentages or fractions were the dominant form of response for older students, *Yes/No* responses were common at grades 5, 6 and 7, whereas chance words were used by between 10-20% of students until grade 11. Over all students, 23% responded "a=0.50, b=0.50" (using various expressions), increasing with grade from 12% at grade 5 to 33% at grade 11; this trend was observed even after removing undefined responses from analysis. A further 2% of all students gave responses to parts (a) and (b) which summed to the total frequency but were not equal to 0.50.

Table 2
Percentage Responses to Item 2 Coded by Numerical Relation and Expression

Response	1993 Grade		1995 Grade					1997 Grade							
category	6	9	5	6	8	9	11	5	6	7	8	9	10	11
Numerical Relation															
b>a	16	29	8	10	25	27	39	11	15	15	23	20	32	41
b=a	44	52	38	46	57	45	50	41	46	51	56	63	56	49
b<a	4	2	6	6	4	1	4	4	6	4	3	1	2	0
b=a/2	0	0	0	1	1	0	1	0	0	0	0	1	0	2
Undefined	35	16	48	38	13	27	7	44	33	30	18	15	9	8
Expression															
Frequency	3	3	5	5	4	1	1	8	6	4	2	4	3	2
Percentage	30	42	23	34	54	36	54	17	31	38	50	45	51	45
Fraction	14	28	10	12	17	23	30	16	13	11	17	11	26	43
Word	15	10	15	12	10	11	4	14	16	15	11	20	9	0
Yes/No	18	4	27	18	4	12	1	27	19	17	5	4	3	0
Mix/Other	20	14	20	19	11	17	9	18	15	16	15	16	8	10
N	307	377	435	321	361	345	215	216	215	305	186	104	292	51

Table 3 shows the performances of female and male students at each grade level (cohorts combined) in a consolidated form (*b>a*, *other*, *undefined*) for the two items. Comparing females with males at each grade for numerical relations to both items, the only significant difference was at grade 10, with males more likely than females to respond *b>a* to Item 1 $(\chi^2_2=12.9, p<0.01)$.

Table 3
Percentage Responses to Items 1 and 2 by Grade and by Sex

Response	Grade 5		Grade 6		Grade 7		Grade 8		Grade 9		Grade 10		Grade 11	
category	f	m	f	m	f	m	f	m	f	m	f	m	f	m
Item 1														
b>a	43	46	48	47	58	55	55	61	61	67	48	64	62	65
Other	52	44	46	46	38	38	43	34	35	28	51	32	33	28
Undefined	5	10	6	6	4	6	2	5	5	5	1	3	5	7
Item 2														
b>a	8	10	12	15	15	16	21	28	24	31	25	39	41	37
Other	43	46	52	50	54	55	64	57	53	52	64	54	51	58
Undefined	50	45	36	35	31	28	15	15	24	17	12	7	8	6
N	317	334	432	411	150	155	273	274	425	401	146	146	143	123

Longitudinal Development

Table 4 presents the percentages of students in each response category for 113 students who were surveyed in each of the three years of the study. This subset of students closely reflects the whole cohort in each year (grade 6 in 1993, grade 8 in 1995, and grade 10 in 1997). For the appropriate response *b>a*, there was improvement from grade 6 to 8 but little after that. To examine within-student longitudinal change, results were consolidated by grouping numerical relations into *b>a*, *other*, and *undefined*. For Item 1, 33 students gave the *b>a* response three times and 12 consistently gave an *other* response. For Item 2, 12 students gave the *b>a* response three times and 28 consistently gave an *other* response. In two 2-year intervals, 1993-1995 and 1995-1997 (the latter in parentheses for the following results) for Item 1, 65 (or 72) students remained in the same grouping, 23 (or 20) students improved their response from *other* to *b>a*, whereas 11 (or 17) reverted from *b>a* to *other*. Similarly, for Item 2, 56 (or 69) students remained in the same grouping, 10 (or 15) students improved their response from *other* to *b>a*, and 10 (or 9) reverted from *b>a* to *other*. Thus overall, the longitudinal study indicates that students from grade 6 to grade 8 developed in appreciating conditional events for Item 1 and in numerically expressing chance for Item 2, but from grade 8 to grade 10 percentages of *b>a* and *other* response were quite stable, with fluctuations of improvement and reversion in similar frequencies.

Table 4
Percentage Responses Assessed Longitudinally in Grade 6, 8 and 10 (n=113)

Response category	Item 1			Item 2		
	Grade 6	Grade 8	Grade 10	Grade 6	Grade 8	Grade 10
Numerical Relation						
b>a	46	61	65	22	30	36
b=a	31	25	26	43	58	52
b<a	9	5	4	4	3	2
b=a/2	4	6	5	1	0	0
Undefined	10	3	1	30	10	10
Expression						
Frequency	64	64	77	2	4	3
Percentage	19	30	16	35	59	58
Fraction	4	0	4	16	16	19
Word	-	-	-	13	7	11
Yes/No	-	-	-	12	3	3
Mix/Other	14	6	4	22	11	7

Cross-Item Analyses

Of 3730 responses, *b>a* was more common for Item 1 (2042 responses) than Item 2 (771 responses), *other* was more common for Item 2 (1967 responses) than Item 1 (1492 responses), and *undefined* was again more common for Item 2 (992 responses) than Item 1 (196 responses). Overall only 16% of responses were classified *b>a* to both items. The relationship of answers to the two questions was statistically significant for the combined data set: 29% of those who responded *b>a* for Item 1 replied similarly on Item 2, whereas only 11% of those who gave an other response to Item 1, responded *b>a* on Item 2.

To explore the association between success on conditional estimate items and more general understanding of chance measurement, 3616 responses to Items 1 and 2 were matched to developmental levels determined by Watson and Moritz (1998), scored from 0 to 6, based on responses to three chance measurement items earlier in the survey. These levels represent increasingly complex cognitive functioning evident across responses to three items concerning (1) likelihood of numbers occurring when a 6-sided die is rolled, (2) likelihood for an outcome drawn from a bag, and (3) comparisons of likelihoods of drawing one colour of marbles from boxes with similar ratios of colours. The distribution of responses to Items 1 and 2 by chance measurement developmental level is shown in Table 5. Higher chance measurement developmental levels were associated with more *b>a* responses and fewer *undefined* responses for both Item 1 and Item 2; *other* responses decreased with increasing developmental level for Item 1, but not for Item 2.

Table 5
Percentage Responses to Items 1 and 2 by Chance Measurement Level

Response	Chance Measurement Level							
category	0	1	2	2.5	3	4	5	6
Numeric Relation (Item 1)								
b>a	33	43	47	53	55	58	67	78
Other	53	49	46	41	41	38	28	20
Undefined	14	8	6	5	4	4	5	2
Expression (Item 1)								
Numerical	86	92	93	94	96	96	95	98
Numerical Relation (Item 2)								
b>a	10	6	11	14	19	30	31	47
Other	31	41	49	53	60	56	53	45
Undefined	59	53	39	33	21	14	16	8
Expression (Item 2)								
Numerical	27	35	47	54	65	74	78	85
N	49	177	543	889	653	914	122	269

DISCUSSION

Conditional probability poses a number of difficulties for school students, including expressing chance and identifying the nature of conditional events in different grammatical constructions and contexts. With respect to expressing chance, the use of "Yes", "No" and chance words diminished with increasing grade for the probability item, with older students more likely to express probability numerically. Not surprisingly, numerical expression of chance for this item was also associated with chance measurement level based on other traditional simple probability items. For the frequency item, about 90% of students at all grades responded with numerical expressions.

Identifying the nature of conditional events (*b>a*) for each item improved only marginally with grade, with sex differences favouring males at grade 10. There was, however, clear improvement with increasing chance measurement level. This association is particularly interesting for the frequency item where the association is not confounded by effects of numerical expression for the probability item. Identifying the nature of conditional events is related to responses to probability tasks that involve identifying relevant numbers to measure chance as a fraction, namely the subset as a proportion of the total set.

Identifying the nature of conditional events also depended on the context and grammatical structure of the items. Even those students who could readily express probabilities numerically where more likely to respond *b=a* for Item 2 than for Item 1. These results differ markedly from the results of Pollatsek et al. (1987). When asked to give estimates for the same question as Item 2, Pollatsek et al. found 75% of college students responded *b>a* and 17% responded *b=a*, whereas in the current study, only 39% of grade 11 students responded *b>a* and 50% responded *b=a*. It may be that in the probability form of the questions, younger students fail to distinguish the language indicating the two conditional events as being distinctive. The sentences are also more succinct for Item 2 than for Item 1. Item 1 differed from Pollatsek et al. in dealing with left-handedness rather than having green eyes, and in asking for a frequency rather than a probability or percentage. In the green-eyed setting, college students estimated *b>a* in 45% of cases. Over 50% of students grade 7 or higher in the current study responded *b>a*, reaching a maximum of 63% in grade 11. It is impossible to speculate on the effect of the left-handed/ green-eyed distinction, but it appears more likely that the better performance was associated with the frequency form of Item 1 (cf. Gigerenzer & Hoffrage, 1995) that provides the cue "out of" to make clear the subset conditional relation.

Pollatsek et al. (1987) also noted a consistently large percentage of students (from 18% to 44% on seven different items) giving responses in the open format which summed to 100%, possibly indicating a belief that the events were complementary. It is not clear whether they distinguished the half-half responses from others as done in this study. The combined data from the current study produce values consistent with Pollatsek et al. but further research is required to determine whether these students are responding in the belief that the events are the same, or that the events are complementary.

Classical probability problems, such as involving balls in urns, have long been used in classrooms, and these contexts cue students to elements that are countable. A strong relationship has been shown to exist for reasoning on such problems (Watson & Moritz, 1998) and success on conditional problems. Questions in social contexts, however, ask students to use their contextual knowledge of the environment rather than numbers provided. It is not surprising that many responses about conditional probabilities in a social context were expressed in words and did not appropriately distinguish conditional events. By asking for frequencies in a social context, however, student responses improved markedly. Educational programs in this area should include exposure to social contexts if useful learning is to occur. Overall it appears likely that without a concentrated effort in the school curriculum to assist students to transfer their probabilistic understanding from countable situations to more social settings involving estimation, reasoning or appropriate intuition (Fischbein, 1975) it will not continue to develop.

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Acknowledgement

This research was funded by the Australian Research Council, Grant Numbers A79231392 and A79800950.

MATHEMATICS: MORE TIME, MORE TICKS!

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The Harristown State High School project investigated the effect of more time via voluntary after-school, core mathematics tutorials on Year 8 student learning outcomes. Six factors (additional time, assessment structure, colour booklets, supportive learning environment, rewarded competitions and qualified tutors) were identified from a pilot study as critical and were chosen to be central to the study. Pleasing outcomes included strong parental support, the number of student volunteers (115), and improved confidence and test results.

Different groups of students learn in different ways and for different individuals the drive to succeed in the quest for knowledge and understanding varies in both strength and source. In countries such as Japan and Korea, noted for outstanding performance in global comparative mathematics assessment (Australian Council for Educational Research 1994; Wu 1999), the immediate and extended family play a crucial role in nurturing this drive to succeed by setting high expectations, taking interest in and supporting students. Western countries differ in that parents and extended families increasingly spend less time with children and provide less structure and guidance to children compared with previous generations (Kane 1994; Stevenson & Nerison-Low 4/20/1999). In the United States, many families where both parents work or the child is living with a single parent who works, much of the child's discretionary time is unsupervised (Task Force on Youth Development and Community Affairs 1992). Similar trends exist at Harristown State High School (HSHS).

Educational research in mathematics has demonstrated a strong link between performance and the time spent studying and revising. A study by Keith (1982) found that the grades of low-ability secondary school students who did ten hours or more of homework and study per week were as good as the grades of high-ability students who did no extra work outside of the classroom. The Carnegie Council on Adolescent Development also noted that a Japanese student spends over 400% more time studying than an American counterpart (Task Force on Youth Development and Community Affairs 1992). For many Junior students at HSHS the comparison is even less favourable. Students do want to succeed (Mamary 1997, 1999) and if given appropriate opportunities will voluntarily extend their school day to spend more time doing mathematics. Students experiencing difficulty in mathematics do not like to be always identified for withdrawal and remediation purposes but often like to work with a mix of other students of varying abilities.

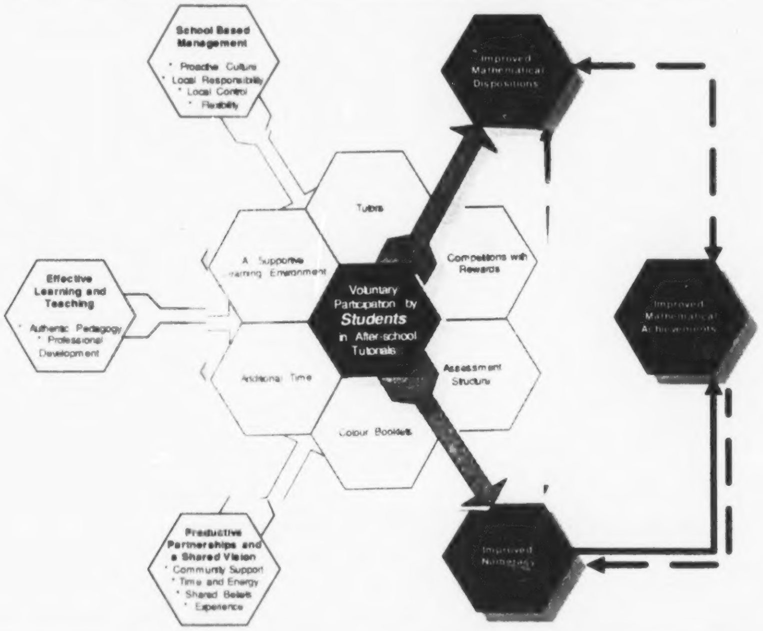
Adolescent development in Asian countries is influenced significantly through activities run at school in a school day of considerably greater length than that in Western Countries. In comparison, in Western Countries such as America, adolescents spend almost one third of their waking time with friends, suggesting that peer-interactions significantly influence adolescent development (Fuligni & Stevenson 1993). The problem for Western teachers and educators would seem to be the ability to develop innovative practices that encourage adolescents to voluntarily incorporate significant learning activities as part of their discretionary time. This, it appears, will only occur if friends and peer interactions are basic to the learning activity.

At HSHS, one particular issue is how to assist Year 8 students achieve improved outcomes in mathematics. Previous experience provided the impetus for a solution. In Term 1 of

1998, weekly two-hour night tutorials for senior mathematics students were begun in a context of competition. As one of three state high schools competing for clientele amongst eleven private secondary schools within Toowoomba, the provision of these tutorials compares with programs offered by the private schools for boarders. These tutorials were completely unstructured and driven by individual student problems and questions. In Term 2 this strategy was tentatively trialed with Year 9 students on a more structured basis commensurate with general cooperative learning principles (Johnson & Johnson 4/29/1999; Burnett 1995). The tutorials were student-centred and involved authentic learning opportunities (NCTM 1998; Scott 4/20/1999; Newmann & Wehlage 1995). Evidence of improved student participation and learning as a result of these tutorials appeared to be similar to that found in previous cooperative learning research (Balkcom 1992). However, the popularity of the HSHS Year program exceeded all expectations. High participation rates and many improved scores together with positive attitudes were replicated. Evaluation endeavoured to identify, and in some cases quantify important curriculum and organisational aspects of the Year 9 project. These aspects together with support structures and expected outcomes and their interrelationships are summarized in Figure 1 below.

Figure 1

Relationships between Key Inputs, Features and Outcomes of the Project.


School Based Management
* Proactive Culture
* Local Responsibility
* Local Control
* Flexibility
Improved Mathematical Dispositions
Tutors
A Supportive Learning Environment
Competitions with Rewards
Effective Learning and Teaching
* Authentic Pedagogy
* Professional Development
Voluntary Participation by Students in After-school Tutorials
Improved Mathematical Achievements
Additional Time
Assessment Structure
Colour Booklets
Productive Partnerships and a Shared Vision
* Community Support
* Time and Energy
* Shared Beliefs
* Experience
Improved Numeracy

To be able to measure the effectiveness of a similar Year 8 program, the subject of this study, the following key and subsidiary research questions were formulated for investigation.

Key Research Question - In what ways and to what extent does the voluntary after-school program result in increased student achievements in mathematics? Specific factors underlying this question include additional time, assessment structure, colour booklets, a supportive learning environment, competitions with rewards and tutors.

Subsidiary Research Question - In what ways and to what extent does the voluntary after-school program affect student mathematical dispositions?

Method

Approximately 90 Year 8 students initially volunteered to participate in the program. This number swelled to 115 within the first fortnight as favourable student and parent comments spread. Numbers stabilised in the range of 90 to 100 students for the Term 3 tutorials. Term 4 tutorials had an average attendance of about 70 students.

The timeline of the study is presented in Figure 2 below. It extends over the ten weeks of Term 4, 1998 showing when the tutorials and assessments were conducted and when permission letters were issued. Also shown are the times of the student and parent surveys and focus group discussions.

Figure 2
HSHS Year 8 Project Timeline

	Term 4 1998									
School Week	1	2	3	4	5	6	7	8	9	10
Date	7-9 Oct	14 Oct	21 Oct	28 Oct	4 Nov	11 Nov	18 Nov	25 Nov	30 Nov	7-10 Dec
Program		Term3 Tute ←	Term3 Tute Study 1	Term3 Tute	→	Term4 Tute ←	Term4 Tute Study 2	Term4 Tute	→	
Assessment (Tests)					Term 3 Resit				Term 4	
Publicity	Assembly, In-class & Letters to Parents				Letters to Parents					
Surveys		Survey One - Study 1				Survey One- Study 2		Survey Two		Parents-Phone Survey
Focus Group Discussions								Tutors		Sample Staff & Students

Two sets of tutorials named Study 1 and Study 2 in this paper were organised for after school from 3:15 pm to 4:45 pm each Wednesday. The Term 3 and Term 4 tutorials covered Term 3 and Term 4 material respectively, yielding 4.5 hours of extra time for each. The Term 3 sessions reviewed Term 3. Students were then given an opportunity to resit the Term 3 test, if they had failed previously, and so a chance to upgrade their result. In the Term 4 sessions, the material for Term 4 was reviewed during the latter half of Term 4 in preparation for the Term 4 test.

The tutorials allowed timetabled classes to operate without segregation. Further, the tutorials utilised some important aspects of cooperative learning involving individual, small group and large group activities. Organisers of the program endeavoured to:

- provide a voluntary learning opportunity that is appealing;
- utilise University of Southern Queensland students as tutors and role models;
- strengthen the social aspect of 'doing maths' incorporating 'friendship groups',
- make spending time on maths socially acceptable and 'normal' for Year 8 students;
- extend the contact time available to students by 1.5 hours per week;

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- provide a catalyst for the substantive conversation that fosters higher order thinking;
- show students the link between success and time on task;
- help students learn how to learn.

Each tutorial began with a brief, whole-group focus activity. Participants then broke into smaller groups of about ten students, each led by one of the ten tutors. Each tutor's brief was to promote a supportive working environment that was friendly and conducive to learning maths. The tutor guided the students through the specially prepared, review material. A couple of competitions involving the whole group added variety and interest during each session. The tutorials concluded with a whole-group pep talk on strategies for, and the importance of, the frequency of review.

The teacher/coordinator wrote the review booklets and was responsible for the general organisation and success of the program. She conducted the whole-group sessions, including the interactive review competitions on selected mathematical topics. Also, the coordinator acted as a freelance tutor, encouraging on-task behaviour, reassuring mathematically anxious students and promoting the program at Year 8 assemblies.

Booklets written were based on the core topic objectives and consisted of key concepts and procedures presented in a colourful, clear and easy-to-read format. The text was not used as a source of ideas, examples, questions or tasks so that the teachers' autonomy within the classroom was not compromised. Key concepts and facts were identified and summarised in coloured 'clouds of knowledge'. Recall of these was fundamental to success in the competitions. No worked solutions were given. Rather students and peers helped those with difficulties to complete the skeletal outlines to important facts, generalisations and problems. Students were encouraged to write their own summaries of work on the facing pages as well as rework problems and pose related problems for their peers to solve. Not only did the booklets help students in their revision, allowing them to work at different rates and aiding their organisation, but also allowed the tutors to rapidly obtain an overview and knowledge of the depth of treatment.

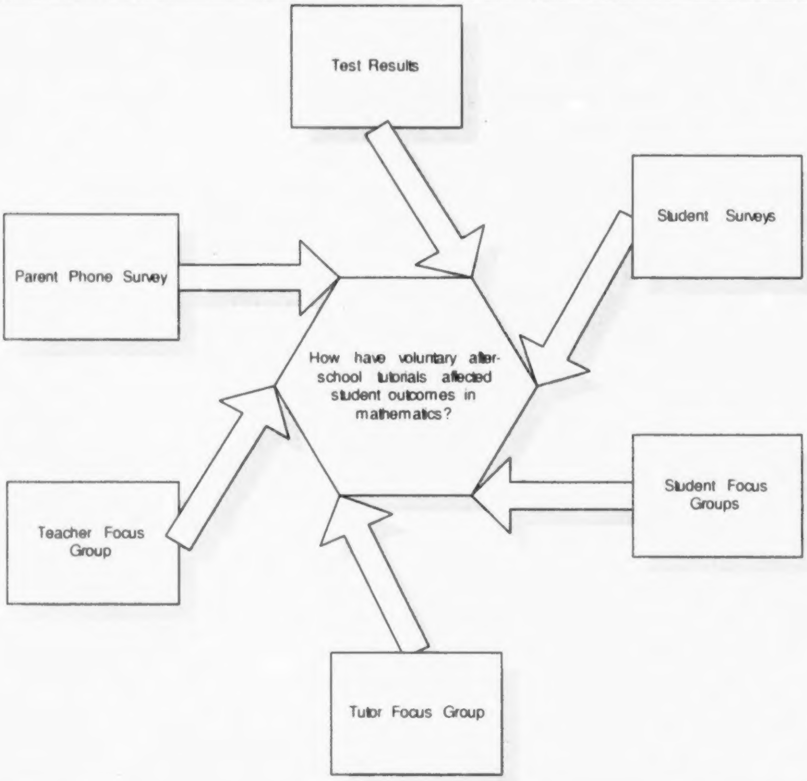
Rewards were instigated to motivate and encourage students to recall reviewed material accurately and quickly. At a school assembly, the Year 8s witnessed the presentation of major prizes (walkmans) to those Year 9s who had participated in the pilot program and had achieved greater than a 35% improvement on their test results. The coordinator confirmed a similar reward system would operate for the Year 8s. Within the tutorials, prizes of mini-sized chocolate bars were awarded to individuals and groups for success in the oral quizzes. These group competitions helped forge a group identity.

Since participation by students in the program was voluntary, a friendly and enjoyable working environment was more easily established. The use of friendship groups, peer tutoring, one-on-one tutoring, sharing of ideas, and many successful, mathematical experiences were all encouraged and facilitated by the tutors. In fact, they became leaders of their groups and acted as role models, promoting a positive and enthusiastic learning environment. Many students could now see that working mathematically need not be a chore but could be enjoyable, even a social event!

Results

The research design of the Year 8 program is presented in Figure 3. It shows the sources of information used to answer the research question, *How have voluntary after-school tutorials affected student outcomes in mathematics?* Both quantitative and qualitative data are used. Analyses of the results of the tests enabled judgement to be made on the extent of improvement or otherwise in student achievement in mathematics. Feedback from the surveys and focus-group sessions provided information on mathematical dispositions.

Figure 3
HSHS Year 8 Project Research Design


Test Results
Student Surveys
Parent Phone Survey
How have voluntary after-school tutorials affected student outcomes in mathematics?
Student Focus Groups
Teacher Focus Group
Tutor Focus Group

Study 1. Study 1 focused on Term 3 (T3) work. For the purposes of analyses, students were grouped according to attendance. Those students who attended all three tutorials and sat the resit test are labelled Group A (n=56); those who attended twice and sat this test are labelled Group B (n=21) while those who attended just once and sat this test are labelled Group C (n=6). The three groups together are called the Whole Group (n=83). *All Year 8s* refers to all students in Year 8 (N=284). Not all students attending the tutorials of Study 1 resat the test (n=32) and these students have been included in *All Year 8s*. The resit test was the same paper as the original test, however, two months elapsed before students sat the resit test. The means of the test scores of these groups are reported in Table 1.

Table 1
Term 3 Test Results by Student Group

Student Groups	T3 Test Mean (%)	T3 Resit Mean (%)	Failed T3 Test Mean (%)	Failed T3 Test Mean (%)	Failed T3 Test Pass Resit (%)
Group A	45.0	50.9	30.5	38.6	26.5
Group B	45.4	52.9	33.0	43.8	33.3
Group C	52.7	53.7	38.3	42.0	16.7
Whole Group	45.6	51.6	31.6	40.1	26.9
All Year 8s	49.0	-	30.1	-	-

Of interest are the results of Group A. On the resit, this group increased their mean by about 6% on the T3 test scores. The Whole Group made a similar gain. On average, this latter group scored below all the Year 8s on the original test but after participating in the program were able to exceed the average on the T3 Resit test scores.

Of special interest are those students who had failed the T3 test for these students had an opportunity to have a failing grade upgraded to a pass. The mean results of these students on the two tests are presented in Table 1, grouped as before. Group A achieved an 8% gain with just over a quarter of the students who failed the T3 Test now passing the resit test. The Whole Group obtained similar results. It is worth noting that the gain for Group C was only about half that for Group A and that only one of the six students in Group C passed the resit test. It appears that frequency of tutorial participation was a major factor in increased achievement by many students.

The absolute and relative increases on the T3 resit test by Groups A, B and C and the Whole Group are presented in Table 2. An absolute increase of at least 20% of the total mark was investigated as the Project team considered that a gain of 10 or more marks out of 50 represented excellent progress. Almost one third of the students in Group A and one quarter of the Whole Group achieved this benchmark.

Table 2
Term 3 Absolute and Relative Increases by Groups of Students

GROUPS OF STUDENTS	T3 Absolute Increase Personal Improvement	T3 Relative Increase Personal Improvement	
	At least 20%	At least 20%	At least 100%
Group A	32.4	79.4	17.6
Group B	20.0	45.0	5.0
Group C	0.0	16.7	0.0
Whole Group	25.0	61.7	11.7

The relative increase was defined as the proportion by which students have changed their original scores. Once again the benchmark was set at a 20% or better increase on the test scores. A 20% relative gain would be less significant than a 20% absolute gain, however, it was considered to be still a substantial achievement. Almost 80% of the students in Group A attained this benchmark, as did about 60% of the Whole Group. Of some interest was the number of students who were able to at least double their initial test score. This doubling would yield an increase of, at least, a grade level. One in six students in Group A and about 12 percent overall achieved this distinction. Comparisons of both the absolute and relative columns indicate that achievements rose the most for students participating in more tutorials.

Study 2. Study 2 focused on Term 4 (T4) work. As in Study 1, students in Study 2 are divided into three groups according to the number of tutorials attended. Group X (n=54) attended three tutorials, Group Y (n=30) two and Group Z (n=31) one. The three groups together are labelled the Whole Group (n=115). The Control Group (n=169) consisted of those students who did not attend any of the tutorials. The results of Study 2 are reported in Table 3.

Table 3
Means of Terms 3 and 4 Test Scores and Relative Personal Improvement by Groups

Student Groups	All Mean T3 (%)	All Mean T4 (%)	Failed T3 Mean T3 (%)	Failed T3 Mean T4 (%)	Relative Personal Improvement	
					At least 20%	At least 100%
Group X	49.2	62.1	29.6	45.0	57.4	7.4
Group Y	48.3	55.2	34.3	43.6	46.7	13.3
Group Z	35.9	42.2	30.2	37.0	45.2	6.5
Whole Group	45.5	55.2	31.1	41.8	51.3	8.7
Control Gp.	51.7	53.7	29.1	36.6	23.7	4.1

The Term 4 test means for all groups increased on those of the Term 3 test, the greatest gain being achieved by Group X. The Whole Group achieved a gain in mean of almost 10% whilst that of the Control Group was only 2%. In term 3, the Group X mean was 49.2%, below the Control Group mean of 51.7%. In the Term 4 test the Group X mean exceeded the Control Group mean by about 8%. Of interest were those students who had failed the Term 3 test. Their average gain on the Term 4 test is 15.4%, more than double that for the Control Group. Of even greater interest were the number of students who had a personal improvement of at least 20% on their Term 4 test scores as compared with their Term 3 test scores. Group X did best with approximately 57% achieving this benchmark, while about 51% of the Whole Group did so. This is more than double the 23.7% of the Control Group.

Surveys and focus groups were used to gather data on the factors underlying the research questions. The six underlying factors (Figure 1) were all supported with both tutor and friends seemingly crucial to the success of the tutorials. Nearly all the participants rated the booklets useful to very useful. Some interesting results were found concerning the amount of out-of-school time spent on mathematics (Figure 4). Over two thirds of the participants reported that they spent more time on mathematics each week during fourth term. However, some spent less time because they were able to work more effectively as a result of the tutorials. Parents in the phone survey supported this.

Responses to the student surveys indicated that the tutorials helped increase students' confidence in doing mathematics. The data is displayed in Figure 5. Eighty-five percent of students felt more confident about the maths topics studied.

Figure 4
Percentage of Students
Spending More Time on Maths in Term 4

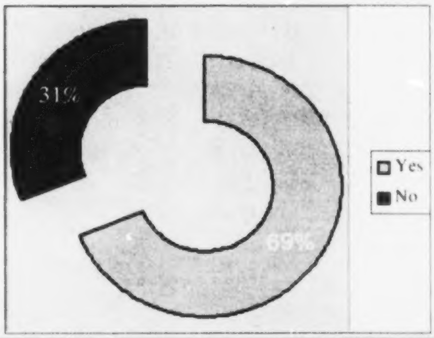
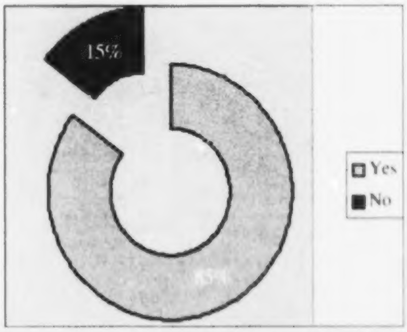


Figure 5
Percentage of Studentswith
Improved Confidence in Maths



CONCLUSION

The Year 8 voluntary, after-school tutorial program in core mathematics attracted a most pleasing number of students who wished to improve their mathematical results. A high percentage of these students participated in all six sessions or, because of a severe electrical storm, missed just one.

As expected, in both Studies 1 and 2 those students who improved the most were those who had the most to gain, that is, those who had failed the Term 3 test. Furthermore, the students who attended all or nearly all the set tutorials achieved the best. Their personal improvement was also greater than those who attended only one of the sessions.

The results from the research vindicated the importance of the specific factors identified in the Key Research Question. Each of these had an important influence on student learning/ achievement. The extent and degree varied between individuals.

Several unexpected outcomes have been noted as a result of the project. It has helped to enhance the image of the Mathematics Department and that of the school generally. Many students who gained in confidence have started the new school year with a more positive attitude to mathematics.

The project team is encouraged by the gains in student achievement and confidence, and by the continuing support of students as well as parents in 1999.

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Acknowledgement

This study was a part of the Innovation and Best Practice Project (1998-99). It was fully funded by the Department of Employment, Education, Training and Youth Affairs.

TEACHER ACTION THEORIES AND THE USE OF GROUP WORK IN UPPER PRIMARY MATHEMATICS CLASSROOMS

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Although group work is widely used by primary teachers in most Key Learning Areas, this is not always the case in mathematics. Why do teachers behave in this way? Argyris and Schon (1974) claimed that behaviour was driven by individual action theories. This study used the Theory of Planned Behaviour (Ajzen, 1988) to uncover teacher action theories with a sample of NSW upper primary teachers (N = 115). It will discuss these theories in order to assist teachers in their critical reflection of current practice.

In America there had been considerable interest in small-group instruction (Good, Grouws, & Mason, 1990; Good, Mulryan, & McCaslin, 1992). Although group work was widely used by primary teachers in most Key Learning Areas, this was not always the case in mathematics. For example, researchers reported that "for reading, the teacher will work with three (or more) groups, but in mathematics the mode is one group - all children are given the same instruction despite individual differences in ability, achievement, needs and interests" (Reys, Suydam, and Lindquist, 1984).

Why do teachers behave differently when teaching mathematics? The beliefs held by teachers concerning the use of groupwork might provide the answers. Good, Grouws, and Mason (1990) surveyed 1509 elementary school teachers and used six categories to describe grouping strategies used in teaching mathematics. Small-groups were regarded as consisting of 1 to 12 members. When asked the organisational structure used throughout the year 55% indicated whole class instruction and only "5% of the teachers indicated that they primarily used a grouping plan in which students worked cooperatively with their peers" (p. 6).

In 1994, a pilot study was conducted with a sample of 89 randomly selected NSW upper primary teachers. A phenomenographical approach (Marton, 1986) was used to analyse the results and to produce a set of descriptive categories which encapsulated the variety of conceptions held by the teachers and to indicate the differences in understanding. Group work was defined as students being divided into small groups of approx 3-6 students, to work independently and together on a task set by the teacher. The results of the study identified and described a range of teacher beliefs for using and for not using group work in an upper primary classroom. However this study was unable to indicate which of these beliefs provided the basis for teacher intentions or behaviour.

While some research efforts had described consistencies between beliefs and classroom actions (Peterson, Fennema, Carpenter, & Loef, 1989), others had described inconsistencies between the two (Thompson, 1992). It was claimed that the link between beliefs and behaviour was a complex mix of intentions, attitudes, social influences, evaluations and motivations (Ajzen & Fishbein, 1980). Argyris and Schon (1974) claimed that behaviour was driven by individual action theories. They identified and described espoused theories and theories-in-use and expected that individuals would not design and implement a theory-in-use which was significantly different from their espoused theory. What they discovered was quite the opposite. Not only were there significant differences between the two types of theories but "individuals develop designs to keep them unaware of the mismatch" (Argyris, 1993, p. 51). Thus, change could not occur until these theories were uncovered and made available for examination and reflection. This current study sought to uncover the microtheories behind teacher classroom behaviour by applying the constructs and instruments of the model titled the Theory of Planned Behaviour (TPB) to the specific teaching behaviour of group work. The results were called teacher action theories to distinguish them from Argyris and Schon's work.

THEORETICAL ISSUES

The Theory of Planned Behaviour (TPB; Ajzen, 1985, 1987, 1988) is an extension of the Theory of Reasoned Action (TRA; Ajzen & Fishbein, 1980). Both have been very successful in using a small number of beliefs to predict behaviour across a wide range of contexts and both come from an objectivist (positivist) stance. This stance has been the target of a great deal of criticism (see for example Lather, 1991) and it is beyond the scope of this paper to address this criticism.

According to TRA the immediate determinant of behaviour is intention and two major factors determine a teacher's behavioural intention: a personal or attitudinal component; and a social or normative component. The attitudinal component measures the teacher's attitude towards performing the behaviour and "is simply a person's general feeling of favorableness or unfavorableness for that concept" (Ajzen & Fishbein, 1980, p. 54). The second or normative component of the theory deals with the influence of the social environment upon intention and behaviour. This is a measure of the teacher's perception of whether most important people support or don't support the performance of the behaviour. This implies that a teacher will usually intend to perform a behaviour that is positively evaluated and has the support of significant others. If TRA was to stop at this point we would have very little information to assist our understanding of behaviour. However, both components can be investigated further. Attitude towards a behaviour is determined by the product of the teacher's beliefs about performing the action and the evaluation of the outcomes of the action. Subjective norm is determined by the product of the teacher's beliefs of social expectations to perform the behaviour and the motivation to comply with these expectations. In the final analysis then, TRA attempts to explain a teacher's behaviour in terms of beliefs which represent the information (be it correct or incorrect) that the teacher has about their situation. Successful predictions of behaviour using TRA have been done of such diverse areas as voting behaviour (Fishbein, Ajzen & Hinkle, 1980), family planning (Fishbein, Jaccard, Davidson, Ajzen & Loken, 1980), consumer behaviour (Ajzen & Fishbein, 1980), and seatbelt use (Budd, North, & Spencer, 1984).

TPB expanded TRA in order to include the construct of perceived behavioural control (PBC). Ajzen (1988) proposed that it would be important as a determinant of intention when the individual had previous knowledge or experience of the behaviour in question. In this study it is probable that teachers would have had knowledge or experience of the behaviour included in the questionnaire. Other researchers have had similiar views such as Budd, North, and Spencer (1984) and Wittenbraker, Gibbs, and Kahle (1983) who included past behaviour as a determinant of behavioural intention when using TRA. Fewer studies have been published using TPB, however Schifter and Ajzen (1985) studied perceived control and weight loss and Ajzen and Madden (1986) used the theory to predict undergraduates' course performance. Parker, Manstead, Stadling, Reason and Baxter (1992) made an assessment of the ability of TPB to account for the intentions of drivers to commit driving violations and concluded that "the addition of perceived behavioral control led to significant increments in the amount of explained variance of intentions, thereby supporting the theory" (p. 94). For the purposes of this paper, only the salient beliefs used in regard to the constructs of attitude, subjective norm and perceived behavioural control which form the teacher action theory will be examined and reported.

METHOD

TPB requires behaviour to be carefully defined. This was done by means of the scenario below which was taken from the phenomenographical pilot study as it produced a high level of agreement among teachers.

> *Scenario*: You are teaching mathematics to your class at the usual time and place. You consider a range of options before using group work. (In this study, group work refers to students being divided into small groups of approx 3-6 students, to work individually and together on a task set by the teacher).

A questionnaire was constructed according to TPB and consists of 28 semantic differential scales and was applied to a sample chosen randomly from across the population of primary teachers who worked at a NSW Department of School Education primary school and who taught mathematics to years 5 and/or 6. A total of 119 teachers returned the questionnaire and the data were entered into a spreadsheet and the software program Statview 4.0 was used to analyse the data. A decision was made not to include four returns because of the amount of missing data.

The questionnaire asked teachers to imagine behaving in the manner described by the scenario and to rate their response on each of the 28 scales. There was one scale for the direct measure of intention to perform the behaviour (endpoints likely/unlikely) and three for a direct measure of attitude (endpoints good/bad, beneficial/harmful, wise/foolish) which were later combined into a single index (Direct attitude index - DAI). A belief-based attitudinal index (BBAI) was also calculated. The behavioural beliefs were selected from across the continuum developed in the pilot study and are listed below.

Table 1
Teacher Beliefs about Using Group Work

My decision to use group work:

1. would place too great a demand upon resources, space and equipment.
2. would increase the efficient organisation of resources and space.
3. wouldn't cater for all individual needs.
4. would allow more capable children to help the less able children.
5. would result in pupils not working well together as they lacked the necessary social skills.
6. would allow students to work together, bouncing ideas off each other and being exposed to each others' view.

According to the model TPB, the strength of the belief was measured on a bipolar 7 point scale (endpoints likely/unlikely) with a score of 3 signifying a strong positive belief strength and -3 signifying the opposite. The evaluation outcome was also measured on a similar scale (endpoints good/bad) giving a product score range of 9 to -9. The BBAI was constructed using the summed products of the six beliefs and the resultant scores ranged from 54 which indicated a very positive attitude towards the behaviour in question to a score of -54 which indicated the opposite (refer to Table 2).

There was one scale to measure teacher perceptions of the overall influence of important others and a corresponding scale to measure teacher willingness to comply with this influence. According to the model TPB, the product of these two scales produced one direct measure of subjective norm (SND). A belief-based index of subjective norm (SNB) was similarly obtained by summing the products of each of four normative beliefs about salient referents (Principal/supervisor, Parents, Students, Other Teachers) with their corresponding motivations to comply. To complete the TPB model there were two scales (endpoints of full-control/no-control, easy/difficult) to measure perceived behavioural control and these were combined to give one index (PBC). Finally there were four questions which gathered information on the teacher's sex, teaching experience in years, approximate age, and year level now taught.

RESULTS

The demographic results of the sample if summarised using modal characteristics indicated that the teacher was likely to be female; aged between 36 and 45; with 11 to 15 years teaching experience; and with a class of only year 6 students.The descriptive statistics of the constructs of TPB listed in Table 2 show that the majority of teachers intended to use group work within the clasroom.

Table 2

Constructs of the Theory of Planned Behaviour: Mean Values and Standard Deviations

	Mean	Standard deviation
Attitude toward behaviour		
Direct. DAI	5.95	2.34
Belief product. BBAI	10.19	9.2
Subjective norm.		
Direct. SND	0.63	2.98
Belief product. SNB	3.91	7.01
Control belief. PBC	3.66	1.74
Behavioural intention	1.78	1.18

Note: The table contains both the direct measures and the summed product measures of attitude and subjective norm. For attitudes the direct measure range is -9 to 9 and the product measure is -54 to 54. The subjective norm direct measure range is -3 to 3 and the product measure range is -36 to 36. The behavioural intention range is -3 to 3. The control belief range is -6 to 6.

The teacher action theory for group work is illustrated pictorially in Figure 1 which shows the correlations between the components of the TPB model. These correlations were mostly positive with a moderate linear association between intention and DAI ($r = 0.52$). There was a very weak association with SND ($r = 0.01$) and a slightly moderate one with PBCI ($r = 0.45$). The correlation results involving the belief based measures show a moderately strong linear relationship between SNB and SND ($r = 0.61$) and a very weak one between BBAI and DAI ($r = 0.27$). Of interest, but not included in the diagram is the moderate linear relationship between the control belief ($r = 0.48$) and PBCI and the very strong one between the self-efficacy belief and PBCI ($r = 0.95$).

Multiple regression analysis of the full sample (N=115) revealed that taken together, attitudes, subjective norm and PBCI accounted for 35.1% (adj R^2) of the variance in intention to use group work in the classroom. The partial standardised regression coefficients show a moderate influence between intention and DAI ($\beta = 0.43, p < 0.0001$), a slightly moderate influence by PBCI ($\beta = 0.33, p < 0.0001$) and a very weak non-significant influence by SND ($\beta = 0.03, p = 0.6938$).

Attitude was the strongest contributor to intentions for the teacher action theories. In an examination of both components of attitude the high intent group revealed that they believed that it was:

(a) unlikely that group work would place too great a demand upon resources, space and equipment, and if it did then it was a very bad outcome;

(b) quite likely that group work would increase the efficient organisation of resources and space and this was a slightly good outcome;

(c) slightly unlikely that group work wouldn't cater for all individual needs, and if it didn't cater for these needs then this was a very good outcome;

(d) likely that group work would allow more capable children to help the less able children and this was a very good outcome;

(e) slightly unlikely that groupwork would result in the pupils not working well together because they lacked the necessary social skills and if it did produce discord then it was neither a good nor bad outcome;

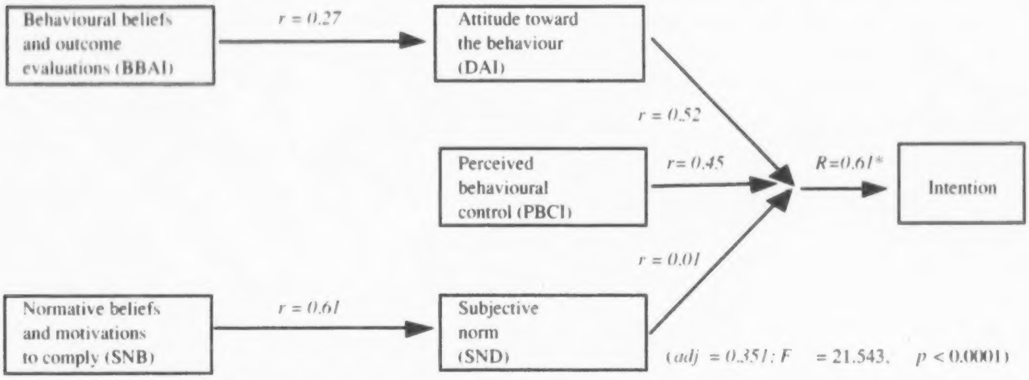
(f) very likely that group work would allow students to work together, bouncing ideas off each other and being exposed to each others views and this was an extremely good outcome.

The influence of subjective norm upon intentions wasn't strong. However, the high intent group believed they received support from all four significant referents and felt encouraged to use group work. They believed that they had full control over the behaviour and that it was a very easy behaviour to perform.

Not surprisingly the teacher action theory of those not intending to use group work was quite different. The components of attitude for the low intent group revealed they believed that it was:

(a) likely that group work would place too great a demand upon resources, space and equipment and this was an very bad outcome;
(b) slightly unlikely that group work would increase the efficient organisation of resources and space and this was a bad outcome;
(c) likely that group work wouldn't cater for all individual needs and this was a very good outcome;
(d) likely that group work would allow more capable children to help the less able children and this was a good outcome;
(e) likely that groupwork would result in the pupils not working well together as they lacked the necessary social skills and this was a bad outcome;
(f) likely that group workwould allow students to work together, bouncing ideas off each other and being exposed to each others views and this was a very good outcome.

Figure 1
Basic Model for Teacher Action Theory: Relations among beliefs, attitude, subjective norm, control and intention


Behavioural beliefs and outcome evaluations (BBAI)
r = 0.27
Attitude toward the behaviour (DAI)
r = 0.52
Perceived behavioural control (PBCI)
r= 0.45
R=0.61*
Intention
r = 0.01
Normative beliefs and motivations to comply (SNB)
r = 0.61
Subjective norm (SND)
(adj = 0.351; F = 21.543, p < 0.0001)

(Note: N = 115; *= p<0.0001)

The social environmental influence revealed that the low intent group believed they received a slight discouragement from the Principal, Students and Parents, and only very slight encouragement from the Other Teachers. While they believed that they had full control over the behaviour they indicated they felt that it was very difficult to carry out. Thus this group of teachers believed that group work although providing opportunities for collaborative learning and peer tutoring, wasn't worth the effort. It was too demanding of time and resources, didn't cater for all students, and was difficult when the students lacked the necessary social skills. This teacher action theory lends itself to speculation concerning its formation. One possibility is that the teachers lack the necessary skills to organise and conduct group work lessons. Either through poor preparation or by not developing the necessary social skills within their students, attempts at group work had failed. Failures in classroom teaching quickly attract attention which may reinforce the perception of the social environment discouraging the use of group work. Only other teachers would fully understand the pressures and be supportive of further attempts.

DISCUSSION

How can this information be used to inform current teaching practice? When considering the various teacher action theories it becomes apparent that some teachers have beliefs that are contrary to the spirit of the mandatory syllabus documents. The task of encouraging teachers to alter their beliefs to mesh with the goals of the K-6 Mathematics Syllabus (NSW Department of Education, 1989) and more recent documents is a challenge for all teacher educators as well as educational leaders and administrators. The stated aim of this paper was to contribute to the understanding of teacher classroom behaviour via teacher action theories and not behaviour change. Yet the results of such reflection may well involve behaviour change.

Stephens, Lovitt, Clarke and Romberg (1989, pp. 223-4) examined recent reform movements in the teaching of mathematics and extracted four principles they claimed were the basis of successful change. The four principles for introducing a new program were:

(i) Any new program must convey to teachers, in practical terms, a clear image of what the changes might mean in the classroom;

(ii) Impediments to change must be addressed;

(iii) Exemplary curriculum materials must be provided; and

(iv) Access to a well-structured environment must be provided for professional growth.

However, Siemon (1989, p.252) disagreed and questioned whether the four point plan would be enough. She claimed that it was predicated on the belief that if the majority of teachers expanded their repertoire of how to teach mathematics more effectively, then a much better state of affairs would come to exist. However there was no guarantee of change. Instead she argued change should follow the 'constructivist view' of learning. It must make sense to those who have to implement it and so must recognise the implementers as learners in their own right. This means that the implementers' theories need to be recognised, understood and challenged in order to facilitate effective and meaningful change. "Until these underlying beliefs, attitudes and knowledge bases are meaningfully challenged, the change effort is in real danger of becoming yet another 'bandwagon'" (p. 254). This study agrees with Siemon claiming that teacher action theories can provide a vehicle for assisting a process of reflection and change. Others such as Thompson (1992) also agree with Siemon.

The authors of TRA and TPB felt that by understanding why teachers perform as they do it would contribute to identifying possible aids and obstructions to change (Ajzen &

Fishbein, 1980). Content was regarded as the most important factor influencing the effectiveness of any communication intended to result in change. However, this content was special because it had to address the primary beliefs that determined the target variable to be changed. Studies that ignored content when examining other presentation variables such as credibility, self esteem, distraction, and others were regarded as being flawed. Thus any manipulations of source, message, or receiver factors may influence the effectiveness of a persuasive communication only to the extent that they affect the message's direct or indirect effects upon the primary beliefs. Thus the limits to the amount of change in the primary beliefs of the target variable were dependent upon the content of the message, and the other variables work within the limits. Teacher action theories clearly identify the beliefs associated with a particular behaviour. The success of both TRA and TPB in predicting behaviour across a wide range of contexts is dependant upon the models ability at tracing the influence of a small selection of salient beliefs upon intention and behaviour.

Argyris and Schön (1974) were also interested in research aimed at putting theory into practice. When applying their research to how people learn, they used the terms 'single loop' and 'double loop' learning to make a crucial distinction. They provide a simple analogy to explain the distinction: "a thermostat that automatically turns on the heat whenever the temperature in a room drops below 68 degrees is a good example of single loop learning. A thermostat that could ask, 'Why am I set at 68 degrees?' and then explore whether or not some other temperature might more economically achieve the goal of heating the room would be engaging in double-loop learning" (Argyris, 1991, p. 100).

Argyris and Schön claimed that a single learning loop was commonly used by professional practitioners, which would also include teachers. However single loop learning was found to have problems concerning the effectiveness of those who used it and its influence upon the individual's ability to learn about their own behaviour. In fact, they claimed that single loop learning worked against effective and productive change because associated with single loop learning were a number of strategies to overcome or hide feelings of embarrassment and threat. This was true for both individuals and groups. Thus "whenever undiscussibles exist, their existence is also undiscussible... These cover-ups, and their cover-up, are indications of organisational *defensive routines* , which may be defined as any policy or practice that prevents organizations (and their agents) from experiencing embarrassment or threat *and* at the same time prevent them from identifying and reducing the causes of embarrassment or threat" (Argyris,1993, p.621).

Single loop learning helps to explain how professionals avoid learning. It also indicates how defensive reasoning blocks learning despite a high individual commitment to learn. Argyris and Schön proposed that effective change needed a double learning loop and they offered a method to assist the transition from a single loop to a double learning loop model. They claimed their method presents a theory of action to enhance human activity, responsibility, learning, effectiveness and self actualization. They argued that people can be taught to recognise the reasoning they use in designing and implementing certain behaviour. They proposed to help subjects identify the inconsistencies between their espoused and actual theories of action. Their method has been successfully adapted to helping successful people learn more effectively (Argyris,1991), and promoting effective organisational change (Argyris,1993). This current study differs from their approach in that it uses the TPB to model the teacher action theories. Whereas, their approach relied upon the researcher to uncover the theory-in-use through the use of observation.

CONCLUSION

In helping current and future teachers work towards practices that are consistent with the wider goals of the educational system, teacher action theories could play an important part. Teachers are exposed to many factors that influence their classroom practice. Teachers would benefit from a self-examination of the relationships between their beliefs, attitudes, intentions and behaviours. Teacher action theories provide a framework for understanding and discussing teacher behaviour. Early and continued reflection of teacher action theories and practice in both preservice and inservice programs may provide a further strategy in the process of improving the quality of teacher classroom practice in the teaching of mathematics.

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EXPLORING DIVERSITY: YEAR 2 STUDENTS' RESPONSES TO QUESTIONS CONCERNING SIMPLE 2D SHAPES

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Recent developments in the Level theory of van Hiele have provided a framework in which the thinking of young students can be explored. This study involved 12 Year 2 students and explores in detail the diversity in responses of six students as they explained their understandings of simple 2D shapes. The focus of the research was on interpreting and exploring the implications of students' language and it highlighted important issues relating to the relevance of the van Hiele model to primary school education.

BACKGROUND

The van Hiele theory (van Hiele, 1986) has been subjected to extensive research in English-speaking countries for nearly 20 years. As a result, there is a large body of empirical support for the hierarchical levels (particularly Levels 1 to 4) identified with the theory and the testing of senior primary, secondary and tertiary students.

Despite this overall positive support for the level aspect, the theory has not been useful in explaining the thinking of younger students at primary school (Clements and Battista, 1992; Lehrer, Jenkins & Osana, 1998). This has been due in part to the fact that the van Hieles were secondary mathematics teachers and hence the intent of their theory was directed towards the group that was most relevant to them. However, recent work on the theory (Pegg, 1997) has offered the potential to address this issue. Of value, has been the splitting of van Hiele's Level 2 (where the student identifies a figure in terms of its properties but these are seen as being independent) into two categories referred to as Level 2A where the focus is on a single property and Level 2B where the focus is on more than one property. This development offers the opportunity for a more focused exploration of the movement from Level 1 (where shapes are known by their global appearance) to Level 2A. While a developmental pattern has been hypothesised (Pegg & Davey, 1998) these writers have also called for the need to explore student diversity during this period of transition between levels. This paper takes up this challenge by considering the detailed responses of six Year 2 students who were interviewed concerning their understanding of simple 2D shapes following the completion of a written questionnaire.

RESEARCH DESIGN

The research question which guided the study was: What diversity can be identified in young students' descriptions of, and language concerning, basic 2D shapes?

In order to examine responses between Level 1 and Level 2A, and thus explore the development of property concepts, a study was carried out with 12 Year 2 students aged between 7.5 and 8.5 years. This age cohort was chosen as this appears to be the time at which responses in geometry are more likely to move from Level 1 towards Level 2.

Sample

The sample for the study was taken from a co-educational primary school in an inner-city suburb of Sydney. The teacher of a Year 2 class nominated twelve students to participate in the study. The teacher selected four students in who were considered to be above average, four who were considered to be average and four who were considered to be below average in their spatial ability.

The sample chosen by the teacher consisted of five girls and seven boys from a variety of socioeconomic and cultural backgrounds. All students had completed units of work on 2D shapes prior to the investigation, with a particular focus on investigating triangles.

Data Collection

The first phase of the study centered on the 12 students' responses to a written questionnaire containing five items. Following this, continuing access to the sample meant that six students from the original sample participated in an individual interview session. These sessions took an average of 30 minutes and were audio taped, and later transcribed. Both the questionnaire and the interview were designed to explore the students' understandings of squares and oblongs.

In this paper, only data pertaining to those children who participated in both the questionnaire and interview phases of the study are discussed.

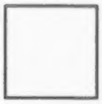
Questionnaire Items

Items 1- 4: Required students to name the figure drawn and to explain their reasoning in naming it. The figures presented in each item appear below. The same questions accompanied each figure, these were:

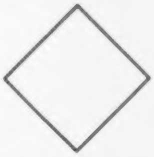
This is a ________.

Why?

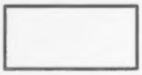
Item 1



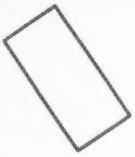
Item 2



Item 3



Item 4



Item 5: This question was included to give students the opportunity to both add to their descriptions of the individual shapes in Items 1-4 and/or to express similarities and/or differences which they saw between figures. The students were asked:

Can you tell me anything else about these shapes?

Comprehension Procedures: It was important to ensure that the students were not being assessed on reading comprehension ability and that a consistent structure was used with all students. As such, if a student was unable to name a figure presented, s/he was encouraged to "think about what (they) would call it". If a student was unable to explain their naming of a figure, the student's name for the figure was used in asking the question "how did you know this was a ________?".

Interviews

The interview consisted of two parts. In the first part students were given the opportunity to discuss and/or explain their responses to the written questionnaire they had completed. This consisted of clarifying students' written responses by asking questions of the type "what did you mean by ________ ?".

In the second part of the interview, students were asked a series of eight questions (with some probes and prompts) using concrete materials in order to explore further their understandings of squares and oblongs. The interview schedule followed is provided below.

Question Number	*Question (notes in italics)*	*Prompt/Probe (if necessary)*
q1	Can you draw me a square?	
q2	What sort of things did you have to think about when you were drawing your square?	
q3	How did you know that you had a square when you were finished drawing?	
q4	*(Place example drawing of a square next to student's square.)* Is your drawing the same as mine? Why/why not?	Is it the same shape? What is the same/
q5	What else can you tell me about your drawing?	
q6	If I do this to your drawing *(add length measurement in centimetres to one side of student's square)*, what else can you tell me about your square?	
q7	*(Make square out of paddlepop sticks.)* □ What shape is this? Why? ↓ *(Move paddlepop sticks, to change orientation of* ◇ *square.)* What is it now? Why?	Is it still a square? Why/why not?
q8	*(Make square out of paddlepop sticks.)* □ What shape is this? Why? ↓ *(Move paddlepop sticks and add one to make* *a trapezium.)* What is it now? Why? ⏢	Is it still a square? Why/why not?

RESULTS

Below are the case studies which summarise the responses of the six students who completed questionnaires and participated in interviews. The case studies are listed in order of increasing sophistication of the responses.

In order to distinguish between written responses and conversation, conversation is reported in italics. Questionnaire responses are reported as written by the students, as such, the meanings of any words that may not be obvious have been included in brackets.

Case Study 1: 'Liam'

Liam used the words "sqwer" (square), "dimnd" (diamond), "reptangle" (rectangle) and "oblonge" (oblong) to name the figures presented in items 1-4, respectively. His reason for each of the names was exactly the same – "because it look like one". He did not write any further information at item 5.

In the first part of the interview, Liam was unable to explain his written answers any further. He named the shapes in the same way as he had done on the written test and simply reiterated *"because it looks like one"*.

In the second part of the interview, Liam drew a square (q1) and, when prompted, recognised that the sample square was the *"same shape"* as his drawing (q4). He was unable to explain any features of squares (q2 - q6). Liam also identified a square made out of paddlepop sticks. When the orientation of the square was changed, he called it a *"diamond"* (q7). Liam was unable to give a name for the trapezium presented at question 8.

Case Study 2: 'Rhimi'

Initially, Rhimi did not understand the written task, due to the demands it placed on his reading skills. Therefore, the questions were read to him.

Rhimi used the words "scwer" (square), "dimen" (diamond), "reek tangoll" (rectangle) and "reek tangoll" (rectangle) to name the figures presented in items 1-4, respectively.

Item 1 was a square because it had "for sids" (four sides); item 2 was a diamond because "it has the same corns (corners)"; item 3 was a rectangle because "it is Loug (long)" and item 4 was a rectangle because "it has for sids" (four sides). At item 5, Rhimi compared item 1 to a box, item 2 to a necklace, item 3 to a present and a mirror, and item 4 to a mirror. He explained that the figures in items 1, 2 and 3 each had four corners: "the dimen has for coones and the scwer to ~~ad~~ and the reektangoll".

In the first part of the interview, Rhimi repeated his written answers in the same form as he had in his writing as well as clarified that when, at item 2, he had written "it has the same corns" he had meant *"the diamond has the same corners as the square"*.

In the second part of the interview, Rhimi drew a square (q1). Rhimi also recognised and named the sample square and the squares in standard orientations presented at questions 4, 7 and 8. He restated that a square had four sides (q2-3, q7-8), and was able to add that a square had *"four corners"* (q5). When the orientation of the square was changed (q7), Rhimi called it a *"diamond"* (q7). He named the trapezium a *rocket* and did so consistently regardless of orientation because *"the rocket's just flying the other way"*.

Case Study 3: 'Angela'

Angela used the words "squr", "diemend", "rectangel" and "dieagenul rectangel" to name the figures presented in items 1-4, respectively. Item 1 was a square "because it has four sides and it has four coners (corners)"; item 2 was a diamond "because it has four coners (corners) and it looks like a triangel"; item 3 was a rectangle "because it has four sides like a squer"; and item 4 was a diagonal rectangle "because it has four sides like a rectangel and it is strayt (straight)". At item 5, Angela commented that the diamond and the diagonal rectangle also both "look like" the square, that the square "looks like" the diamond, and that the "rectangel has four coners (corners)".

In the first part of the interview, Angela clarified that the diamond *"looks like a triangle because there is one at the top and one at the bottom"* and that a rectangle *"is like a square because it has four sides and four corners"*. Angela explained that she named the figure in item 4 a diagonal rectangle because *"it looks like that rectangle"* (item 3) *"but on a diagonal"* and that when she wrote that the diagonal rectangle "is strayt (straight)" she was referring to the fact that the sides were straight.

In the second part of the interview, Angela drew a square (q1). She also recognised and named the sample square and the squares in standard orientations (q 4, 7 & 8). Angela restated that a square had four sides and four corners (q2-3, q7-8), and added that a square had *"straight sides"* (q5). When the orientation of the square was changed (q7), Angela called it a *"diamond"* (q7) because *"it's on a diagonal"*. Angela was unable to give a name for the trapezium presented at question 8 but described it as *"a bit like a square"*.

Case Study 4: 'Nicole'

Nicole used the words "square", "dianglar", "rectangular" and "oblong" to name the figures presented in items 1-4, respectively. Item 1 was a square "because it has four sides and it has Pointy edges on each side"; item 2 was a "dianglar" "because it looks Like a hexagon"; item 3 was a "rectangular" "because it's big and long"; and item 4 was an oblong "because it looks like a rectangle and it looks like a stick if it[illegible]as skinny". At item 5, Nicole drew comparisons between the square and the "dianglar", and between the "rectangular" and the oblong:

> The square and the dianglar ~~are~~ look like the same and I think they are because the diangler is the same but the diangler is been tipped over.
>
> The rectangular and the oblong are the same and they are the same ~~shape But~~ the rectangular and the oblong ~~aren't and~~ are the same Height.

In the first part of the interview, Nicole explained that her written comment about the "Pointy edges on each side" referred to the place *"at the end of the sides where they join"*, and that when she described the "rectangular" as "big and long" in the written questionnaire she had meant that *"it's like a big long square"*. She also explained that the oblong in item 4 was "like a stick if it was skinny" because skinny sticks go *"up and down"*, while *"fat sticks go across"*.

In the second part of the interview, Nicole drew a square (q1) and was able to recognise and name the sample square and the squares in standard orientations (q 4, 7 & 8). She reiterated that a square had four sides (q3), and went on to explain that a square *"has four sides with pointy edges"* (q7-8) and had to have four *"straight"* sides (q2). When the orientation of the square was changed (q7), Nicole again called it a *"dianglar"* because *"it's like a diagonal rectangular"* which has four sides but is *"crooked"*. Nicole was unable to give a name for the trapezium presented at question 8 but noted that it was *"pointy"*.

Case Study 5: 'Sam'

Sam used the words "sqaure", "diamond", "rectangle" and "dieagnal rectangle" to name the figures presented in items 1-4, respectively. Item 1 was a square "because it has four sides and four corners"; item 2 was a diamond "because it is a turn up side down square"; item 3 was a rectangle "because it is a wider sqaure"; and item 4 was a "dieagnal rectangle" because "it is ~~short~~ diffent to the other ~~Oo~~ One because it is tiped over" At item 5, Sam noted:

I realised they all have four sides and four corners And One of them is not strate (straight). Two of them ar both sqaure shaped. Except One of them are diamand shaped. Not all of them are eaqual achaul (actually).

In the first part of the interview, Sam explained that the figure described as "not strate" was that presented in item four. After much discussion it became clear that Sam was trying to describe that the oblong in non-standard orientation was the only shape on the questionnaire which did not have a vertical axis of symmetry, though these terms were not used by the student. The squares in items 1 and 2 were both seen to be *"square shaped"* but the square in a non-standard orientation (item 2) was still *"diamond shaped"*. The reference to not all of the shapes being equal was explained by Sam to be a comparison between squares and rectangles - both have four sides but the sides are *"not all the same"*.

Sam also explained his descriptions of the diamond and the rectangle (items 2 & 3). He explained his description of the diamond as a "turn up side down sqaure" to mean that *"it's got four sides and corners like a square but it's been turned upside down"*. Similarly, the description of the rectangle as a "wider sqaure" meant that *"it's got four sides and corners but it's wider than the square"*.

In the second part of the interview, Sam drew a square (q1) and was able to recognise and name the sample square and the standardly oriented squares (q4, 7 & 8). He continued to describe a square as having *"four sides and four corners"* (q2-4, 7-8) and noted that the sides also had to be *"straight"* (q5). When the orientation of the square was changed in question 7, Sam continued to call it a *"diamond"*. He was unable to give a name for the trapezium (q8) but observed that *"it still has four sides"*.

Case Study 6: 'Sean'

Sean used the words "square", "diamond", "rectangle" and "oblong" to name the figures presented in items 1-4, respectively. Item 1 was a square "because it a (has) four sides and angles and it has all sides the same"; item 2 was a diamond "because if you know what a square is this is a diagonal square"; item 3 was a rectangle "because it has two sides shorter then top and bottom"; and item 4 was an oblong because "it is the same as the rectangle except the shorter parts are at the top". Item 5 allowed Sean to express that if the diamond was "cut in half you would have an isosceles triangle", that all four figures presented could be used together to make a picture of a man, and that if you took a square and "turned it diagnaly (diagonally) you would have a diamond".

In the first part of the interview, Sean explained that his description of the diamond (item 2) as "a diagnal square" included the previous description he had written of a square in that the diamond has *"four sides the same and four angles but it's on a diagonal"*. Similarly, his description of the oblong (item 4) as "the same as the rectangle except the shorter parts are at the top" meant that *"it still has four sides but the top and bottom are shorter than the sides"*.

In the second part of the interview Sean drew a square (q1). He also recognised and named the sample square and the squares in standard orientations (q4, 7 & 8). His description of a square as having *"four sides the same and four angles"* remained constant (q2-5, 7 & 8) but at question 5, Sean added that the *"sides are straight"* and the *"angles all look the same"*. When the orientation of the square was changed at question 7, Sean named it a *"diamond"* but continued to move around the shape on the floor. When he sat down, he said *"you would think it's a square"* (and from my perspective the square was standardly oriented), though to Sean it was still *"really a diamond"*.

Sean named the trapezium (q8) a *"squashed square"* because it *"looks like someone has pushed it down"*. Although Sean was able to explain that the sides of a square had to be equal or the *"same"*, the addition of a side-length measurement at question 6 prompted no further explanation of this equality.

DISCUSSION

The case studies above illustrate effectively the enormous range of responses which Year 2 children provide when answering questions about and discussing simple 2D shapes. Of the six cases reported, only one student was unable to explain or describe any features of the figures presented. All of the five remaining students recognised that a square has four sides. From this point, however, explanations and understandings diverge.

While Rhimi was able to explain that a square has *"four sides"*, Angela, Nicole, Sam and Sean all made some reference to the fact that those sides also had to be *"straight"*. Only Sean, however, was able to articulate that the sides of a square are *"the same"*. Even Sean, however, was unable to interpret the implications on the length of the sides of a square given that one was 3cm long (q5). Thus, while these students all understand that a feature of a square is that it has four sides, the development of the individual aspects of 'straightness' and 'equality' which is characteristic of those sides provides us with insight into the complexity of their understandings.

The word *"straight"* was used by four students to describe sides. One of these students, Sam, also used the term "not strate (straight)" to convey his idea that the oblong presented in item 4 differed from the other three items on the questionnaire in that it did not have a vertical axis of symmetry. This highlights the necessity of interviews in the correct interpretation of students' written responses.

Another area in which considerable diversity was observed involved the concept of angles. Four students were able to combine the number of sides and the existence of angles in their descriptions of a square. The idea of angles first emerged in Nicole's description of the "pointy edges" which are *"at the end of the sides where they join"*. Sam and Angela both spoke of *"corners"*, but only Sean used the term *"angle"*. Sean was also the only student to acknowledge that the angles in a square also *"all look the same"*.

Students' descriptions of *"diamonds"*, *"rectangles"* and *"oblongs"* during interviews highlighted the supplementary nature of their written responses. If the student had written a description of a shape and then referred to another shape as being "like" it, the student often did not see the need to repeat themselves. For example, Sam wrote that a square "has four sides and four corners" and that a rectangle is a "wider sqaure". When interviewed, Sam explained that a rectangle is a "wider sqaure" because it has *"got four sides and corners but it's wider than the square"*. Similarly Sean's description of a diamond as "a diagnal square" included his description of a square and therefore meant that the diamond has *"four sides the same and four angles but it's on a diagonal"*. While these two examples come from students who were able to provide comparatively more sophisticated responses than their peers, it appears that it is not only these high ability students who may see their descriptions as cumulative. For example, Nicole described her "rectangular" as "big and long" in the written questionnaire but clarified this by saying *"it's like a big long square"* in her interview. While this does not add any new features to the description, it does clarify it. This implies that Nicole sees links between squares and *"rectangulars"*. This issue of a number of written descriptions having a cumulative nature is not unequivocal, however, and may be traced to the working memory demands, which a particular task places on students. For example, "a rectangle is a long square" may not imply that the student is aware of any features of either shape.

Interestingly, no students gave the same name to the two squares presented in different orientations (items 1 & 2), and only one student, Rhimi, gave the name *"rectangle"* to both oblongs presented (items 3 & 4). Four students used the name *"diamond"* to describe the square in non-standard orientation even after they had saw the entire square moved in front of them, of these, three students made some reference to the diamond being *"like"* a square or *"square shaped"* but *"on a diagonal"* or *"like a square but it's been turned upside down"*. Some students used non-conventional names of their own composition that indicated the similarities between figures presented and their conviction that, due to their orientation, they were still *"not the same"*. These non-conventional names include Nicole's use of the word *"dianglar"* for the square in a non-standard orientation (item 2), and the use of *"diagonal rectangle"* by both Angela and Sam for the non-standardly oriented oblong (item 4). Three students consistently used the word *"rectangle"* for the oblong in standard orientation, and the word *"oblong"* for the oblong in a non-standard orientation. Again, these students saw similarities between the two figures but were not convinced that they were the same shape and, as such, they could not be given the same name. This difficulty, with orientation being seen as a feature of the shape itself, was clearly illustrated by Sean in his interview. At question 5, Sean named the square in standard orientation as a *"square"* but when the orientation of the figure was changed he named it a *"diamond"*. After viewing the *"diamond"* from various locations, Sean remarked that *"you would think it's a square"* (due to my different perspective) even though it was still *"really a diamond"*.

CONCLUSION AND IMPLICATIONS

This study has illuminated the diverse range of responses to questions on 2D shapes provided by the sample of Year 2 students. The discussion above has emphasised key aspects of this diversity. However, this study has also brought to the fore two issues, which require a deeper analysis. These issues appear central to our knowledge of how young children acquire geometric competencies.

First, the descriptions "four sides and four corners" and *"four sides the same and four corners"* while not contradicting the developmental descriptors hypothesised by Pegg and Davey (1998), are not easily accommodated into their model.

The second issue concerns the possible differentiation between global attributes which may build towards a Level 1 understanding and features which are building blocks towards the notion of a property at Level 2A. The clarification of this issue is of fundamental importance to the appropriateness of the van Hiele model if it is to be relevant in describing the development of young children's geometric understandings.

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NOVICE STUDENTS' CONCEPTUAL KNOWLEDGE OF STATISTICAL HYPOTHESIS TESTING

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Examination of the statistical literature shows that consensus on definition, terminology, and interpretation of some hypothesis testing concepts is elusive. This makes hypothesis testing a difficult topic to teach and learn. This paper reports on the results of a study of novice students' conceptual knowledge of four hypothesis testing concepts through talking aloud and interview methods. While some students seemed to have a reasonable understanding of some concepts, many students seemed to have more limited understanding. The study explores students' faulty conceptual knowledge.

It is widely recognised that statistics, and in particular hypothesis testing, is a difficult subject to teach and learn (Garfield & Ahlgren, 1988; Hawkins, 1991). In fact, the statistical literature shows evidence of misconceptions at all ages and levels of expertise. It is believed that the nature of the subject itself presents problems because "*the important concepts of statistics are quintessentially abstract* [italics in original]" and open to interpretation (Watts, 1991, p. 290), and are "unlike anything the student has thought of before" (Garfield & Ahlgren, 1986, p. 271). Moreover, it has been suggested that past teaching practices have contributed to difficulties because of an overemphasis on formulae and routine techniques (Ehrenberg, 1990), which encourage rote learning and computation, rather than reflection and interpretation (Hawkins, Jolliffe, & Glickman, 1992).

A study of the literature on particular concepts in hypothesis testing demonstrates that many inconsistencies associated with definitions, terminology, and interpretations exist. With respect to definitions, Truran (1998) observed a lack of consistency and comprehensiveness in his study of text book definitions of the null hypothesis. Freund and Perles (1993) found four different definitions of p-value. With respect to terminology, level of significance, for example, has been termed significance level or *alpha*, and is represented by the symbol a or (less commonly) by the symbol $P_{critical}$ (see Thompson, 1994). P-value has been referred to as "significance probability," "probability level" (Huberty, 1985), "*p*" (Carver, 1978), "$P_{calculated}$" (Thompson, 1994), "prob-value," "tail probability," "P," "P-value," and "descriptive significance level" (Freund & Perles, 1993). Interpretation of concepts presents an even greater problem. For example, there are many stances regarding the issue of "acceptance" of an hypothesis (e.g., Frick, 1996; Hagen, 1997; Serlin, 1993). Gigerenzer (1993) reported finding nine different interpretations of the level of significance in a single textbook. Misinterpretations of this concept were attributed to confusion of the conditional probabilities associated with it (Falk, 1986). In particular, Carver (1978) criticised the interpretation of p-value as the probability of the sample result occurring by chance, the "Odds-Against-Chance Fantasy". In addition, misinterpretations of the significance concept have included perceiving it as a way to establish validity or reliability (Menon, 1993), a means of providing confidence (Chandler, 1970) or importance (West, 1990), or as the goal of research (Carver, 1993).

In summary, the literature highlights the range and complexities of problems associated with defining, representing, and interpreting hypothesis testing concepts, but provides relatively little empirical research on students' understanding of these concepts. It would be expected that students have some knowledge of the issues discussed in the literature in order to gain conceptual understanding of hypothesis testing. This paper reports the results of a study of novice students' conceptual knowledge of hypothesis testing concepts after the completion of a semester's study of statistics.

THE STUDY

Eighteen volunteer students from a large class enrolled in a university-level introductory statistics subject were interviewed after their final examination in the subject. The aim was to explore their knowledge of elementary hypothesis testing (one- and two-sample z and t tests only), a major component of their subject. These students were asked to talk aloud while they worked through three tasks, a Concept Mapping Task and two typical Hypothesis Testing tasks. In the Concept Mapping Task, students were provided with a number of labels marked with concept names associated with hypothesis testing. They were asked to arrange them in such a way as to show the relationships between them, and were encouraged to label the connecting links. *Typical* Hypothesis Testing tasks meant that summary data was provided, numerical calculations could be performed, and data exploration was limited. After the completion of each task, a semistructured interview was conducted in order to encourage the expansion and exploration of ideas touched on during the performance of the tasks themselves.

This paper reports on the students' conceptual knowledge of hypothesis testing evident from the talking aloud tasks and the individual interviews. Given the large number of concepts associated with hypothesis testing, the results reported here concentrate on four major concepts, namely hypothesis, significance level, p-value, and significance. Conceptual knowledge is defined as (a) the knowledge about concepts (revealed in definitions, examples/non-examples, and discussion of issues, features, properties, uses, or limitations associated with concepts); and (b) the knowledge about relationships between and among the concepts. It is assumed that the greater the knowledge about the concepts and their integration with others, and the greater the number and strength of these relationships, then the better the level of conceptual knowledge is, and the more accessible it is for higher-level thinking (Chi & Koeske, 1983). As students found difficulty applying linking words between the concept labels, it was the students' dialogue, rather than the concept maps, that was more useful in revealing their conceptual knowledge. As only half of the students completed the first Hypothesis Testing Task (requiring a large one-sample z test), and two students the second (requiring a small two-sample t test), evidence of students' conceptual knowledge again depended very much on what they said, rather than what they did during the tasks.

FINDINGS AND DISCUSSION

Initial examination of the talk aloud and interview results showed that while some students had a reasonable conceptual knowledge of some concepts, many seemed to have a more limited understanding. A widespread difficulty was associated with the expression of statistical ideas. These initial findings were examined further, influenced by the studies of McKeown and Beck (1990) and Perkins and Simmons (1988), who each characterised novices' knowledge of different subject areas, believing that students' problems could not simply be attributed to a lack of knowledge. For example, novices often have conceptual gaps in their repertoires of knowledge, but these repertoires also contain much incorrect knowledge.

Table 1 summarises the main responses relating to conceptual knowledge of the four concepts. Predominantly, students' protocols were definitions or statements about relationships. Sometimes, definitions were expressed in terms of relationships. In Table 1, acknowledgement of an idea meant that a student offered a definition of the particular concept, and this was classified as correct, partially correct, or incorrect. A correct classification meant that the idea was conveyed with statistical accuracy, for example, "you could have as the null hypothesis the mean of the population being equal to 180... so the alternative hypothesis could be that the mean equals less than 180" (correct example). A partially correct classification usually included statistically imprecise information or gaps in the student's knowledge about the concept, for example, "p-value is the probability of statistic correct if null hypothesis is correct" (partially correct definition). An incorrect classification meant that a student had the wrong idea, possibly a misconception, for example, "I would say the significance level is ultimately say a variance ... an ultimate line on the *x* and *y* axis" (incorrect definition). Acknowledgement of a relationship meant that a connection between a particular concept and another one provided on the Concept Mapping Task was noted by the student, for example, "p-value is a probability" (relationship between p-value and probability, no elaboration). When the connection was elaborated upon, and not simply acknowledged, this too was rated as correct, partially correct, or incorrect (with the same meanings as before). For example, in the following statement, "the statistic might be very significant if you have a high p-value," the significance and p-value concepts are linked incorrectly (relationship with incorrect elaboration).

Table 1
Summary of Number of Acknowledgements of the Major Ideas and Relationships, and their Classifications

Concepts		Number of Acknowledgements	Classification of Acknowledgements or Elaborations			
			Correct	Partially Correct	Incorr.	**Totals**
Hypothesis	Major Ideas	**71**	17	48	6	**71**
	Relationships	**76**	19	49	7	**75**
Significance level	Major Ideas	**34**	13	18	3	**34**
	Relationships	**32**	8	13	6	**27**
P-value	Major Ideas	**19**	0	3	5	**8**
	Relationships	**34**	11	7	6	**24**
Significance	Major Ideas/ Relationships	**16**	1	8	1	**10**
TOTALS		**282**	**69**	**146**	**34**	**249**
				249		

Table 1 exhibited three major features. First, the best-known concept was the hypothesis concept. Collectively, there were more statements made about the hypothesis concept (71+75) than any other concept in the table. This may account for the poor responses on the Hypothesis Testing tasks, many of which did not progress beyond the hypothesis statements. The table shows that the students found the other concepts considerably more difficult to discuss, with the total number of statements becoming fewer for concepts progressively lower in the table. The least known concept was the significance concept, and information contributed about this concept was mainly in the form of relationships. Several students could contribute no information at all about a concept (respectively 3, 3, and 7 students for significance level, p-value, and significance).

Second, there were more "Acknowledgements" classifications than "Correct," "Partially Correct," or "Incorrect" classifications (282 versus 249). This difference was evident mainly in the Relationships statements, but also in the Major Ideas associated with the p-value and significance concepts. The implication from this finding was that while students had some sense of the concepts and those with which they are related, on many occasions they had difficulty elaborating the nature of the relationships.

Third, "Partially Correct" or "Incorrect" statements were more common than "Correct" ones (146+34 versus 69). This implied that even when students could discuss an idea or elaborate upon a relationship, it was rarely statistically accurate.

In order to understand the nature of students' conceptual difficulties with the above concepts, further analysis was undertaken. When statements about a particular concept were not classified as "Correct," the student's responses associated with that concept were examined across all tasks. A combination of the categories used by McKeown and Beck (1990) and Perkins and Simmons (1988) was found to be useful in characterising students' faulty knowledge. It was found that in this study, faulty conceptual knowledge could be classified as *incomplete knowledge* (no elaboration, gaps in knowledge), *garbled knowledge* (knowledge about one concept is mixed up with knowledge about another concept), *more sophisticated knowledge* (but with some errors), and *misconceptions* (wrong ideas). Another difficulty, lack of precision in expressing statistical ideas, was usually classed as garbled knowledge, but sometimes it could also be associated with incomplete knowledge. That is, it was sometimes difficult to determine conclusively which category was most appropriate when students were statistically imprecise in their explanations. Table 2 summarises these classifications, provides a comment explaining each one, and gives the number of students with each type of problem for all four concepts.

Table 2

Conceptual Problems Associated with Hypothesis Testing Concepts and the Number of Students Exhibiting the Problem for Each Concept

Problems and comments	Concepts			
	H	SL	P	S
Incomplete knowledge	18	12	8	11
• no qualifications in using words such as "accept", "prove", "true", "opposite"	*			
• lack of precision in expressing statistical ideas	*			
• no elaboration	*	*	*	*
• gaps in knowledge	*	*	*	*
Garbled knowledge	8	13	10	5
• confused associations	*	*	*	*
• lack of precision in expressing statistical ideas		*	*	*
• confusing statistical language			*	
• problems with interpreting numbers		*	*	*
More sophisticated knowledge	6	7	3	0
• generally good conceptual knowledge, but with some lapses in the above (often expression)	*	*	*	
• visual representations			*	
Misconceptions	5	2	1	0
• incorrect ideas	*	*	*	

Note: H – hypothesis; SL – significance level; P – p-value; S – significance; * - difficulty evident for this concept

Incomplete knowledge was a major problem for the hypothesis and significance concepts, slightly less so for the significance level and p-value concepts. For the hypothesis concept, two concerns were that words such as "accept", "prove" and "true" were unqualified, and that statistical expression concerning the null and alternative hypotheses and the relationships described with other concepts were poor. Both are perhaps unavoidable in novices, because of the newness of the language and the abstract nature of the concepts. However, instruction should highlight the statistical inaccuracies associated with the first concern in particular, because it is part of the knowledge pertaining to the hypothesis concept. With respect to the second concern, imprecise statistical expression detracted from the ideas being presented, and usually consisted of partially correct statements. For the hypothesis concept, this usually meant that relationships were not initially explained correctly, nor were they enhanced through additional explanations or actions at another time. This meant that there were gaps in the knowledge base. For all concepts, lack of elaboration of ideas, and gaps in knowledge, were problems associated with incomplete knowledge. This usually meant that the particular concept was underdeveloped, or there was no evidence of any knowledge about it at all. Furthermore, gaps in knowledge usually meant that the concept was not included on the students' concept map.

Garbled knowledge was the main problem associated with the significance level and p-value concepts, less so for the hypothesis and significance concepts. One difficulty associated with garbled knowledge was confused associations between concepts. Examples include failure to differentiate between the separate roles of the p-value, z value, and t value, and mixing the meaning of significance level with significance, p-value, critical region, and confidence interval. This may be a result of the number of concepts associated with hypothesis testing, whose definitions and roles become jumbled together because of the pace of the subject and the large amount of content in statistical subjects. Care must be taken by instructors that text books do not cause this confusion through inadequate definitions. Again, problems with the language of statistics may be avoided by creating an environment where statistical ideas and concepts are openly discussed.

Another difficulty associated with garbled knowledge was imprecise statistical expression. When this was included in garbled knowledge, it usually meant that relationships were not initially explained correctly, yet at some later stage in the interview, a student showed understanding of the relationships. This normally occurred through performance on the Hypothesis Testing Tasks or through demonstration on a distribution diagram, a visual representation. As the Hypothesis Testing Tasks presented a major problem to many students, this avenue for explaining or highlighting relationships was closed to them. Hong and O'Neil's (1992) study on statistics students concluded that emphasising distribution diagrams in instruction facilitated students' understanding of the concepts. The present study provides some support for this conclusion, particularly in its facility for clarifying relationships between concepts when language is inadequate.

Confusing statistical language was another difficulty included in the classification of garbled knowledge, and this was concerned with the different meanings associated with extreme ideas such as high p-values, high significance, and high significance level. Visual representation in the form of distribution diagrams would assist in establishing the differences between the three concepts of p-value, significance, and significance level. Creating an environment for discussing concepts could help reduce such incorrect usage.

Garbled knowledge also included evidence of difficulties associated with interpreting the numerical values, and this applied mainly to the p-value concept. One example suffices as an illustration: "if it's very close to zero then it's good, but if it's getting, usually over the 20 mark, above that I think is kind of shady on whether to accept it or not." Undoubtedly, this difficulty is linked to other difficulties such as confused associations with other concepts.

More Sophisticated Knowledge was present for the hypothesis, significance level, and, to a lesser extent, the p-value concepts. It meant that students demonstrated a better grasp of the concept than other students, particularly in terms of the elaboration of the relationships with other concepts. Nevertheless, there were still occasional lapses in statistical precision and incidences of garbled or incomplete knowledge. For example, for the p-value concept, students had difficulty explaining the link with probability, but students in this category represented p-value diagramatically, aptly demonstrating and explaining the numerical values associated with p-value.

Misconceptions were the least common problem. It is arguable whether some of these really were misconceptions, or just erroneous attempts to say something to the researcher at the time. For the significance level concept, one misconception was associated with its definition as the probability of being wrong, and another as being the same as Type I error. These misinterpretations could be a result of confusion of the probabilities associated with the concept (Falk, 1986). For the p-value concept, the single misconception was associated with the contention that p-value is always low. This misinterpretation existed because the examples this particular student actually performed, or was shown as models, resulted in low p-values. Such a misconception can easily be eradicated through carefully chosen problems which result in a mixture of low and high p-values, and through the discussion of such results.

Given the lack of consensus on definitions, terminology, and interpretations cited in the statistical literature, it is hardly surprising that many problems of this nature exist in novice students after a one-semester course of statistical study. In response to why students do not seek out ways of improving their knowledge, Perkins and Simmons (1988) suggested that students' inquiry frame is not cultivated. First, they are often not encouraged to seek out and ponder upon problems that might improve their knowledge. Second, the applications of the topic to be learned are situated solely in an academic environment where students master text book abstractions. Third, there is little encouragement to develop knowledge beyond the boundaries of the content area with "What if?" and "What if not?" questions.

CONCLUSIONS

The results in this study revealed that students have a major problem in expressing statistical ideas with accuracy (see Table 1). Sometimes this masks their otherwise good conceptual knowledge of hypothesis testing concepts, and at other times it is representative of their inadequacies. A solution to this problem by lecturers may mean facilitating situations (seminars, reflections, discussions, explorations, critical evaluations, writing tasks, concept mapping) where students have no option but to conquer their fears about statistics and immerse themselves in the utilisation of statistical language. This would encourage the organisation of ideas, and improve the clarity of explanations. As Knapp (1996) noted, one of the annoying things about hypothesis testing is the jargon associated with it, and the sooner it is taken on board the better.

Furthermore, concept mapping proved to be useful for investigating conceptual knowledge. It was effective in studying how people link and structure ideas and relationships relating to concepts, in highlighting the main problems, and in establishing whether key concepts in a topic were known. These functions make the concept map a powerful tool. In particular, it can invoke discussion about concepts and their relationships, especially when combined with talking aloud and interview methods. However, as indicated above, the concept map used on its own may not be as useful, because students have difficulty adding the linking words on the map itself. Its true potential is realised when combined with other methods. Then it also becomes a powerful learning tool.

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STUDENTS' OPTIMISM, PESSIMISM AND ACHIEVEMENT IN MATHEMATICS: A LONGITUDINAL STUDY

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Optimism, pessimism and achievement in mathematics in primary and lower secondary students were measured on two occasions separated by almost three years. The students' Grade level and gender were also considered. The hypothesis that, relative to more pessimistically oriented students, those with a more generally optimistic outlook on life would evidence a higher level of achievement in mathematics was confirmed. Students' Grade level and gender were also significant variables. The implications of these findings for mathematics education are discussed.

INTRODUCTION

The influence of attitudes, values and personality characteristics on achievement outcomes and later participation in the learning of mathematics are important considerations for mathematics educators. The *National Statement and Profile on Mathematics for Australian Schools* (AEC, 1990) stresses attitudes and appreciations. The development of optimism is included in the second significant educational aim for all students in the 1989 *National Goals for Australian Schooling* (The Hobart Declaration).

For centuries optimism, the belief that positive events outweigh the negative, has been regarded as a natural, normal and healthy personality attribute. Differences between optimists and pessimists abound within popular culture, literature and humour. Recently, studies have established that these differences encompass significant aspects of personal adjustment and influence motivation, learning and health (Peterson & Bossio, 1991). Optimistic and pessimistic tendencies, referred to as an explanatory style, are formed during childhood (Nolen-Hoeksema & Girgus, 1995; Yates, 1998) and impact on children's health, motivation and achievement (Seligman, 1990). In the present study, relationships between students' optimism, pessimism and achievement in mathematics were explored over a period of almost three years. In a pilot study, explanatory style was implicated as a determinant of mathematics performance (Yates, Yates & Lippett, 1995), but it was unclear whether this relationship was stable or predictive. More substantive, extensive data were therefore collected, using a longitudinal approach so that these relationships could be investigated more fully.

Explanatory style has been defined as the trait-like characteristic way in which people explain the causes of events in their lives, particularly when these events are negative and occur under ambiguous circumstances (Seligman, 1990). Optimists attribute good events to permanent, personal and pervasive causes and bad events to unstable, external and specific causes. For pessimistically oriented people, bad events are attributable to causes which are universal, unchangeable and due to their ineptitudes and good events to factors that are transitory, specific and due to outside factors such as luck. Explanatory style interacts with achievement to create self-fulfilling prophesies, which either enhance or undermine performance (Seligman, 1990; Schulman 1995). Explanations that individuals habitually make for their successes and failures lead to expectations that affect their reactions to future successes and failures. In turn, these expectations affect performance through a variety of behaviours. Individuals with a more optimistic explanatory style are more likely to take the initiative, persist under adversity, take risks, be decisive, engage in quality problem-solving strategies and be more assertive (Schulman, 1995).

Since 1980, research has supported the theory that explanatory style is related to achievement in various domains including education, work, and sports (Schulman, 1995). In the workplace people with an optimistic explanatory style have greater productivity than pessimists (Seligman & Schulman, 1986). The deleterious effects of a pessimistic style have been implicated also in studies of athletic performance and illness (see Seligman, 1990) and academic performance, with Peterson and Barrett (1987) reporting a correlation of 0.36 between tertiary academic grades and negative attributional style. In mathematics, a small but significant correlation was found between college students' pessimistic style and final grade in algebra ($r = 0.08$, $p < 0.05$) (Pierce & Henry, 1993).

Peterson, Maier and Seligman (1993) have asserted that after depression the best known influence of explanatory style is on achievement. However, surprisingly few studies have actually examined this phenomenon in school-aged students. Two studies have investigated relationships between explanatory style and general academic achievement. Significant moderate correlations were found between concurrent measures of academic achievement, as assessed by the *California Achievement Test* (California Testing Bureau, 1982) and explanatory style ($r = 0.26$, $p < 0.05$) (Nolen-Hoeksema, Girgus & Seligman, 1986). In a five year longitudinal study, a weak relationship was found between primary school children's explanatory style and academic achievement measured six months later, with pessimistic children being somewhat less successful on standardised achievement tests (Nolen-Hoeksema, Girgus & Seligman, 1992; Nolen-Hoeksema & Girgus, 1995). Boys were consistently more pessimistic than the girls, particularly in terms of their explanations for negative events. Relationships over the longer term were not examined.

Relationships between explanatory style and achievement in mathematics has been explored in a single pilot study of 145 primary school-aged students (Yates, Yates and Lippett, 1993; 1995). Significant correlations were found between students' optimistic or pessimistic explanatory style and concurrent achievement in mathematics ($r = 0.20$, $p < 0.05$ and $r = -0.30$, $p < 0.01$ respectively). Pessimism also correlated significantly with students' achievement in mathematics two and three years previously ($r = -0.29$, $p < 0.05$ and $r = -0.35$, $p < 0.05$ respectively). The girls were significantly more optimistic ($F(1, 130) = 4.31$, $p < 0.05$) and less pessimistic ($F(1,130) = 13.66$, $p < 0.001$) than the boys.

Children develop characteristic patterns of explaining causes of events from the myriad of experiences of their lives (Peterson and Bossio, 1991). In their everyday interactions in the classroom all students encounter successes and failures. Students who have developed pessimistic cognitive frameworks are at risk for doing less well in school (Seligman, 1995) as they are likely to view failure as being a permanent state over which they have no control. Furthermore, failure in a subject area is likely to pervade all other aspects of their schooling. For such students, failure is not seen as part of the fabric of learning but is likely to be recast, leading them to expect further negative outcomes, thus setting up a vicious circle. This is particularly likely to occur in mathematics where such strong views about the nature of the subject matter abound.

In Western cultures people often hold very strong negative views of mathematics, with many considering it a subject to be feared. Studies in the United States have reported repeatedly that people believe that they cannot do mathematics, as the subject is too difficult (McLeod, 1992), a belief more commonly espoused by females. People not only admit readily to failure in mathematics, but also see it part of a familial pattern and therefore outside of their personal control (McLeod, 1992). Furthermore, failures in school mathematics are often interpreted as a confirmation of a lack of ability, with the very fact of having to make an effort to master mathematics further proof of this lack of accomplishment. Many students in the United States prefer to not try than to try and fail (Covington & Omelich, 1979).

In a longitudinal study over a period of 12 months, in a causal model earlier attitudes towards mathematics were related to later achievement in mathematics (Keeves, 1972; 1986). Clearly beliefs that students hold about themselves and about mathematics play an important role in their achievement in mathematics in the United States, particularly when these perceptions lead students to expect to do well (McLeod, 1992). While there has been some indication that in comparison with males, female students are more likely to attribute their success to effort and their failure to lack of ability, gender differences in attributions in mathematics have not been clear cut (McLeod, 1992).

These equivocal findings between gender differences in the United States in selected internal belief variables and gender differences in learning mathematics suggest the need to consider gender in this study. If boys in primary school are more pessimistic than girls, and if girls become more pessimistic as they enter their teenage years, as suggested by Nolen-Hoeksema and Girgus (1995), then this might have differential effects on their achievement. Of particular interest in this study therefore, is not only whether the findings of Nolen-Hoeksema *et al.* (1986) would be affirmed for the concurrent measures of achievement in the specific subject area of mathematics, but also whether explanatory style would be related to subsequent achievement in mathematics. In general, attitudes towards mathematics appear to be related to age (Fraser, 1980), so Grade level too is an important variable.

The use of a longitudinal design facilitates the examination of relative gain over time and the factors that are hypothesised to influence both this gain as well as change over time. In order to investigate relative rather than absolute change over time, and to identify some of the factors that influence this relative change, two student measurement points were planned, with an objective measure of achievement in mathematics and student self-report indices of explanatory style collected on both occasions.

METHOD

The fundamental hypothesis of this study is that possessing an optimistic explanatory style leads to enhanced achievement levels in mathematics. Thus optimistic children are more likely than pessimistic children to show increases in their relative mathematical achievement levels over time. The aims of this study therefore are:

1. to examine the relationships between students' optimism, pessimism and achievement in mathematics over time; and
2. to consider the influence of student's gender and Grade level on these relationships.

Subjects The study commenced in Term 1, 1993 with a non-random, non-representative sample of 335 students in Grades 3 to 7 in two government primary schools in South Australia. In Term 4, 1995, 243 of these students were traced to 26 primary and 24 lower secondary schools in the government and non-government sectors. The Grade level and gender of the 243 students in 1995 is presented in Table 1.

Table 1
Numbers of Students by Grade Level and Gender in 1995

Gender	Grade 5	Grade 6	Grade 7	Grade 8	Grade 9	Total N
Male	8	28	21	28	24	109
Female	10	34	22	38	30	134
Combined	18	62	43	66	54	243

Mathematics Achievement Test Students' achievement in mathematics was measured with Form A of the *Progressive Achievement Tests in Mathematics* (*PATMaths*) (ACER, 1984) which consisted of three timed standardised multiple choice format tests. In 1993 students were administered Test 1, 2 or 3 in accordance with the recommendations made in the *Teachers Handbook* (1984). In 1995 students in Grades 5, 6 and 7 were administered Test 1 or 2, Grade 8 Test 2 or 3 and Grade 9 Test 3.

Explanatory Style Questionnaire Students' optimistic or pessimistic explanatory style was measured by the *Children's Attributional Style Questionnaire* (CASQ), (Seligman, Peterson, Kaslow, Tanenbaum, Alloy, & Abramson, 1984). The questionnaire consisted of 48 hypothetical statements about good and bad events, with students choosing between two possible explanations for each event.

Procedure The *PATMaths* and the CASQ were administered to students in their own schools in Term 1, 1993 and again in Term 4, 1995.

RESULTS

As the *PATMaths* had been Rasch scaled (ACER, 1984) the results for all students could be placed on the single standard scale of achievement (*Teachers Handbook*, ACER, 1984), irrespective of the level or the time at which the students took the test. Rasch analysis of the CASQ indicated that it contained two separate scales of optimism (CASQCP) and pessimism (CASQCN). Students' results for the 1993 and 1995 CASQCP and CASQCN were equated concurrently (Yates, 1998).

Relationships between students' optimism, pessimism and achievement in mathematics in 1993 and 1995 were examined with correlational statistics with the Grade level and gender variables initially included and then omitted. In Table 2, weak but significant negative correlations were evident between the measures of optimism in 1993 and 1995 and achievement in mathematics in 1993 and 1995. While pessimism in 1993 was negatively correlated with concurrent achievement in mathematics at a significant but weak level, the 1995 CASQCN did not correlate with *PATMaths* in either 1993 or 1995. Gender was correlated significantly with pessimism in both years, but not with achievement.

Table 2
Correlations Between Mathematics Achievement, CASQCP and CASQCN in 1993 and 1995, Grade Level and Gender

N = 243	2	3	4	5	6	7	8
1. 1993 Maths achievement	0.74***	-0.18**	-0.17**	-0.14*	•	0.62***	•
2. 1995 Maths achievement	-	-0.19**	-0.17**	-0.21***	•	0.43***	•
3. 1993 CASQCP		-	0.35***	•	-0.15*	-0.15*	0.13*
4. 1995 CASQCP			-	•	•	•	•
5. 1993 CASQCN				-	0.32***	•	-0.19**
6. 1995 CASQCN					-	•	-0.19**
7. 1993 Grade level						-	•
8. Gender							-

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$, • correlation not significant

The nature and extent of the relationships between the 1993 optimistic and pessimistic explanatory style measures and subsequent achievement in mathematics were of particular interest since it had been hypothesised that a pessimistic explanatory style in 1993 would predispose students to poorer achievement in mathematics in 1995. These relationships were therefore tested with multiple regression analysis, with the Grade level and gender variables initially retained as presented in Table 3.

Table 3
Regression Analysis: Predicting Mathematics Achievement in 1995 by 1993 CASQCP, 1993 CASQCN, Grade Level and Gender

1995 Mathematics achievement				
Variable N = 243	r	Beta	t	Significance of t
1993 CASQCP	-0.19	-0.15	-2.58	0.01
1993 CASQCN	-0.21	-0.27	-4.81	0.00
1993 Grade level	0.43	0.43	7.66	0.00
Gender	-0.06	-0.08	-1.47	NS
Multiple R = 0.52		F = 22.47		
R square = 0.27		Significance of F = 0.00		

The 1993 CASQCN score was strongly related to subsequent mathematics achievement ($t = -4.81$, $p < 0.000$), as was the 1993 CASQCP ($t = -2.58$, $p < 0.01$). Grade level was also significantly related to achievement ($t = 7.66$, $p < 0.000$). This strong correlation was not unexpected as it was not only evident in Table 2 for both 1993 ($r = 0.62$, $p < 0.001$) and 1995 ($r = 0.43$, $p < 0.001$), but also had been found in correlational and regression analyses reported elsewhere (Yates, 1998). Over time students show incremental increases in achievement as they are exposed increasingly to the mathematics curriculum in schools (ACER, 1984). Gender was not a significant variable.

In order to consider the influence of the prior measures of explanatory style on subsequent achievement in mathematics, it was necessary to control for the influence of Grade level on achievement with partial correlation. Table 4 shows the negative correlation between the 1993 CASQCN and the 1995 achievement in mathematics ($r = -0.27$, $p < 0.000$) at a moderate level. The 1993 CASQCP was correlated also with achievement in mathematics in 1995, but this correlation was weaker ($r = -0.14$, $p < 0.05$) than that of the 1993 CASQCN.

Table 4
Partial Correlations between 1993 CASQCP and CASQCN and 1995 Achievement in Mathematics Controlling for 1993 Grade Level

Variables Df = 240	2	3
1. 1995 Mathematics achievement	-0.14*	-0.27***
2. 1993 CASQCP		-0.09
3. 1993 CASQCN		-

* $p < 0.05$ *** $p < 0.000$

The relationship between the 1993 CASQCP and CASQCN and achievement was then examined with direct entry multiple regression analysis, as shown in Table 5, with the Grade level and gender variables omitted. Both of the 1993 CASQ measures were related to mathematics achievement in 1995.

Table 5
Regression Analysis: Predicting Mathematics Achievement in 1995 by 1993 CASQCP and 1993 CASQCN

1995 Mathematics achievement

Variable N = 243	r	Beta	t	Significance of t
1993 CASQCP	-0.19	-0.22	-3.50	0.0005
1993 CASQCN	-0.21	-0.23	-3.72	0.0002

Multiple R = 0.30 F = 11.82
R square = 0.09 Significance of F = 0.000

Summary of the Results

Both positive and negative explanatory style in 1993 were related to achievement in mathematics in 1995. As this study commenced when all of the students were in primary school, it is evident that those who held a more pessimistic explanatory style in 1993 experienced a lower rate of achievement in 1995, as measured by the *Progressive Achievement Test in Mathematics*. While boys were significantly more pessimistic than girls in both 1993 and 1995, gender was not a significant factor in relation to achievement in mathematics. Grade level was a significant factor in achievement.

DISCUSSION

Schulman's (1995) assertion that explanatory style was related to achievement in a number of domains including achievement in school was borne out in this study. The finding of Nolen-Hoeksema *et al.* (1986; 1992) that explanatory style was related to achievement was not only confirmed, and extended to the specific subject area of mathematics, but it was also clearly linked, particularly for negative explanatory style, with subsequent achievement. While the pilot study had indicated a relationship between students' explanatory style and concurrent achievement in mathematics (Yates *et al.*, 1995), this study affirmed this relationship over time. Both the CASQCP and CASQCN in 1993 were significantly related to achievement in mathematics in 1995, with the CASQCN measure showing a slightly stronger effect than the 1993 CASQCP.

Seligman and other researchers have frequently claimed that optimistic children achieve more at school, and that conversely pessimistic children achieve less. While these claims were substantiated with correlational and regression analyses in this study, achievement in mathematics was strongly related to prior achievement in mathematics. Notwithstanding the magnitude of this relationship, explanatory style, particularly the CASQCN, was related to achievement in mathematics. Students who reported a pessimistic explanatory style experienced a lower rate of achievement, with boys consistently more pessimistic than girls over time.

Implications for Mathematics Education

While attitudes towards mathematics have been found to be related to achievement in mathematics (Keeves, 1972), the nature of the relationship between attitude and achievement remains largely unexplored. Explanatory style is a motivational characteristic that might conceivably impact upon a disposition to maintain effortful responding over extended time sequences. Skill development in the area of mathematics is likely to demand a continuing level of high motivation. The deleterious effects of a pessimistic explanatory style on academic performance at the tertiary level (Peterson & Barrett, 1987), and general achievement at school (Nolen-Hoeksema *et al.* 1986; 1992) were linked clearly with achievement in mathematics in this study.

Failure and success are considered to be highly salient in mathematics classrooms (McLeod, 1992). Students bring to the classroom a rich range of experiences which influence how they account for their successes and failures. While teachers have been exhorted to provide successful learning experiences in mathematics for their students (Berliner, 1987), the attributions that students make are equally important. In particular, pessimistic students who believe that their failures are likely to be long-lasting, to pervade all aspects of their work and to be due to their own ineptitudes need to be identified as early as possible in their primary school years, so that their trajectory towards poorer achievement can be interrupted and reversed. Intervention studies have been conducted for pessimistic adolescents (Peterson, 1988; Jaycox, Reivich, Gillham & Seligman, 1994) and college students (DeRubeis & Hollon 1995). This study would suggest that such interventions should begin when students are in primary schools and should target attributions in specific subject areas, particularly as attributions have been found to be subject specific (Marsh, 1986).

Mathematics teachers need to be cognisant of the attributions that they make about students' work, particularly in relation to failures. Seligman (1990) has suggested that teachers, together with parents and coaches, have an important part to play in children's development of an optimistic or pessimistic explanatory style. Since mathematics is a subject which is likely to have negative connotations for many students, teachers need to be aware of pessimistic students who attribute their mistakes and errors to permanent, personal and pervasive factors. Explanatory style is learned and is therefore changeable.

Future research could be directed at the measurement of optimism and pessimism as it is operationalised in the regular classroom. Interviews with teachers, students and their peers could not only be used to investigate their own explanatory styles but also that of each other. Such data would shed light on some of the factors that influence the development of optimism and pessimism and its relationship with achievement in mathematics .

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Acknowledgments

Deepfelt thanks are extended to Professor John Keeves for his expert guidance and supervision throughout the study, and to students, parents and staff from the 50 schools for their co-operation and participation. The study was supported by a Flinders University Research Board Establishment Grant

ABSTRACTS OF SHORT COMMUNICATIONS

MERGA 22
adelaide 99

MATHEMATICAL INVESTIGATIONS AS AN ALTERNATIVE ASSESSMENT MOTIVATOR: AN ISRAELI-BASED STUDY

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This paper describes a doctoral project, conducted nationally in Israel, which uses investigative tasks for assessment purposes. While elementary school teachers in Israel believe that understanding and thinking are their most important goals in teaching mathematics, many continue using traditional tests to assess mathematical understanding. Workshops were organised in order to encourage teachers to use investigative mathematics tasks as assessment tools. Evidence from the study indicates that teachers' participation in these workshops had an impact on their instruction and assessment.

A LONGITUDINAL STUDY OF STUDENTS' UNDERSTANDING OF TRIANGLE CLASS INCLUSION CONCEPTS

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Class inclusion, is an important characteristic of Level 3 thinking in the van Hiele theory. Researchers have highlighted the difficulties associated with understanding this idea, and have described class inclusion as a prerequisite for deductive reasoning. This study, involving in-depth interviews with 12 secondary students, considers the qualitative similarities and differences in students' attempts at linking and grouping seven different triangle types over a two-year period.

TEACHERS ASSESSING CHILDREN'S PERFORMANCES IN MATHEMATICS: HOW RELIABLE ARE THEIR JUDGEMENTS?

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The purpose of this investigation was to assess the degree of agreement between teachers when judging the arithmetical ability of young children on a performance-based assessment instrument. It was found that while there was some degree of inter-rater variability, it was not significant.

GRAPHICS CALCULATORS IN MATHEMATICS EDUCATION: A STUDY OF STUDENT LEARNING

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Although graphics calculators are being increasingly used in mathematics teaching, research has yet to identify fully those aspects of graphics calculators which best facilitate student learning. This short communication reports on some preliminary results of a research project in which clinical interviews were conducted with 25 Year 10 students as they used graphics calculators to study graphs. The interviews focused specifically on the procedural (remembering which buttons to press) and conceptual (interpreting the calculator's output) difficulties which the students experienced.

PRELIMINARY ASSESSMENT OF A TRIAL FLEXIBLE DELIVERY COURSE IN PRE-SERVICE TEACHER EDUCATION

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An initiative by Charles Sturt University (CSU) involves delivery of first semester subjects in the BEd (Primary) courses to students at both the home (Bathurst) campus and a new remote campus in Dubbo. The latter receive tuition via a recently established satellite video-conferencing facility by academic staff already involved in first semester teaching. Lecturers in the subject EMM111 (Exploring Numeracy & Technology) have supplemented flexible delivery through fortnightly visits to Dubbo and the establishment of a trial on-line version of the subject. EMM111 on-line has been enhanced by weekly postings of relevant weblinks (many interactive & dynamic) and a variety of electronic communication functions. Formative evaluation of the overall initiative was undertaken by CSU during the semester break (April, 1999), whilst specific evaluation of EMM111 will have commenced at the end of Autumn Session (June, 1999). Research dimensions of the pilot study include delivery mode impact, affective domain factors, cognitive growth and the relationship between presage, process and product variables.

CHILDREN'S VISUALISATION OF THREE-DIMENSIONAL SHAPES

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This paper describes a pilot study involving 30 students in Grades 1, 3 and 5, which investigated the students' ability to visualise three-dimensional shapes. In particular, the study examined whether, in their visualisations, students focus on critical or non-critical aspects of shapes and whether any differences exist between students' visual images, verbal descriptions and drawn representations. An interview-based assessment was devised to address the research questions. Results showed that although the quality of some students' visualisations may improve with grade level, non-mathematical or non-critical qualities feature strongly in most children's visualisations.

A STUDY OF THE IMPACT OF GRAPHICS CALCULATORS ON STUDENT ACHIEVEMENT IN TERTIARY ENTRANCE EXAMINATIONS IN MATHEMATICS

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In November 1998, students in Western Australia had access to graphics calculators when they sat for the Tertiary Entrance Examinations (TEE). This was the first time the use of graphics calculators was allowed in the TEE, and applied to all mathematics examinations and those in the physical sciences. A unique opportunity has been presented for a comparative study of student achievement using the results from the 1997 (pre graphics calculators) and 1998 (post graphics calculators) TEE that will provide data with relevance for teachers of mathematics at both the secondary and tertiary level. The central aim of the study is to establish the extent to which access to graphics calculators has impacted on student achievement in mathematics examinations in the TEE. Subsidiary to this is to identify the components of the syllabus and kinds of items for which the availability of graphics calculators seems of most benefit to students. An opportunity also avails to examine gender and student location (rural versus urban) differences in achievement pre and post introduction of the graphics calculators. Data relating to student achievement on the 1997 and 1998 TEE mathematics examinations are being obtained from the Curriculum Council of Western Australia. These are to be supplemented by those from a survey and interviews conducted with students enrolled in first year mathematics courses at Curtin University and who sat the TEE in 1998.

TEACHING NEGATIVE NUMBER USING INTEGER TILES

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The paper reports some of the findings of a study involving use of an integer tiles embodiment of the neutralisation model. Short and long-term learning outcomes of negative number teaching using integer tiles and normal number-line based teaching methods are compared. The integer tile model appears to be superior to the number line model, as currently interpreted and used in popular textbooks, for teaching of negative number operations and justifying sign rules.

MATHEMATICS EDUCATION FOR EARLY CHILDHOOD: A PARTNERSHIP OF TWO CURRICULUMS

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This paper describes the mathematics education component of one early childhood teacher education programme at the Auckland College of Education. It explores how the school curriculum can provide a meaningful framework for the mathematical learning of very young children. It identifies how the interweaving of this curriculum document with the New Zealand early childhood curriculum can support the early childhood teacher to maximise the provision of mathematically-rich environments. A variety of issues pertaining to the early emergence and development of mathematical concepts will be examined.

A CHANGE IN FOCUS: PROBLEM POSING AND PROFESSIONAL DEVELOPMENT

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This is a report of the second year of a three-year research project that aims to integrate problem posing into the elementary mathematics curriculum. Teachers attended two seminars a term: the first for debriefing and the second for reflection. In between seminars was the implementation period. During the period, teachers implemented problem posing, analysed children's work and mailed findings. Results indicated that when teachers were supported with resources and worked with colleagues, they were able to change as they act as researchers.

ADULTS' NON-OCCUPATIONAL MATHEMATICS: WANDT AND BROWN REVISITED

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A survey was administered to 187 adults of varying backgrounds. The subjects were requested to record mathematical calculations completed over a twenty four hour period. The calculations were analysed in terms of location, level of difficulty, purpose, type and frequency. Seven subjects were interviewed in relation to their survey results. The majority of non-occupational calculations made by adults were performed mentally, were completed inside the home, involved addition or subtraction and were related to time or money.

TEACHING MULTIPLICATION TO YOUNG CHILDREN USING MULTIPLICATIVE RATHER THAN ADDITIVE STRATEGIES

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The teaching of multiplication in the early childhood setting has traditionally taken the form of multiplying using repeated addition or equal groups. This reliance on addition to multiply may have implications in later years of schooling, for example in the development of understanding of decimals and the base ten system. In this session key ideas within the additive model will be compared with a model which treats multiplication primarily as change of measure. An analysis of the frequency and nature of the many types of multiplication situations found in teacher materials for the early years of school will be presented.

RESEARCH INTO VALUES IN MATHEMATICS EDUCATION: AN INTEGRATED THEORETICAL FRAMEWORK

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Several recent initiatives in education have emphasised the need to explore values and beliefs in education in general and mathematics education in particular. The beliefs and values one holds will effect the way in which one learns and teaches mathematics. This paper seeks to develop a research structure or framework that takes into consideration the most critical elements in the learning/teaching process from the perspective of values and beliefs.

DO CHILDREN REALLY HAVE A CHOICE WHEN CALCULATING?

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Year Four to Seven students from three primary schools in the South West of Western Australia were asked to give examples of questions they would complete mentally, with paper and pencil and with a calculator. The results add to the growing body of knowledge about computational choice.

AUTHOR INDEX
AND
TOPIC INDEX

MERGA 22
adelaide 99

AUTHOR INDEX

In this index, the first or single author quoted on the research report is in bold, and, for a report with multiple authors, a presenting author who is not the first author is underlined.

Albert 571
Asp 358
Baker 571
Bana 461
Barkatsas 200
Barnes 53
Baturo 60
Baturo 177
Begg 68
Bicknell 75
Bobis 84
Bobis 571
Boulton-Lewis 3
Bramald 91
Brew 98
Brinkworth 105
Brown 113
Burgess 120
Buzeika 128
Callingham 135
Cavanagh 572
Charles 143
Chick 151
Chinnappan 159
Chinnappan 167
Clarke 365
Cobb 351
Cooper 60
Cooper 177
Crameri 529
Dengate 572
Diezmann 185
Dix 192
Dole 43
Everett 572
Forgasz 200
Forster 207
Forster 396
Galbraith 215
Galbraith 223
Geiger 223
Gervasoni 231
Goos 238
Goos 223
Gould 84
Gould 571
Gunn 246
Hadfield 417
Haimes 573
Haines 215
Hannah 343
Hayes 573
Haynes 573
Heirdsfield 253
Higgins 91
Horne 261
Horwood 269
Howard 424
Hyde 275
Jones 501
Kanes 283
Keast 291
Keeves 29
Keeves 484
Kendal 358
Kidman 298
Kidman 60
Klein 306
Lawrie 312
Lawson 167
Leder 98
Leder 200
Leigh-Lancaster 320
Leung 574
Lowrie 328
Malone 571
Maguire 336
Mayo 343
McClain 351
McCrae 358
McCrae 506
McIntosh, A 574
McIntosh, J 574
McRobbie 60
Menzel 365
Miller 424
Millar 529
Mitchelmore 373
Moritz 380
Moritz 521
Mousley 388
Mueller 396
Mulligan 476
Nason 143
Nason 167
Nisbet 514
Northcote 574
Norton 404
Outhred 411
Pegg 417
Pegg 312
Pegg 545
Pegg 571
Pemberton 215
Perry 424
Pierce 430
Power 275
Renshaw 113
Renshaw 223
Routitsky 501
Rowe 320
Rowley 98
Shaw 411
Shield 438
Southwell 575
Sparrow 574
Sparrow 575
Stacey 445
Stacey 573
Steinle 445
Stillman 453
Sullivan 15
Swan 461
Swan 575
Swedosh 468
Thomas, M 159
Thomas, N 476
Tilahun 484
Tirta-Gondoseputro 493
Tobin 501
Vincent 506
Warren 514
Watson 521
Watson 380
Watts 529
White 537
Whitland 545
Williams 553
Williams 177
Yates 560
Zevenbergen 275

TOPIC INDEX

References are to the presenting or first author. The page location of the paper may be found from the Author Index. The official MERGA topic classifications are printed in **bold**; other classifications have been selected from the authors' suggested keywords, and are printed in ***bold italics.*** Authors of full Research Reports are printed in Normal; authors of Short Communications are printed in *Italics*.

Advanced Mathematical Thinking
Chinnappan
Galbraith
Mueller
Pierce

Algebra
Asp
Cooper
Hadfield
Horne
Malone
Swedosh

Assessment
Bobis
Brew
Callingham
Diezmann
Hadfield
Horwood
Lawrie
Leigh-Lancaster
Mueller
Perry
Tilahun
Warren

Beliefs
Brinkworth
Forgasz
Galbraith
Gervasoni
Norton
Shield
Southwell
Warren
White
Yates

Calculators
Asp
Cavanagh
Haimes
McIntosh, A.
Mueller
Swan
Tobin

Concept Maps
Chinnappan
Williams

Co-operative Learning
Barnes
Brown
White

Cross-cultural Studies
Forgasz
Tirta Gondoseputro

Curriculum
Dole
Gunn
Horwood

Early Childhood
Bobis
Gervasoni
Haynes

Gender
Brew
Brinkworth
Tilahun
Yates

Geometry
Baker
Chinnappan
Dix
Everett
Kidman
Lawrie
Mitchelmore
Vincent
Whitland

History
Horwood

Interactions—student/teacher
Kanes
McClain
Forster

Interactions—teaching/learning
Begg
Bragg
Burgess
Barnes
Kanes
Mitchelmore

Language
Bicknell
Goos
Hyde
Perry

Longitudinal Studies
Moritz
Steinle
Watson
Yates

Mathematics Education Research
Begg
Chinnappan
Keeves

Measurement
Kidman

Metacognition
Wilson

Number Concepts
Bobis
Bobis
Buzeika
Hayes
Heirdsfield
McIntosh, J.
Nason
Steinle
Thomas

Numeracy
Bobis
Bobis
Gervasoni
Klein
McIntosh, A.
Sparrow
Swan

Pedagogy
Bramald
Chinnappan
Sullivan
Watts

Politics
Horwood

Post-structuralism
Klein

Primary
Boulton-Lewis
Bramald
Buzeika
Heirdsfield
Lowrie
Maguire

Probability
Moritz
Watson

Problem Solving
Diezmann
Dole
Leung
Lowrie
Maguire
Mitchelmore

Professional Development
Bobis
Gervasoni
Leung
Mayo

Representations
Diezmann

Secondary
Asp
Baker
Brinkworth
Callingham
Cooper
Dole
Galbraith
Goos
Hadfield
Hayes
Keast
Lawrie
Perry
Stillman
Watts

SOLO Taxonomy
Callingham

Statistics
Chick
Gunn
Outhred
Williams

Symbolising
McClain

Teacher Education
Dengate
Lawson

Teacher Perceptions
Tirta Gondoseputro
Buzeika
Chinnappan
Menzel
Malone
Mousley
Shield
Warren
White

Technology
Baturo
Bramald
Dix
Galbraith
Mayo
Mueller
Norton
Pierce
Vincent

Theory
Begg

van Hiele
Lawrie
Vincent

Visual Imagery
Everett
Outhred

Writing
Bicknell